Bridge index for theta curves in the 3-sphere

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Abstract

We define an invariant 'bridge index' for theta curves which are spatial graphs in the 3-sphere. This invariant is a generalization of bridge index for links defined by Schubert. Then we investigate the relationships to other invariants and prove that 2-bridge theta curves are prime. © 1997 Elsevier Science B.V.

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1. Introduction

Notions of bridge presentation and bridge index for links were defined by Schubert [15], and several relations with other notions in knot theory have been investigated. For instance, Jones gave an estimate of bridge index by using Jones polynomial [5].

On the other hand, a theta curve is a spatial graph in the 3-sphere $S^3$ which consists of two vertices and three edges, where each edge joins two vertices. Let $e_i$ ($i = 1, 2, 3$) be an edge of a theta curve $\theta$. Then $e_1 \cup e_2$, $e_2 \cup e_3$ and $e_3 \cup e_1$ become knots in $S^3$, called constituent knots of $\theta$. Thus we can see that a theta curve is a generalization of a knot, and it has some interesting and original characters, e.g., Kinoshita's theta curve (see Fig. 3) has 3 trivial knots as its constituent knots, but is not ambient isotopic to the planar theta curve $\Theta$ [7].

In this paper, we extend the ideas of bridge presentation and bridge index for theta curves, and show some their properties. At first, in Section 2, we define bridge presen-
tation and bridge index for theta curves, and present some examples. In Section 3, we define thin position of theta curves and investigate the relation between thin position and bridge presentation. In fact, we generalize Thompson's result for knots \[171\] to show that, under a condition, thin position of a theta curve is the bridge presentation that realizes bridge index of the theta curve. Note that thin position of knots, which was defined by Gabai \[2\], is a simple and important notion in knot theory. Section 4 is devoted to an estimate of bridge index of theta curves by Yamada's invariant which is an invariant of spatial trivalent graphs. In Section 5, we show that 2-bridge theta curves are prime (for the definition, see Section 2).

2. Preliminaries

Throughout this paper, for a topological space \( B \), \#\( B \) denotes the number of components of \( B \), and \( \hat{B} \) denotes the interior of \( B \). Let \( K \) be a 1-complex contained in \( M \). \( N(K, M) \) or \( N(K) \) denotes a regular neighborhood of \( K \) in \( M \), and \( E(K) \) means the exterior of \( K \) in \( M \).

Let \( \theta \) be a theta curve. Suppose that there are two 3-balls \( B_+, B_- \) such that \( B_+ \cup B_- = S^3 \), both \( \theta \cap B_+ \) and \( \theta \cap B_- \) consist of one unknotted bouquet of arcs and unknotted arcs (see Fig. 1). Then we call this decomposition of \((S^3, \theta)\) a bridge presentation of \( \theta \), and we call the number of components of \( \theta \cap B_+ \) (\( \theta \cap B_- \)) the bridge number. Bridge index of \( \theta \), \( b(\theta) \), is the minimum number of bridge numbers of \( \theta \).

We can easily observe (cf. \[1,15\]):

**Proposition 2.1.** Any theta curve has a bridge presentation.

A theta curve \( \theta \) is trivial if \( \theta \) is ambient isotopic to the planar theta curve.

**Proposition 2.2.** \( b(\theta) = 1 \) if and only if \( \theta \) is trivial.

**Proof.** The 'if' part is trivial. So, suppose that \( b(\theta) = 1 \). Then we obtain a decomposition of \((S^3, \theta)\), \((B_+, \theta \cap B_+) \cup (B_-, \theta \cap B_-)\), ambient isotopic to \((B_1, \theta \cap B_1)\) and \((B_2, \theta \cap B_2)\) as illustrated in Fig. 2. We say that this ambient isotopy is \( f \), and set \( S = B_+ \cap B_- = \partial B_+ = \partial B_- \). Take a small regular neighborhood of \( S \), \( S \times [0, 1] \), such that \( S \times \{1/2\} = S \). We may suppose that \((S \times [0, 1]) \cap \theta = \{3 \text{ points}\} \times [0, 1] \), and that

\[
S^3 = B'_+ \cup_{S \times \{1\}} S \times [0, 1] \cup_{S \times \{0\}} B'_-,
\]

where \( B'_+ \subset B_+ \). By the ambient isotopy \( f : (B_+, \theta \cap B_+) \to (B_1, \theta \cap B_1) \), we can see that \( \theta \cap (S \times [0, 1]) \) are fix in \( \theta \cap (S \times [0, 1/2]) \) and, in \( \theta \cap (S \times (1/2, 1]) \), move with keeping the height of each point of each arc. Thus, after moving \( \theta \) by \( f \), \( \theta \cap (S \times [1/2, 1]) \) becomes a braid. Similarly, \( \theta \cap (S \times [0, 1/2]) \) becomes a braid, too. By Reidemeister moves \( V \) of graphs (Theorem 2.1 in \[6\]), we can get \((B_i, \theta \cap B_i) \) \( (i = 1, 2) \) with fixing \( S \times \{1/2\} \). Then, \( \theta \) is trivial. \( \square \)
Example 2.3. Let $\theta$ be Kinoshita's theta curve (see Fig. 3), rational or pseudo-rational theta curves (see [3]). Then $b(\theta) = 2$. 
Example 2.4. Let \( \theta \) be a Wolcott's theta curve [18], that is, a theta curve whose constituent knots are all trivial knots. (It is a generalization of Kinoshita's theta curve.) Then \( b(\theta) = 2 \).

Example 2.5. Bridge indices of theta curves in Litherland's table [11] are all equal to 2.

Let \( \theta \) be a theta curve and \( S^2 \) a 2-sphere which intersects \( \theta \) in 3 points. If \( S^2 \) decomposes \( S^3 \) into 3-balls \( B_1, B_2 \) such that neither \( \theta \cap B_1 \) nor \( \theta \cap B_2 \) is an unknotted bouquet (Fig. 2), then we say that \( S^2 \) is a decomposing sphere for \( \theta \). If there is no decomposing sphere for \( \theta \), we say that \( \theta \) is prime. Suppose that \( \theta \) is not prime. Then we can see that \( \theta \) is obtained from two theta curves by removing regular neighborhoods of vertices and gluing the remaining 3-balls together in such a way that the 3 points of the theta curve and 3 points of the other theta curve in the boundary of each 3-ball are identified. This operation is called a connected sum. Note that we have 6 different connected sums for any two given vertices and that each of them is well defined up to ambient isotopy (see [18]).

3. Thin position and bridge index of theta curves

Let \( h : S^3 \rightarrow [0,1] \) be a height function, and \( \theta \) a theta curve and \( V = \{v_1, v_2\} \) vertices of \( \theta \), \( \theta \) is in Morse position with respect to \( h \) if

1. on any edge \( e \) of \( \theta \), the critical points of \( h|_e \) are nondegenerate and lie in the interior of \( e \), and
2. the critical points of \( h|_{\theta - V} \) and vertices \( v_1, v_2 \) all occur at different heights.

By usual Morse theory, we have:

**Proposition 3.1.** Any theta curve can be put in Morse position.

**Remark 3.2.** Suppose that \( \theta \) is in Morse position with respect to \( h \). Then we have the following:

1. \( h^{-1}(t) \) \( (t \in (0,1)) \) construct a foliation on \( S^3 - \{2 \text{ points}\} \) by 2-spheres,
2. \( \theta \) intersects all but a finite number of the leaves (2-spheres) transversely,
3. it fails to be transverse to any given leaf at at most one point.

In this article, we suppose that the value \( t \) such that \( v_i \in h^{-1}(t) \) is a critical value. The vertices \( V \) of \( \theta \) can be classified into four types (see Fig. 4):

- **A:** \( v_i \) in \( V \) so that exactly two ends of incident edges lie below \( v_i \);
- **B:** \( v_i \) in \( V \) so that exactly two ends of incident edge lie above \( v_i \);
- **C:** \( v_i \) in \( V \) so that all ends of incident edges lie below \( v_i \);
- **D:** \( v_i \) in \( V \) so that all ends of incident edges lie above \( v_i \).

Note that we can transfer a neighborhood of a vertex of type C (type D respectively) into it of type A (type B respectively) by an ambient isotopy. Then, in what follows, we assume that the neighborhood of a vertex is of type A or type B. A theta curve
in Morse position of which vertices are of type A or B is said to be in \textit{normal form}. Let $e_i$ ($i = 1, 2 \text{ or } 3$) be edges of a theta curve $\theta$. We denote the constituent knots $e_1 \cup e_2$, $e_2 \cup e_3$ and $e_3 \cup e_1$ by $\theta^1$, $\theta^2$ and $\theta^3$ respectively. Note that a vertex $v_i$ ($i = 1, 2$) becomes a critical point for exactly one constituent knot, and is not a critical point for the other two constituent knots.

Now, we define a notion of thin position of $\theta$. Assume that $\theta$ is in Morse position with respect to $h$. For each value of $t$, let $w(t) = \#(Q \cap h^{-1}(t))$, and $W_\theta$ denotes the largest value of $w(t)$. Let $n_\theta = \sum_{j=1}^{3} \sum_{t_{ij}} \#(\theta^j \cap h^{-1}(t_{ij}))$, where $t_{ij}$ is a regular value just prior to the critical values for $\theta^j$. $\theta$ is in \textit{thin position} if among all normal forms obtained from $\theta$ by ambient isotopies, $(W_\theta, n_\theta)$, lexicographically ordered, is minimized.

A meridional planar surface $F$ in the complement of $\theta$, $E(\theta)$, is a planar surface properly embedded in $E(\theta)$ with meridional boundary components. We say that a boundary parallel surface with meridional boundary components is a \textit{trivial} meridional planar surface.

\textbf{Theorem 3.3.} Suppose that a theta curve $\theta$ is in thin position and that the complement of $\theta$ contains no nontrivial incompressible meridional planar surface. Then we have:

$$b(\theta) = \begin{cases} \frac{1}{2}(W_\theta - 1) & \text{if } W_\theta \text{ odd}, \\ \frac{1}{2}W_\theta & \text{if } W_\theta \text{ even}. \end{cases}$$

Before we prove this theorem, we prepare definitions and lemmas.

$S$ is a \textit{thin 2-sphere} of $\theta$ with respect to the height function $h$ if $S = h^{-1}(t_0)$ for some regular value $t_0$ where $t_0$ lies between adjacent critical values $x$ and $y$ of $h$, where $x$ is a minimum of $\theta$ lying above $t_0$ and $y$ is a maximum of $\theta$ lying below $t_0$. Note that a minimum and a maximum of $\theta$ do not contain the heights of vertices of $\theta$ and that vertices may be in between $x$ and $y$.

\textbf{Definition 3.4.} Let $t$ be a regular value, and $P = h^{-1}(t) - \hat{N}(\theta, S^3)$ a meridional planar surface. We say that $D$ is a \textit{strict upper} (lower respectively) \textit{disk} for $P$ if $D$ satisfies the following conditions:
(i) $\partial D = \alpha \cup \beta$ ($\partial \alpha = \partial \beta$), where $\alpha$ is an arc properly embedded in $P$, $\beta$ is an embedded arc in $\partial N(\theta)$, and there is a disk $E$ in $N(\theta)$ such that $\partial N(\theta) \cap E = \beta$ and that $E \cap \theta (= \partial E \cap \theta)$, say $\beta'$, is a subarc of $e_i$ ($i = 1, 2$ or 3) parallel with $\beta$. (see Fig. 5),

(ii) $\hat{D} \cap P = \emptyset$ or simple closed curves,

(iii) $N(\alpha, D)$ lies on the side of $P$ containing $h^{-1}(1)$ ($h^{-1}(0)$ respectively).

**Lemma 3.5** [17, Lemma 2]. If $\theta$ is in thin position and there is a thin 2-sphere $S$ for $\theta$, with respect to the height function $h$. Then, there is no strict upper or lower disk for the meridional planar surface $P = S - \hat{N}(\theta)$.

**Proof.** Suppose that there is a strict upper disk $D$ for $P$ with $\partial D = \alpha \cup \beta$ as in Definition 3.4. $\beta'$ must contain at least one maximum of $\theta$, so we denote the highest maximum point in $\beta'$ by $b$. Consider that a theta curve $\tilde{\theta} = (\theta - \beta') \cup \tilde{\beta}$, where $\tilde{\beta}$ is an arc from the one component of $\partial \beta'$ to the other component of $\partial \beta'$ and has only one maximum point $b$. Then we have $(W_{\theta}, n_{\theta}) \geq (W_{\tilde{\theta}}, n_{\tilde{\theta}})$. Next, we consider that we transform a part of $\theta$ by sliding $\beta'$ to $\alpha$ on $E \cup D$ so that we get a theta curve $\tilde{\theta}$. We may suppose that $\tilde{\theta}$ is in normal form by modifying it slightly. In this isotopy, $W_{\theta}$ never increase. Let $x$ be a minimum point of $\theta$ for $S$ (we abbreviate the minimum point ($\in \theta \cap h^{-1}(x)$) to $x$). If $x \in \beta'$, $(W_{\theta}, n_{\theta}) > (W_{\tilde{\theta}}, n_{\tilde{\theta}})$: a contradiction. Then we assume that $x \notin \beta'$.

**Case 1.** $x \in e_1$ and $\beta' \subset e_1$. In this case,

$$\sum \#(\tilde{\beta} \cap h^{-1}(t_{i_j})) > \sum \#(\beta \cap h^{-1}(t_{i_j}))$$

for $j = 1, 3$, and

$$\sum \#(\beta^2 \cap h^{-1}(t_{i_2})) = \sum \#(\tilde{\beta}^2 \cap h^{-1}(t_{i_2})).$$

Then $(W_{\tilde{\theta}}, n_{\tilde{\theta}}) > (W_{\tilde{\theta}}, n_{\tilde{\theta}})$. Since $\tilde{\theta}$ is ambient isotopic to $\theta$, this contradicts the assumption.
Fig. 6.

Case 2. $x \in e_1$, $\beta' \subset e_2$. In this case, we have:
\[
\sum \#(\tilde{\theta}^1 \cap h^{-1}(t_{i1})) > \sum \#(\tilde{\theta}^1 \cap h^{-1}(t_{i1})),
\]
\[
\sum \#(\tilde{\theta}^2 \cap h^{-1}(t_{i2})) \geq \sum \#(\tilde{\theta}^2 \cap h^{-1}(t_{i2})),
\]
\[
\sum \#(\tilde{\theta}^3 \cap h^{-1}(t_{i3})) = \sum \#(\tilde{\theta}^3 \cap h^{-1}(t_{i3})).
\]
Thus, $n_{\theta}$ is reducing and we have a contradiction.

Case 3. $x \in e_1$, $\beta' \subset e_3$. By the same argument as in Case 2, we have a contradiction.

In any case, there is no problem if there is a vertex of $\theta$ between the height of $P$ and that of $b$.

**Definition 3.6.** Let $t$, $P$ be as in Definition 3.4. We say that $(D^1, D^2)$ is a strict upper (lower respectively) disk pair of type $I$ for $P$ if $(D^1, D^2)$ satisfies the following conditions:

(i) $\partial D^i = \alpha^i \cup \beta^i$ ($\partial \alpha^i = \partial \beta^i$), where $\alpha^i$ ($i = 1, 2$) is an arc properly embedded in $P$, $\beta^i$ ($i = 1, 2$) is an embedded arc in $\partial N(\theta)$, and there is a disk $E$ in $N(\theta)$ such that $\partial N(\theta) \cap E = \beta^1 \cup \beta^2$ and that $E \cap \theta$ is a subarc of $e_1$, $e_2$ and $e_3$ (two of them are contained in $\partial E$) and $E \cap \theta \supset v_j$ ($j = 1$ or $2$) (see Fig. 6),

(ii) $D^1 \cap P = \phi$ or simple closed curves, and $D^1 \cap D^2 = \phi$,

(iii) both $N(\alpha^1, D^1)$ and $N(\alpha^2, D^2)$ lie on the side of $P$ containing $h^{-1}(1)$ ($h^{-1}(0)$ respectively).
Lemma 3.7. If $\theta$ is in thin position and there is a thin 2-sphere $S$ for $\theta$, with respect to the height function $h$. Then, there is no strict upper or lower disk pair of type I for $P = S - \hat{N}(\theta)$.

Proof. Suppose that there is a strict upper disk pair of type I, $(D^1, D^2)$, for $P$ with $\partial D^i = \alpha^i \cup \beta^i$, and $E$ as in Definition 3.6. Let $v_1$ be a vertex of $\theta$ contained in $\partial E$. We suppose that $e_1$ starts above from $v_1$, a subarc of $e_2$ is contained in $\hat{E}$ and a subarc of $e_3$ and a subarc of $e_3$ are contained in $\partial E$ (see Fig. 6). We denote the subarcs of $e_i$ by $\beta_i (i = 1, 2, 3)$. We consider that we transform a part of $\theta$ by sliding $\beta^1 \cup \beta^2$ to $\alpha^1 \cup \alpha^2$ on $D^1 \cup D^2$ together with $\beta_1, \beta_2$ and $\beta_3$ so that we get $\hat{\theta}$. We may suppose that $\hat{\theta}$ is in normal form by modifying it slightly.

Case 1: $v_1$ is of type A. $\beta_1$ must contain at least one maximum of $\theta$, so we denote the highest maximum point in $\beta_1$ by $b_1$. Consider that a theta curve
\[
\hat{\theta} = \left( \theta - \left( \bigcup_{i=1}^{3} \beta_i \right) \right) \cup \left( \bigcup_{i=1}^{3} \hat{\beta}_i \right)
\]
such that $\hat{\beta}_i$ is an arc from $v_1$ to $\beta_1 \cap P$ and has only one maximum point $b_1$ and that $\beta_i (i = 2, 3)$ is an arc from $v_1$ to $\beta_i \cap P$ and has no critical point with respect to $h$. Then we have $(W_{\theta}, n_{\theta}) \geq (W_{\hat{\theta}}, n_{\hat{\theta}})$, and by similar argument to in the proof of Lemma 3.5, we have $(W_{\theta}, n_{\theta}) > (W_{\hat{\theta}}, n_{\hat{\theta}})$; a contradiction. Here, there is no problem if there is the vertex $v_2$ of $\theta$ between the height of $P$ and that of $b_1$.

Case 2: $v_1$ is of type B. $\beta_2$ or $\beta_3$ must contain at least one maximum of $\theta$. We suppose that $\beta_3$ contains the maximum point, and denote the highest maximum point in $\beta_3$ by $b_4$. The case that $\beta_2$ contains the maximum point can be proved by the same method as follows. Let $\tilde{\beta} = (\theta - \beta_3) \cup \beta_3$ such that $\tilde{\beta}_3$ is an arc from $v_1$ to $\beta_3 \cap P$ and has only one maximum point $b_3$. Then we have $(W_{\theta}, n_{\theta}) \geq (W_{\tilde{\beta}}, n_{\tilde{\beta}})$. Next, let $\tilde{\theta} = (\theta - \beta_3) \cup \beta_3$ such that $\tilde{\beta}_3$ is an arc from $v_1$ to $\beta_3 \cap P = \beta_3 \cap P$ and has no critical point. The vertex $v_1$ of $\tilde{\theta}$ becomes of type A, and we have $(W_{\theta}, n_{\theta}) > (W_{\tilde{\theta}}, n_{\tilde{\theta}})$. Note that we can obtain $\hat{\theta}'$ from $\theta$ by sliding $\beta_3$ on $E_{1} \cap D^2$. Hence we have a contradiction by the argument of Case 1. \qed

Definition 3.8. Let $t, P$ be as in Definition 3.4. We say that $(D^1, D^2, D^3)$ is a strict upper (lower respectively) disk triple for $P$ if $(D^1, D^2, D^3)$ satisfies the following conditions:

(i) $\partial D^i = \alpha^i \cup \beta^i (\partial \alpha^i = \partial \beta^i)$, where $\alpha^i (i = 1, 2, 3)$ is an arc properly embedded in $P$, $\beta^i (i = 1, 2, 3)$ is an embedded arc in $\partial N(\theta)$, and there is a disk $E$ in $N(\theta)$ such that $\partial E \cap E = \bigcup_{i=1}^{3} \beta^i, E \cap \theta$ consists of $e_i$ and subarcs of $e_j$ and $e_k (i, j, k) = (1, 2, 3) \sigma$ so that $\partial E$ contains $v_1$ and $v_2$ (see Fig. 7),

(ii) $\tilde{D} \cap P = \phi$ or simple closed curves, and $D^1, D^2$ and $D^3$ are mutually disjoint,

(iii) all $N(\alpha^i, D^i) (i = 1, 2, 3)$ lie on the side of $P$ containing $h^{-1}(1)$ ($h^{-1}(0)$ respectively).

By the same argument as in the proof of Lemma 3.7, we have:

Lemma 3.9. If $\theta$ is in thin position and there is a thin 2-sphere $S$ for $\theta$, with respect to the height function $h$. Then there is no strict upper or lower disk triple for $P = S - \hat{N}(\theta)$.
Proof of Theorem 3.3. Suppose that $\theta (\subset S^3)$ is in thin position with respect to a height function $h$. If we suppose that there is no thin 2-sphere with respect to $h$, we can get a decomposition $(S^3, \theta) = (B_1, \theta \cap B_1) \cup (B_2, \theta \cap B_2)$ as in Fig. 8:

**Case A:** this is a bridge presentation, and $b(\theta) = (W_\theta - 1)/2$ since $\theta$ is in thin position.

**Cases B, C, D:** let $\gamma$ be a subarc of $e_i$ ($i = 1, 2$ or 3) such that one component of $\partial \gamma$ is in $\partial B_1$ and another is $v_j$ ($j = 1$ or 2) (see Fig. 8), and let $B'_i = B_1 - \hat{N}(\gamma, B_1)$ and $B'_2 = B_2 \cup N(\gamma, B_1)$. Then $(D'_1, \theta \cap D'_1) \cup (D'_2, \theta \cap D'_2)$ becomes a bridge presentation. Further, $b(\theta) = W_\theta/2$ (or $(W_\theta - 1)/2$ in Case D) since $\theta$ is in thin position.

Next, we suppose that there is a thin 2-sphere with respect to $h$, and set $P = S - \hat{N}(\theta)$. Compress $P$ as much as possible in $E(\theta)$. Then we get the collection of meridional planar surfaces, $\tilde{P}$, and each component of $\tilde{P}$ is incompressible in $E(\theta)$. Let $P'$ be a component of $\tilde{P}$ such that $P'$ bounds a submanifold of $S^3 - \hat{N}(\theta)$ with interior disjoint from $\tilde{P}$. Here we can consider the following four cases:

**Case 0:** $P'$ is a nonboundary parallel incompressible meridional planar surface in $E(\theta)$.
**Case 1:** $P'$ is a boundary parallel once punctured disk.
**Case 2:** $P'$ is a boundary parallel two punctured disk.
**Case 3:** $P'$ is a boundary parallel three punctured disk.

If Case 0 occurs, we are done. Assume that Case 1 occurs. Then we can consider a disk $D$ such that $\partial D$ consists of two arcs $\alpha$ and $\beta$, $\partial \alpha = \partial \beta$, with $\alpha$ a properly embedded essential arc in $P'$, $\beta$ an arc embedded on $\partial \hat{N}(\theta)$. Moreover, $\hat{D}$ is disjoint from $P'$. By reversing the compressions on $\tilde{P}$ to re-assemble $P$, $D$ becomes a strict upper or lower disk for $P$. This contradicts Lemma 3.5.

By similar reasons, Cases 2, 3 contradict of Lemmas 3.7, 3.9 respectively. Hence, we have Theorem 3.3. □
4. An estimate by Yamada’s invariant

In this section, we give an estimate of bridge index by Yamada’s invariant. Firstly, we recall some notations and definitions following [12,19].

A triple \((a, b, c)\) of nonnegative integers is said to be \textit{admissible} if

\[
|a - b| \leq c \leq a + b, \quad a + b + c \in 2\mathbb{Z}.
\]
The condition is equivalent to the condition that there are nonnegative integers \( x, \ y, \) and \( z \) such that

\[
x + y = a, \quad y + z = b, \quad z + x = c.
\]

Let \( A \) be a unit complex number. We put

\[
\Delta_m = (-1)^m \frac{A^{2(m+1)} - A^{-2(m+1)}}{A^2 - A^{-2}},
\]

\[
\Gamma(x, y, z) = \frac{\Delta_{x+y+z}! \Delta_{x-1}! \Delta_{y-1}! \Delta_{z-1}!}{\Delta_{y+z-1}! \Delta_{x+z-1}! \Delta_{x+y-1}!}.
\]

Here, \( \Delta_m! = \Delta_m \Delta_{m-1} \cdots \Delta_0 \) and \( \Delta_{-1} = 1 \). Moreover, we put \( \Delta_{a,b,c} = \Gamma(x, y, z) \).

Let \( \theta \) be a theta curve with edges \( e_1, \ e_2 \) and \( e_3 \), and \( \omega \) a weight of \( \theta \), that is, \( \omega \) is a map from the set of edges to the set of nonnegative integers. We denote by \( Y_{\theta, a, b, c}(A) \) Yamada’s invariant for \( \theta \) with \( \omega(e_1) = a, \ \omega(e_2) = b \) and \( \omega(e_3) = c \). Here, \( (a, b, c) \) must be admissible. Then we have:

**Theorem 4.1.** Suppose that \((B_1, \theta \cap B_1) \cup (B_2, \theta \cap B_2)\) is a bridge presentation realizing \( b(\theta) \) and that \( \#(e_i \cap B_1) = n_i + 1 \) \((i = 1, 2, 3) \). (Hence, \( b(\theta) = \sum_{i=1}^{3}(n_i) + 1 \).) Then we have:

\[
|Y_{\theta, e_1, e_2, e_3}(A)| \leq \left| \Delta_{\ell_1, \ell_2, \ell_3} \prod_{i=1}^{3} (\Delta_{\ell_i})^{n_i} \right| \text{ for } |\psi| < \pi / \left( \sum_{i=1}^{3} ((2n_i + 1)\ell_i) + 2 \right).
\]

This theorem is a generalization of Theorem 3.2(2) in [12], and is obtained by the same method as its proof. So, we omit the proof here.

**Theorem 4.2.** Let \( \theta \) be a theta curve obtained from two theta curves \( \theta_1 \) and \( \theta_2 \) by a connected sum. Then we have:

\[
Y_{\theta, e_1, e_2, e_3}(A) = \frac{1}{\Delta_{\ell_1, \ell_2, \ell_3}} Y_{\theta_1, e_1, e_2, e_3}(A) Y_{\theta_2, e_1, e_2, e_3}(A).
\]

**Proof.** We consider a resolution of \( \theta_1 \) under the defining equation of Yamada’s invariant:

\[
\begin{array}{c}
\includegraphics{resolution_1} \\
\end{array} = A \begin{array}{c}
\includegraphics{resolution_2} \\
\end{array} + A^{-1} \begin{array}{c}
\includegraphics{resolution_3} \\
\end{array}. 
\]

By Lemma 1(i) in [9], we may have that each terminal of the resolution is only one component \( \hat{\theta} \) as in Fig. 9. If we get a diagram as in Fig. 10 on the way of the resolution, we shall ignore the crossing until the end and use the formula

\[
\begin{array}{c}
\includegraphics{crossing_1} \\
\end{array} = -A^3 \begin{array}{c}
\includegraphics{crossing_2} \\
\end{array}.
\]

Then, there is a polynomial \( f(A) \) such that \( Y_{\hat{\theta}, e_1, e_2, e_3}(A) = f(A) \cdot \Delta_{\ell_1, \ell_2, \ell_3} \) since \( Y_{\hat{\theta}, e_1, e_2, e_3} = \Delta_{\ell_1, \ell_2, \ell_3} \) by Lemma 1 in [10]. We have this theorem by considering a resolution of \( \theta \) similarly. \( \square \)
Example 4.3. Let \( \theta \) be a theta curve as illustrated in Fig. 11, then \( b(\theta) = 3 \).

Proof. We can see that \( \theta \) is obtained from two Kinoshita’s theta curves \( \theta_1, \theta_2 \) (see Fig. 3) by a connected sum.

\[
Y_{\theta_1,2,1,1}(A) = Y_{\theta_2,2,1,1}(A) = (A^{11} - A^7 + A^3 - A^{-5} + 2A^{-9} - A^{-13}) \Delta_2.
\]

Then, by Theorem 4.2, we have:

\[
Y_{\theta,2,1,1}(A) = (A^{11} - A^7 + A^3 - A^{-5} + 2A^{-9} - A^{-13})^2 \Delta_2.
\]
since $\Delta_2 = \Delta_{2,1,1}$. Suppose that $A = c^{\pi i/10.2}$, then

$$|Y_{2,1,1}(A)| = 2.83 \ldots > |\Delta_{2,1,1}\Delta_2| = 2.77 \ldots, \quad |\Delta_{2,1,1}\Delta_1| = 2.71 \ldots.$$

Thus we have $b(\theta) > 2$. On the other hand, we can verify that $b(\theta) \leq 3$ by a straightforward observation. Then we have this example. $\square$

5. 2-bridge theta curves are prime

In this section, we show:

**Theorem 5.1.** 2-bridge theta curves are prime.

Before we prove this theorem, we prepare some definitions and lemmas.

Let $M$ be an oriented 3-manifold and $F$ a properly embedded surface in $M$. $\theta$ denotes an embedded 1-complex in $M$. An arc or simple closed curve in $F$ is $\theta$-essential if it is essential in the punctured surface $F - \theta$. If there is a $\theta$-essential simple closed curve $c$ in $F$ which bounds a disk in $M - F$ disjoint from $\theta$, the $F$ is called $\theta$-compressible. Otherwise, $F$ is $\theta$-noncompressible. $M$ is $\theta$-irreducible if any 2-sphere disjoint from $\theta$ in $M$ bounds a 3-ball disjoint from $\theta$. $F$ is $\theta$-compressible if there is a disk $D$ in $M - \theta$ for which $D \cap (\partial M \cup F) = \partial D$ is the union of an arc in $\partial M$ and a $\theta$-essential arc in $F$. $\theta$-compression means $\partial$-compression along this disk $D$. Let $c$ and $c'$ be simple closed curves in $\partial M$. $c$ is parallel with $c'$ if there exists an annulus $A$ embedded in $\partial M - \theta$ such that $\partial A = c \cup c'$. Let $A$ be an annulus properly embedded in $M$. We call that $A$ is boundary parallel if there exists an annulus $A'$ embedded in $\partial M - \theta$ such that $A \cup A'$ bounds a solid torus disjoint from $\theta$ in $M$.

Let $\theta$ be a 2-bridge theta curve, and $(B_1, \theta \cap B_1) \cup (B_2, \theta \cap B_2)$ denotes a 2-bridge presentation of $\theta$. Put $\theta_i = \theta \cap B_i \ (i = 1, 2)$ and we label each edge $e_1$, $e_2$, $e_3$ as illustrated in Fig. 12, i.e., $\#(e_1 \cap B_1) = \#(e_1 \cap B_2) = 2$.

Assume that there is a decomposing sphere $S$ for $\theta$.

**Lemma 5.2.** We may suppose that each component of $B_1 \cap S$ is a disk disjoint from $\theta_1$ by moving $\partial B_1 \ (= \partial B_2)$ by an isotopy.
Proof. Let $\gamma$ be an unknotted arc in $B_1$ joining with the vertex of bouquet component of $\theta_1$ and a point of the arc component of $\theta_1$ (see Fig. 12). By standard general position argument, we may suppose that $\gamma$ meets $S$ transversely. Then we may also suppose that $N(\gamma, B_1) \cap S = N(\gamma \cap S, S)$. $\partial B_1$ is isotopic to $\partial N(\gamma, B_1)$ with keeping this 2-bridge presentation. Hence, if $\gamma \cap S \neq \emptyset$, we have this lemma. If $\gamma \cap S = \emptyset$, $S$ is contained in $B_2$. This contradicts the definition of $S$. \hfill \Box

We assume that $\sharp(B_1 \cap S) \geq 1$ is minimum by an isotopy of $S$.

Lemma 5.3. $B_2 \cap S$ is $\theta_2$-incompressible in $B_2$.

Proof. If not, there is a $\theta_2$-compressing disk $D$ in $B_2$. We cut $B_2 \cap S$ along $\partial D$ so that we can have two planar surfaces $S_1$ and $S_2$. Since $S$ is a decomposing sphere, $\theta_2 \cap (B_2 \cap S)$ consists of 3 points, called $p_i \in (e_i (i = 1, 2, 3))$.

Claim. We may suppose that $S_1 \supset \bigcup_{i=1}^{3} p_i$.

Proof. If $S_2 \supset \bigcup_{i=1}^{3} p_i$, we change the labeling. Suppose that $p_1 \subset S_1$ and $p_2 \subset S_2$. Then we have a 2-sphere $S^2$ which consists of $S_1$, $D$ and some components of $B_1 \cap S$ such that $S^2 \cap \theta = p_1$ or $p_1 \cup p_3$. This contradicts the definition of the theta curve $\theta$. Hence we have this claim. By this claim, we can see that $S_2 \cup (\text{some components of } B_1 \cap S)$ becomes a disk $\tilde{D}$ disjoint from $\theta$ so that $D \cup \tilde{D}$ is a 2-sphere disjoint from $\theta$. Since $E(\theta)$ is irreducible, we get an isotopy of $S$ which reduces $\sharp(B_1 \cap S)$. This contradicts the assumption. \hfill \Box

Let $E_i \ (i = 1, 2, 3)$ be an embedded disk which is bounded by the components of $\theta$ and $\partial B_2$ in $B_2$, and $E$ a properly embedded disk in $B_2$ which contains $\bigcup_{i=1}^{3} E_i$ as illustrated in Fig. 12. Further, we denote points of $\partial(\theta \cap B_1)$ and $\partial(\theta \cap B_2)$ by...
q₁, q₁', q₁'' (∈ e₁), q₂ (∈ e₂), q₃ (∈ e₃) as in Fig. 12. By using innermost circle argument and Lemma 5.3, we may suppose that there is no circle component of \( E \cap (B_2 \cap S) \) since \( B_2 \) is \( \theta_2 \)-irreducible. If the circle meets \( \theta_2 \), we can get a contradiction by using the disk of the frontier of \( N(E, B_2) \) and Lemma 5.3 since \( \#(B_1 \cap S) \geq 1 \). Here, we denote \( B_i \cap S \) by \( S_i^0 \) \((i = 1, 2)\). By using disk \( E' \) (and the disk of frontier of \( N(E, B_2) \), if necessary), we have a hierarchy of \( \partial_{\theta_2} \)-compressions:

\[
S_1^0 \rightarrow S_1^1 \rightarrow \cdots \rightarrow S_1^m, \quad S_2^0 \rightarrow S_2^1 \rightarrow \cdots \rightarrow S_2^m,
\]

where each component of \( S_i^m \) is a disk.

Suppose that \( \#S_2^m = 1 \). Then, by two more times \( \partial_{\theta_2} \)-compressions along disk which meets \( E \) transversely (see Fig. 13), we get a hierarchy \( S_2^m \rightarrow S_2^{m+1} \rightarrow S_2^{m+2} \) such that each component of \( S_2^{m+2} \) is a disk and \( \#S_2^{m+2} = 3 \). Then, we assume that each component of \( S_2^m \) is a disk and that \( \#S_2^m > 1 \) from now on.

**Lemma 5.4.** There is a number \( \ell \) \((1 \leq \ell \leq m)\) such that \( S_1^{\ell-1} \) contains a disk component and \( S_1^\ell \) contains no disk component (see Fig. 14).

**Proof.** If not, \( S_1^m \) contains a disk since each component of \( S_1^0 \) is a disk. Then we have a contradiction since \( S \) is a 2-sphere, and each component of \( S_2^m \) is a disk and \( \#S_2^m > 1 \).

**Lemma 5.5.** Each component of \( S_1^\ell \) is an annulus or a disk with two holes.

**Proof.** \( S_1^{\ell-1} \) contains a disk component and this disk separates two components of \( \theta_1 \). Then, under the partial hierarchy \( S_1^0 \rightarrow \cdots \rightarrow S_1^\ell \), we may suppose that a band which appears in each \( \partial_{\theta_2} \)-compression joins one component. If not, we can reduce \( \#S_1^0 \) by considering the reversing \( \partial_{\theta_2} \)-compressions or we get a punctured torus. Both of them contradict the assumption. Thus we have proved this lemma.
Lemma 5.6. Each boundary of $S_1^k$ separates $\partial B_1$ into two disks; either one contains only one point of $\partial \theta_1$ and the other contains the remaining points of $\partial \theta_1$, or one contains $q_i$ and $q_j$ ((i, j) = (1, 2), (2, 3), (3, 1)) and the other contains the remaining points of $\partial \theta_1$.

By Lemma 5.5, we can see that $S_1^k$ contains a disk component since $S$ is a 2-sphere.

Lemma 5.7. Each disk component of $S_2^k$ contains $p_i$ ($i = 1, 2$ or 3).

Proof. If not, this disk, say $D$, separates two components of $\theta_2$. $\partial D$ separates $\partial B_2$ into two disks; one disk contains $q_1'$, $q_2$, $q_3$ and the other contains $q_1$, $q_1'$. Then, by Lemma 5.6, we have a contradiction.  

If $S_2^k$ has a boundary parallel annulus component, we can reduce $\#S_1^0$. So, we have:

Lemma 5.8. $S_2^k$ has no boundary parallel annulus.

Let $I$ be a disk in $\partial B_2$ whose boundary is the boundary of a disk component $D$ of $S_2^k$. Suppose that $\#(D \cap \theta_2) = 1$ and that $I$ contains only one point of $\partial \theta_2$.

By Lemmas 5.3, 5.8, and Lemma 3.1 in [8], we can prove the next lemma (see also III.13 in [4]):

Lemma 5.9. $I \cap S_2^k = \emptyset$.

Lemma 5.10. The disk $I$ does not contain the point $q_1'$ or $q_1''$.

Proof. By Lemma 5.9, we can push the whole $D$ towards $\partial B_2$ so that we get the component $F$ of $S_1^k$ such that $F \cap \theta_1 = 1$ pt. By considering the reversing $\partial$-compressions
of $S^f_1$, we have $\#S^0_1 - 1$ disk components disjoint from $\theta_1$ and one component disk which meets arc component of $\theta_1$ once. By pushing this disk toward $\partial B_1$, $\#S^0_1$ can be reduced. This contradicts the assumption. \qed

By Lemmas 5.5, 5.7 and the fact that $S$ is a 2-sphere, we have:

**Lemma 5.11.** $S^f_1$ has at most 1 disk with two holes component.

**Proof of Theorem 5.1.**

**Case A:** $S^f_1$ has just 1 disk with two holes. In this case, the components of $S^f_2$ are annuli and three disks $D_1$, $D_2$ and $D_3$, say. We may suppose that $D_i$ contains $p_i$ ($i = 1, 2, 3$) by Lemma 5.7. Here we have three cases as follows:

**Case 1:** $\partial D_1$ bounds a disk $I_1$ on $\partial B_2$ such that $I_1 \cap \theta_2 = q_1$, and $\partial D_j$ bounds a disk $I_j$ on $\partial B_2$ such that $I_j \cap \theta_2 = q_j$ ($j = 2, 3$).

**Case 2:** $\partial D_1$ bounds a disk $I_1$ on $\partial B_2$ such that $I_1 \cap \theta_2 = q_1$, $\partial D_2$ ($\partial D_3$ respectively) bounds a disk $I_2$ ($I_3$ respectively) on $\partial B_2$ such that $I_2 \cap \theta_2 = q_2$ ($I_3 \cap \theta_2 = q_3$ respectively), and $\partial D_3$ ($\partial D_2$ respectively) bounds a disk $I_1$ ($I_2$ respectively) on $\partial B_2$ such that $I_3 \cap \theta_2 = q_3 \cup q_1 \cup q'_1$ ($I_2 \cap \theta_2 = q_2 \cup q_1 \cup q'_1$ respectively).

**Case 3:** $\partial D_1$ bounds a disk $I_1$ on $\partial B_2$ such that $I_1 \cap \theta_2 = q_1 \cup q'_1 \cup q''_1$, and $\partial D_j$ bounds a disk $I_j$ on $\partial B_2$ such that $I_j \cap \theta_2 = q_j$ ($j = 2, 3$). (See Fig. 15.)

**Case 1.** By Lemma 5.9, we can push the whole $D_1$, $D_2$ and $D_3$ toward $\partial B_2$ so that we get the 2-sphere $S$ in $B_1$. This contradicts the definition of $S$.

**Case 2.** $\partial D_3$ ($\partial D_2$ respectively) separates $\partial B_2$ into two disks; one contains $q''_1$, $q_2$ ($q''_1$, $q_3$ respectively) and the other contains $q_1$, $q'_1$, $q_3$ ($q_1$, $q'_1$, $q_2$ respectively). This contradicts Lemma 5.6.

**Case 3.** By cutting $B_2$ along $D_1$, we get the 3-ball $B_2'$ which contains two components $\theta_1$, $\theta_2$ of $\theta_2$. We can see that $B_2' - \hat{N}(\theta_1, B_2')$ is a genus two handlebody by the existence of disks $E_1 \cap B_2'$ and $E_3$. Nonboundary parallel, incompressible annuli properly embedded in a genus two handlebody are classified in [8], and then we can see that an annulus component of $S^f_2$ is parallel to either the frontier of $N(\theta_1, B_2')$ or that of $N(\theta_2, B_2')$. By the construction of $S^f_1$, the circles $\partial S^f_1 = \partial S^f_2$ satisfy the following:

$\#(\text{the circle which bounds a disk } I'_1 \text{ on } \partial B_1 \text{ such that } I'_1 \cap \theta_1 = q'_1)$
and
\[ \#(\text{the circle which bounds a disk } I'_{1} \text{ on } \partial B_{1} \text{ such that } I'_{1} \cap \theta_{1} = q'_{1}) \]
\[ \#(\text{the circle which bounds a disk } \bar{I}_{1} \text{ on } \partial B_{1} \text{ such that } \bar{I}_{1} \cap \theta_{1} = q_{1}) \]
\[ \#(\text{the circle which bounds a disk } I_{23} \text{ on } \partial B_{1} \text{ such that } I_{23} \cap \theta_{1} = q_{2} \cup q_{3} \]

\[ (\text{the circle parallel with } \partial D_{1})) + 1. \]

Then, we have
\[ \#(\text{the circle which bounds a disk } \bar{I}_{1} \text{ on } \partial B_{1} \text{ such that } \bar{I}_{1} \cap \theta_{1} = q_{1}) \]
\[ \#(\text{the circle which bounds a disk } I'_{1} \text{ on } \partial B_{1} \text{ such that } I'_{1} \cap \theta_{1} = q'_{1}) \]
\[ \#(\text{the circle which bounds a disk } I'_{23} \text{ on } \partial B_{1} \text{ such that } I'_{23} \cap \theta_{1} = q'_{1}) \]
\[ \#(\text{the circle parallel with } \partial D_{1})) + 1; \]

a contradiction since
\[ \#(\text{the circle which bounds a disk } I''_{1} \text{ such that } I''_{1} \cap \theta_{1} = q''_{1}) \]
\[ < \#(\text{the circle which bounds a disk } I_{23} \text{ such that } I_{23} \cap \theta_{1} = q_{2} \cup q_{3}) \text{ on } \partial B_{2}. \]

Case B. $S_{1}^{f}$ has no disk with two holes. In this case, we have two cases as follows:

Case I: the components of $S_{2}^{f}$ are a disk with two holes, annuli and three disks.

Case II: the components of $S_{2}^{f}$ are annuli and two disks $D_{1}$ and $D_{2}$, say.

Case I. We can have the same cases as in Case A. We can get a contradiction in Case 2 for the same reasons as in Case A. Cases 1 and 3 contradict that $S_{1}^{f}$ has no disk with two holes.

Case II. Here, by Lemma 5.7, we have two cases as follows:

Case II-1. $D_{1} \cup D_{2}$ contains just two points of $p_{i}$ ($i = 1, 2, 3$).

Case II-2. $D_{1} \cup D_{2}$ contains all $p_{i}$ ($i = 1, 2, 3$).

Case II-1-1. $D_{1} \ni p_{1}, \ D_{2} \ni p_{2} \ (D_{2} \ni p_{3}$ respectively).

By Lemmas 5.6, 5.10 and the assumption that $S_{1}^{f}$ has no disk with two holes, $\partial D_{1}$ bounds a disk $I$ on $\partial B_{2}$ such that $I \cap \partial \theta_{2} = q_{2} \ (q_{2}$ respectively) or $q_{1} \cup q_{3} \ (q_{1} \cup q_{2}$ respectively). Both of them contradict that $D_{1} \cap \theta_{2} = p_{1}$.

Case II-1-2. $D_{1} \ni p_{2}, \ D_{2} \ni p_{3}$.

This case contradicts the assumption that $S_{1}^{f}$ has no disk with two holes.

Case II-2-1. $D_{1} \ni p_{1}, \ D_{2} \ni p_{2} \cup p_{3}$.

If $\partial D_{1}$ bounds a disk $I_{1}$ on $\partial B_{2}$ such that $I_{1} \cap \partial \theta_{2} = q_{1} \cup q_{1}' \cup q_{1}''$, we get a contradiction for the similar argument to that in Case A-3. Therefore, we may suppose that $\partial D_{1}$ bounds a disk $I_{1}$ on $\partial B_{2}$ such that $I_{1} \cap \partial \theta_{2} = q_{1}$ by Lemma 5.10. Here we have two cases, that is,

(i) $\partial D_{2}$ bounds a disk $I_{2}$ on $\partial B_{2}$ such that $I_{2} \cap \partial \theta_{2} = q_{2} \cup q_{3}$.

(ii) $\partial D_{2}$ bounds a disk $I_{2}$ on $\partial B_{2}$ such that $I_{2} \cap \partial \theta_{2} = q''_{1}$.

Suppose that (i) occurs. Since we may suppose that

$(E_{2} \cap (\text{the ball bounded by } D_{2} \cup I_{2})) \cap S_{2}^{f} = \phi$
by Lemma 5.6, we have \( \tilde{I}_2 \cap S^2_I = \emptyset \) for the same reason as Lemma 5.9. Then, we can push the whole \( D_1 \) and the whole \( D_2 \) toward \( \partial B_2 \) so that we have the 2-sphere \( S \) in \( B_1 \). This contradicts the definition of \( S \). Next, suppose that (ii) occurs. By cutting \( B_2 \) along \( D_2 \), we have the 3-ball \( B'_2 \) which contains three components \( \theta_1, \theta_2, \theta_3 \) \((\theta_i \in e_i (i = 1, 2, 3)) \) of \( \partial_2 \). By Lemma 5.6, we may suppose that \((E_2 \cap B'_2) \cap S^2_I = \emptyset \). Then, we can see that \( B'_2 \cap N(\theta_2 \cup (E_2 \cap B'_2)), B'_3 \) is a genus two handlebody by the existence of disks \( E_3 \cap B'_2 \) and \( E_3 \). For the similar argument to that in Case A-3, we get a contradiction.

Case II-2-2. \( D_1 \ni p_2 \) \((p_3 \text{ respectively})\), \( D_2 \ni p_1 \cup p_3 \) \((p_1 \cup p_2 \text{ respectively})\).

In this case, by Lemma 5.6, \( \partial D_2 \) bounds a disk \( I \) on \( \partial B_2 \) such that \( I \cap \partial \theta_2 = q_2 \) or \( q_1 \cup q_3 \) \((q_1 \cup q_2 \text{ respectively})\). If \( I \cap \partial \theta_2 = q_1 \cup q_3 \) \((q_1 \cup q_2 \text{ respectively})\), by the same reason as in Case II-2-1(i), we have a contradiction. Suppose that \( I \cap \partial \theta_2 = q_2 \). By the construction of \( S^2_I \), there is the component of \( \partial S^2_I \) which bounds a disk \( I_{13} \) on \( \partial B_1 \) such that \( I_{13} \cap \theta_1 = q_1 \cup q_3 \). This contradicts Lemma 5.8.

This completes the proof of Theorem 5.1. \( \Box \)

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References


