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A characterization of flip-accessibility for rhombus tilings of the whole plane

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ABSTRACT

It is known that any two rhombus tilings of a polygon are flip-accessible, that is, linked by a finite sequence of local transformations called flips. This paper considers flip-accessibility for rhombus tilings of the *whole plane*, asking whether any two of them are linked by a *possibly infinite* sequence of flips. The answer turning out to depend on tilings, a *characterization* of flip-accessibility is provided. This yields, for example, that any tiling by Penrose tiles is flip-accessible from a Penrose tiling.

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1. Introduction

A *rhombus tiling* of $D \subset \mathbb{R}^2$ is a set of rhombus-shaped compact sets, namely *rhombus tiles*, whose interiors are disjoint, which meet edge-to-edge and whose union is D . Fig. 1 depicts celebrated rhombus tilings of $D = \mathbb{R}^2$ (see also [6]).

Then, the *flip* is a well-known local transformation over rhombus tilings which just exchanges three rhombus tiles sharing a vertex (see e.g., [1,2,5,9,11,15], and also Fig. 2). Flips rise the question of *flip-accessibility*: can a given rhombus tiling be transformed into another one by performing a sequence of flips?

A motivation for studying flip-accessibility for rhombus tilings comes from statistical mechanics. Indeed, rhombus tilings appeared to be a suitable model for the structure of recently discovered quasicrystalline alloys (see [14]). Moreover, elementary transformations of real quasicrystal, called *phasons*, seem being efficiently modeled by flips (see [10]). This led to study flip dynamics, thus the preliminary question of flip-accessibility.

In the case of rhombus tilings of a polygon, it is proven in [9] that any two rhombus tilings are linked by a finite sequence of flips. In other words, rhombus tilings of a polygon are all mutually flip-accessible. Many results concerning flip dynamics, in particular random sampling, have been obtained (see e.g., [5,11]). The case of rhombus tilings of the whole plane is more complicated. First, note that it is natural to consider flip-accessibility in terms of *possibly infinite* sequences of flips. Then, even with this definition, tilings turn out to be not always flip-accessible. Thus, answering the question of flip-accessibility amounts to *characterize* flip-accessibility between pairs of tilings.

The paper is organized as follows. In Section 2, we more formally define rhombus tilings of the whole plane and the corresponding notion of flip-accessibility. We also show that rhombus tilings are naturally associated with a useful higher-dimensional notion, namely *stepped surfaces*. Section 3 then states the main result of this paper, that is, a characterization of

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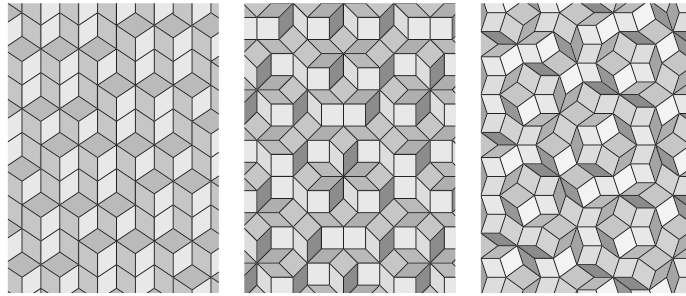


Fig. 1. Rauzy-dual, Ammann-Beenker and Penrose rhombus tilings (from left to right).

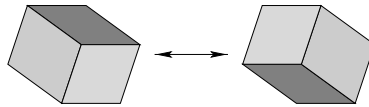


Fig. 2. A flip is a local exchange of three rhombus tiles.

flip-accessibility in terms of *shadows* (Theorem 1). As a corollary, we show that there is a large class of rhombus tilings, namely the *canonical projection tilings*, from which any other rhombus tiling over the same set of rhombus tiles is flip-accessible. The last section is devoted to the proof of this characterization. In particular, we rely on the de Bruijn lines of [3] to introduce *de Bruijn cones*, a tool which could be used for achieving efficient algorithms in the finite case.

2. General settings

Let us first define rhombus tilings of the whole plane. Let $\vec{v}_1, \dots, \vec{v}_d$ be $d \geq 3$ non-colinear unit vectors of \mathbb{R}^2 . *Rhombus tiles* are the $\binom{d}{2}$ compact sets of non-empty interior defined for $1 \leq i < j \leq d$ by:

$$T_{ij} = \{\lambda \vec{v}_i + \mu \vec{v}_j, \quad 0 \leq \lambda, \mu \leq 1\}.$$

Then, for $\vec{x} \in \oplus_i \mathbb{Z} \vec{v}_i$, we denote by $\vec{x} + T_{ij}$ the rhombus tile obtained by translating T_{ij} by \vec{x} . Note that there is no loss of generality by considering rhombus tiles translated in $\oplus_i \mathbb{Z} \vec{v}_i$ (instead of the whole \mathbb{R}^2) because we are here interested in flip-accessibility; this restriction will be useful in Proposition 1, below. Let us now define rhombus tilings of the whole plane \mathbb{R}^2 .

Definition 1. A $d \rightarrow 2$ rhombus tiling is a set of translated rhombus tiles of disjoint interiors, meeting edge-to-edge¹ and whose union is the whole plane.

For example, Fig. 1 depicts $d \rightarrow 2$ rhombus tilings for, respectively, $d = 3, 4, 5$.

Let us now define *flip-accessibility* for $d \rightarrow 2$ rhombus tilings. Introduced in [15] for finite domino or lozenge tilings, flips are similarly defined for rhombus tilings. If three and only tiles meet at a vertex, then their union is a hexagon, and there is a unique way to move these tiles such that their union is still this hexagon. We call *flip* the operation which performs this move (see Fig. 3).

Clearly, performing a flip on a rhombus tiling yields a (new) rhombus tiling. This also holds for a finite sequence of flips, but we need to be more precise in the case of an *infinite* sequence of flips. We define the distance $d(\mathcal{T}, \mathcal{T}')$ between tilings \mathcal{T} and \mathcal{T}' by:

$$d(\mathcal{T}, \mathcal{T}') = \inf \left\{ 2^{-r} \mid \mathcal{T}_{|B(\vec{0}, r)} = \mathcal{T}'_{|B(\vec{0}, r)} \right\},$$

where $\mathcal{T}_{|B(\vec{0}, r)}$ denotes the set of rhombus tiles in \mathcal{T} which belong to the two-dimensional ball of center $\vec{0}$ and radius r . This allows us to indiscriminately consider finite or infinite sequences of flips for defining flip-accessibility:

¹ That is, two intersecting tiles share either a point \vec{x} or an edge $\{\vec{x} + \lambda \vec{v}_i, \quad 0 \leq \lambda \leq 1\}$.

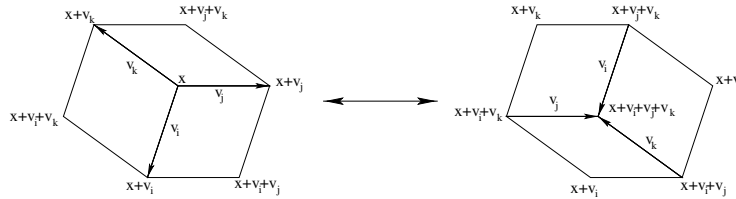


Fig. 3. A flip is a local exchange of three rhombus tiles.

Definition 2. Let \mathcal{T} and \mathcal{T}' be two rhombus tilings of the whole plane. If there is a sequence $(\mathcal{T}_n)_{n \geq 0}$ of rhombus tilings such that $\mathcal{T}_0 = \mathcal{T}$, \mathcal{T}_{n+1} is obtained by performing a flip on \mathcal{T}_n and $d(\mathcal{T}_n, \mathcal{T}')$ tends towards 0, then one says that \mathcal{T}' is *flip-accessible* from \mathcal{T} , and one writes:

$$\mathcal{T} \xrightarrow{\text{flips}} \mathcal{T}'$$

Last, let us show how rhombus tilings and flips can be seen from a higher-dimensional viewpoint. This will be very useful in the following sections.

Let $(\vec{e}_1, \dots, \vec{e}_d)$ be the canonical basis of \mathbb{R}^d . For $1 \leq i < j \leq d$ and $\vec{x} \in \mathbb{Z}^d$, the *unit face of type t_{ij} located at \vec{x}* is the subset of \mathbb{R}^d defined by:

$$(\vec{x}, t_{ij}) = \{\vec{x} + \lambda \vec{e}_i + \mu \vec{e}_j, \quad 0 \leq \lambda, \mu \leq 1\}.$$

Let then $\Psi : \mathbb{R}^d \rightarrow \mathbb{R}^2$ be the linear map defined by:

$$\Psi(x_1, \dots, x_d) = \sum_{i=1}^d x_i \vec{v}_i.$$

We are now in a position to introduce so-called *stepped surfaces*.

Definition 3. A $d \rightarrow 2$ *stepped surface* is a set S of unit faces of \mathbb{R}^d such that Ψ is a homeomorphism from the union of these unit faces onto \mathbb{R}^2 .

A stepped surface is thus a sort of fairly rugged subset of \mathbb{R}^d homeomorphic to a plane. Rhombus tilings and stepped surfaces turn out to be naturally connected.

Proposition 1. If S is a $d \rightarrow 2$ stepped surface, then $\Psi(S)$ is a $d \rightarrow 2$ rhombus tiling. Conversely, if \mathcal{T} is a $d \rightarrow 2$ rhombus tiling, then there is a $d \rightarrow 2$ stepped surface S such that $\Psi(S) = \mathcal{T}$, and S is unique up to a translation in $\ker(\Psi) \cap \mathbb{Z}^d$.

Proof. Let S be a stepped surface. First, Ψ clearly maps unit faces onto rhombus tiles whose vertices belong to $\oplus_i \mathbb{Z} \vec{v}_i$. Then, note that unit faces are of disjoint interiors and meet edge-to-edge: this still holds by applying the homeomorphism Ψ . Last, Ψ is onto \mathbb{R}^2 . This shows that $\Psi(S)$ is a rhombus tiling of \mathbb{R}^2 .

Conversely, let \mathcal{T} be a rhombus tiling of \mathbb{R}^2 . Let \vec{x}_0 be a vertex of \mathcal{T} . Since $\vec{x}_0 \in \oplus_i \mathbb{Z} \vec{v}_i$ (by definition), there is some $\vec{y}_0 \in \mathbb{Z}^d$ such that $\Psi(\vec{y}_0) = \vec{x}_0$, and \vec{y}_0 is unique up to a translation in $\ker(\Psi) \cap \mathbb{Z}^d$. One then define a function h from the vertices of \mathcal{T} to \mathbb{Z}^d as follows:

$$h(\vec{x}_0) = \vec{y}_0 \quad \text{and} \quad \vec{x}' = \vec{x} + \vec{v}_i \Rightarrow h(\vec{x}') = h(\vec{x}) + \vec{e}_i.$$

Actually, h is nothing but a *height function*, and is thus consistent (see e.g., [4]). Here, note that $\Psi(h(\vec{x})) = \vec{x}$ for any vertex \vec{x} of \mathcal{T} , and let us define the following set of unit faces:

$$S = \{(h(\vec{x}), t_{ij}) | \vec{x} + T_{ij} \in \mathcal{T}\}.$$

It follows from the construction of S that the restriction of Ψ to the union of unit faces of S , denoted by $\Psi|_S$, is a bijection onto \mathbb{R}^2 , whose inverse is a piecewise linear map depending on the height function h associated with the rhombus tiling $\Psi(S)$. Since both $\Psi|_S$ and its inverse are continuous, Ψ is a homeomorphism from S onto \mathbb{R}^2 , that is, S is a stepped surface. Last, S is unique up to the initial choice of \vec{y}_0 , that is, up to a translation in $\ker(\Psi) \cap \mathbb{Z}^d$. \square

In other words, stepped surfaces are nothing but rhombus tilings seen from a higher-dimensional viewpoint. Actually, this is just a generalization of ideas introduced in [15] for finite domino or lozenge tilings. Note also that the case $d = 3$ corresponds

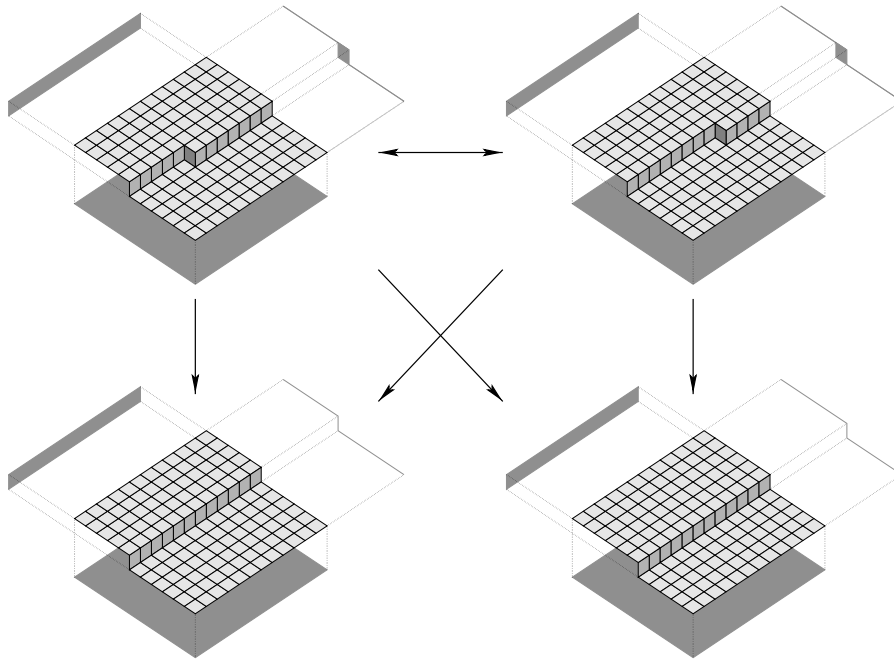


Fig. 4. Four patches of 3 → 2 stepped surfaces and their shadows (see Definition 4, below). Flip-accessibility is represented by arrows: the top two stepped surfaces are mutually flip-accessible (by a finite sequence of flips), and the bottom two stepped surfaces are flip-accessible from them (by an infinite sequence of flips rejecting the “corner” to infinity in one of the two possible directions). The bottom two stepped surfaces are sort of *dead ends*: no flip can be performed on them. It is worth noticing that a stepped surface is flip-accessible from another one if and only if the shadows of the latter are included in the shadows of the former (this illustrates Theorem 1, below).

to the notion introduced in [8], where the 3-dimensional viewpoint is very natural (see, for example, the leftmost tiling of Fig. 1).

The notion of flip is then defined over stepped surfaces so that if a stepped surface S' is obtained by performing a flip on a stepped surface S , then the rhombus tiling $\Psi(S')$ is obtained by performing a flip on the rhombus tiling $\Psi(S)$ (it suffices to replace \tilde{v}_i by \tilde{e}_i on Fig. 3). If, moreover, one says that two stepped surfaces S and S' are at distance less than 2^{-r} if they share the same set of unit faces within the d -dimensional ball $B(0,r)$, then this leads to a notion of flip-accessibility for stepped surfaces which satisfies:

Proposition 2. For two stepped surfaces S and S' , one has:

$$\Psi(S) \xrightarrow{\text{flips}} \Psi(S') \Leftrightarrow \exists \vec{a} \in \ker(\Psi) \cap \mathbb{Z}^d \text{ s.t. } S \xrightarrow{\text{flips}} \vec{a} + S',$$

where $\vec{a} + S'$ denotes the stepped surface obtained by translating S' by \vec{a} .

Fig. 4 illustrates the notion of flip-accessibility. Note that, contrarily to the case of rhombus tilings of a polygon, flip-accessibility does not always holds, and is moreover even not symmetric.

3. Characterization by shadows

The aim of this section is to provide a characterization of flip-accessibility for stepped surfaces (which can be then restated in terms of rhombus tilings according to Propositions 1 and 2). We define the following maps, for $1 \leq i < j \leq d$:

$$\pi_{ij} : \begin{matrix} \mathbb{R}^d & \rightarrow & \mathbb{R}^2 \\ (z_1, \dots, z_d) & \mapsto & (z_i, z_j) \end{matrix}$$

In particular, π_{ij} maps the unit face (\vec{x}, t_{kl}) onto a unit square if $i = k$ and $j = l$, onto a unit segment if $i = k$ or $j = l$ and onto a point otherwise. We then use these maps to define the *shadows* of a stepped surface (see Fig. 4):

Definition 4. The shadows of a $d \rightarrow 2$ stepped surface S are the $\binom{d}{2}$ subsets of \mathbb{R}^2 defined, for $1 \leq i < j \leq d$, by:

$$\pi_{ij}(S) = \bigcup_{(\vec{x},t) \in S} \pi_{ij}(\vec{x},t).$$

A simple but fundamental property of shadows is that they are invariant by performing a flip (this can be easily checked on Fig. 3). This also holds for finite sequences of flips, but we have only a weaker property for infinite sequences.

Proposition 3. If a stepped surface S' is flip-accessible from a stepped surface S , then the shadows of S' are included in the shadows of S :

$$S \xrightarrow{\text{flips}} S' \Rightarrow \forall i, \forall j, \pi_{ij}(S') \subset \pi_{ij}(S).$$

Proof. Let S_n be a sequence of stepped surfaces, obtained by performing flips on S , which tends towards S' . Let $z \in \pi_{ij}(S')$: z belongs to the projection of a face $(\vec{x},t) \in S'$. Let $r \in \mathbb{R}$ such that $(\vec{x},t) \subset B(0,r)$ and $N \in \mathbb{N}$ such that $d(S_N, S') \leq 2^{-r}$. In particular, $(\vec{x},t) \in S_N$. Since S_N is obtained from S by performing a finite number of flips, both have the same shadows. Thus, $z \in \pi_{ij}(\vec{x},t) \subset \pi_{ij}(S_N)$ yields $z \in \pi_{ij}(S)$. This proves $\pi_{ij}(S') \subset \pi_{ij}(S)$. \square

In the previous proposition, inclusions of shadows can be strict (see, for example, Fig. 4). Actually, the main result of this paper is that the converse of this proposition holds:

Theorem 1. A stepped surface S' is flip-accessible from a stepped surface S iff the shadows of S' are included in the shadows of S :

$$S \xrightarrow{\text{flips}} S' \Leftrightarrow \forall i, \forall j, \pi_{ij}(S') \subset \pi_{ij}(S).$$

Theorem 1 is proven in the following section. Before this, let us provide an interesting corollary. We need the following definition:

Definition 5. Given two vectors \vec{u} and \vec{v} of \mathbb{R}^d with non-zero entries, the $d \rightarrow 2$ stepped plane $\mathcal{P}_{\vec{u},\vec{v}}$ is the set of all unit faces which lie (entirely) in the following “slice” of \mathbb{R}^d :

$$\mathbb{R}\vec{u} + \mathbb{R}\vec{v} + [0,1]^d.$$

Roughly speaking, the stepped plane $\mathcal{P}_{\vec{u},\vec{v}}$ is an approximation by unit faces of the real plane $\mathbb{R}\vec{u} + \mathbb{R}\vec{v}$ (this corresponds to a viewpoint developed in discrete geometry, see e.g., [12]). Actually, stepped planes are nothing but the stepped surfaces which are associated by Proposition 1 with so-called *canonical projection tilings*. These are rhombus tilings obtained by the *cut and project* method (see [7,13]). For example, the Rauzy-dual, Ammann-Beenker and Penrose tilings depicted on Fig. 1 are canonical projection tilings associated with $d \rightarrow 2$ stepped planes for, respectively, $d = 3, 4, 5$ (see [6]).

Now, let us note that $\pi_{ij}(\mathbb{R}\vec{u} + \mathbb{R}\vec{v}) = \mathbb{R}^2$. This easily yields that $\pi_{ij}(\mathcal{P}_{\vec{u},\vec{v}}) = \mathbb{R}^2$. In particular, the shadows of the stepped plane $\mathcal{P}_{\vec{u},\vec{v}}$ contain the shadows of any other stepped surface. We thus obtain as an immediate corollary of Theorem 1.

Corollary 1. Any stepped surface is flip-accessible from a stepped plane.

In terms of rhombus tilings, this means that any rhombus tiling is flip-accessible from a canonical projection tiling over the same set of rhombus tiles.

4. Proof of the characterization

This section provides a proof of the characterization stated in Theorem 1. The necessary condition is proven by Proposition 3. Let thus S and S' be two stepped surfaces such that the shadows of S' are included in the shadows of S , and let us prove that S' is flip-accessible from S .

Since the proof is not so short, it is worth giving a brief outline. The general idea is to transform S into S' by moving one by one unit faces. More precisely, for $(\vec{x}',t_{ij}) \in S'$, inclusion of shadows ensure that there is a unit face $(\vec{x},t_{ij}) \in S$ such that $\pi_{ij}(\vec{x}',t_{ij}) = \pi_{ij}(\vec{x},t_{ij})$. We would like to move (\vec{x},t_{ij}) to (\vec{x}',t_{ij}) . We proceed as follows. While there is k such that $x_k < x'_k$, we choose such a k and we define a set $F_k^*(\vec{x},t_{ij})$ such that, by performing a finite number flips over this set.² Similarly, we can

² Although the faces of a given set F of faces are modified by flips, note that the projection $\Psi(F)$ (recall Section 2) is not modified, since a flip exchanges three faces which still project on a hexagon. Thus, by abuse, *performing flips over F* means performing flips over the faces whose images by Ψ lie in $\Psi(F)$.

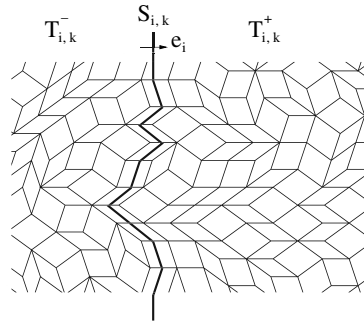


Fig. 5. A de Bruijn section $S_{i,k}$, here represented by a broken line crossing its unit faces, splits a stepped surface into two connected sets of unit faces, $T_{i,k}^-$ and $T_{i,k}^+$.

translate (\vec{x}, t_{ij}) by $-\vec{e}_k$ for k such that $x_k > x'_k$. Hence, we can move $(\vec{x}, t_{ij}) \in \mathcal{S}$ to $(\vec{x}', t_{ij}) \in \mathcal{S}'$ by performing a finite number of flips. The last step will be to show that we can, in this way, obtain unit faces of \mathcal{S}' over growing balls centered in $\vec{0}$ (Lemma 4), that is, that \mathcal{S}' is flip-accessible from \mathcal{S} (see Definition 2). Let us now start the proof. We first define a useful tool.

Definition 6. Let \mathcal{S} be a stepped surface, $k \in \mathbb{Z}$ and $1 \leq i \leq d$. If not empty, the following set of unit faces is the k -th de Bruijn section of type i of \mathcal{S} :

$$S_{i,k} = \{((x_1, \dots, x_d), t_{ij}) \in \mathcal{S} \mid x_i = k\}.$$

It is easily seen that $S_{i,k}$ is an infinite stripe of unit faces two by two adjacent along vectors \vec{e}_i . Then, removing $S_{i,k}$ naturally splits \mathcal{S} into the two following connected sets of unit faces (see Fig. 5):

$$T_{i,k}^+ = \{((x_1, \dots, x_d), t) \in \mathcal{S} \mid x_i > k\} \quad \text{and} \quad T_{i,k}^- = \mathcal{S} \setminus (S_{i,k} \cup T_{i,k}^+).$$

Actually, de Bruijn sections turn out to be the set of unit faces associated by Proposition 1 with the well-known de Bruijn lines introduced in [3]. In other words, $S_{i,k}$ is a de Bruijn section of \mathcal{S} iff $\Psi(S_{i,k})$ is a de Bruijn line of the rhombus tiling $\Psi(\mathcal{S})$. In particular, two de Bruijn sections share at most one face, as well as de Bruijn lines. In such a case, they are said to intersect. Note that, if $(\vec{x}, t_{kl}) = S_{i,n} \cap S_{j,m}$, then $k = i$, $l = j$, $x_i = n$ and $x_j = m$. In particular, only sections of different types can intersect, although they can also not intersect.

We use de Bruijn sections to define so-called de Bruijn triangles.

Definition 7. For $(\vec{x} = (x_1, \dots, x_d), t_{ij}) \in \mathcal{S}$ and $1 \leq k \leq d$, $k \neq i$, $k \neq j$, the de Bruijn triangle $F_k(\vec{x}, t_{ij})$ is the set of unit faces of \mathcal{S} defined by:

$$F_k(\vec{x}, t_{ij}) = (S_{i,x_i} \cup T_{i,x_i}^{\varepsilon_i}) \cap (S_{j,x_j} \cup T_{j,x_j}^{\varepsilon_j}) \cap (S_{k,x_k} \cup T_{k,x_k}^-),$$

where ε_i and ε_j , respectively, denote the signs of entries of \vec{v}_k in the basis (\vec{v}_i, \vec{v}_j) .

Roughly speaking, $F_k(\vec{x}, t_{ij})$ is the triangle defined by the three “lines” S_{i,x_i} , S_{j,x_j} and S_{k,x_k} (see Fig. 6, left). Note that it could be infinite, since the de Bruijn sections S_{i,x_i} or S_{j,x_j} do not necessarily intersect S_{k,x_k} . We will later avoid this case (Lemma 3). Intuitively, for translating (\vec{x}, t_{ij}) by \vec{e}_k , we first need to translate by \vec{e}_k the unit faces in $F_k(\vec{x}, t_{ij})$. However, moving a unit face of $F_k(\vec{x}, t_{ij})$ requires, in turn, to move some others unit faces before. Therefore, we extend de Bruijn triangles by so-called de Bruijn cones (see also Fig. 6, right).

Definition 8. With the convention $F_k(A \cup B) = F_k(A) \cup F_k(B)$, we define:

$$F_k^0(\vec{x}, t_{ij}) = (\vec{x}, t_{ij}) \quad \text{and} \quad F_k^{n+1}(\vec{x}, t_{ij}) = F_k(F_k^n(\vec{x}, t_{ij})).$$

Then, the de Bruijn cone $F_k^*(\vec{x}, t_{ij})$ is defined by:

$$F_k^*(\vec{x}, t_{ij}) = \bigcup_{n \geq 0} F_k^n(\vec{x}, t_{ij}).$$

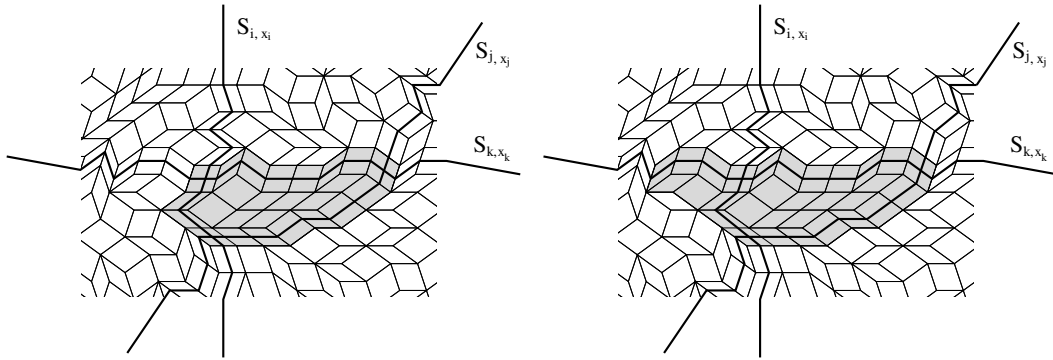


Fig. 6. A de Bruijn triangle $F_k(\vec{x}, t_{ij})$ (the shaded unit faces, left) and its closure, the de Bruijn cone $F_k^*(\vec{x}, t_{ij})$ (right). Recall that one has always $(\vec{x}, t_{ij}) = S_{i, x_i} \cap S_{j, x_j}$.

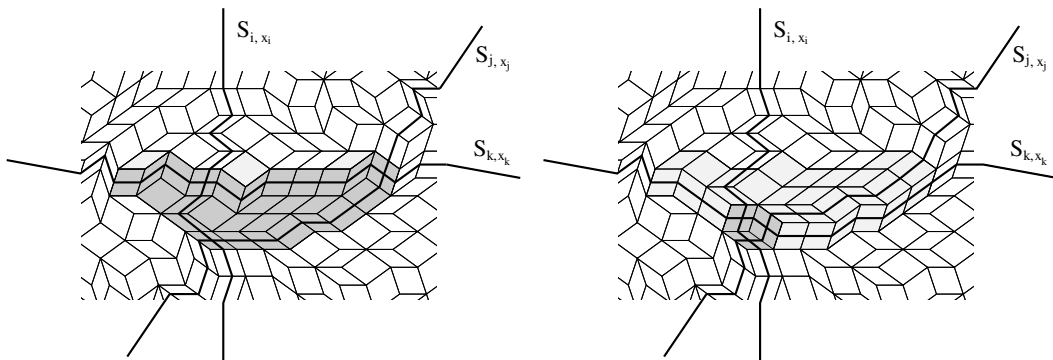


Fig. 7. Three flips have been performed on the minimal elements of the de Bruijn cone of Fig. 6 (left). This can be repeated, reducing the de Bruijn cone up to only three unit faces (right), on which performing a flip will translate the unit face (\vec{x}, t_{ij}) by \vec{e}_k .

Let us now show that (\vec{x}, t_{ij}) can be translated by performing flips over $F_k^*(\vec{x}, t_{ij})$.

Lemma 1. *If $F_k^*(\vec{x}, t_{ij})$ is finite, then one can translate (\vec{x}, t_{ij}) by \vec{e}_k by performing $\text{card}(F_k^*(\vec{x}, t_{ij}) \setminus S_{k, x_k})$ flips over $F_k^*(\vec{x}, t_{ij})$.*

Proof. Definition 8 yields, for any unit faces (\vec{y}, t) and (\vec{y}', t') :

$$(\vec{y}, t) \in F_k^*(\vec{y}', t') \Rightarrow F_k^*(\vec{y}, t) \subset F_k^*(\vec{y}', t').$$

Let us define a sequence $(f_n)_{n \geq 0}$ of faces as follows: $f_0 = (\vec{x}, t_{ij})$, and for $n \geq 1$, $f_{n+1} \in F_k^*(f_n) \setminus S_{k, x_k}$ if this last set is not empty. This sequence is finite ($f_n \in F_k^*(\vec{x}, t_{ij})$), so let (\vec{y}, t) be its last face. By construction, $F_k^*(\vec{y}, t)$ contains only three faces, on which a flip can be performed (see, for example, Fig. 6, right). By performing this flip, (\vec{y}, t) is translated by \vec{e}_k , so that the obtained face does no more belongs to $F_k^*(\vec{x}, t_{ij})$, which thus decreased (Fig. 7, left). This can be inductively repeated, up to translate by \vec{e}_k the face (\vec{x}, t_{ij}) itself (Fig. 7, right). Since there is one flip performed for each translated unit face, there is a total of $\text{card}(F_k^*(\vec{x}, t_{ij}) \setminus S_{k, x_k})$ flips performed. \square

Although the definition of de Bruijn cones by transitive closure suffices to prove the previous lemma, the following stronger property actually holds:

Lemma 2. *One has $F_k^*(\vec{x}, t_{ij}) = F_k^2(\vec{x}, t_{ij})$.*

Proof. Let $(\vec{y}, t) \in F_k^2(\vec{x}, t_{ij})$. If $F_k(\vec{y}, t)$ is not included in $F_k^2(\vec{x}, t_{ij})$, then a case study (relying on the fact that two de Bruijn sections intersect at most once) shows that one of the two de Bruijn sections containing (\vec{y}, t) , say $S_{k', y_{k'}}$, necessarily intersects $F_k(\vec{x}, t_{ij})$. Let thus $(\vec{y}', t') \in S_{k', y_{k'}} \cap F_k(\vec{x}, t_{ij})$. One has $F_k(\vec{y}, t) \subset F_k(\vec{y}', t')$, and $(\vec{y}', t') \in F_k(\vec{x}, t_{ij})$ yields $F_k(\vec{y}', t') \subset F_k^2(\vec{x}, t_{ij})$. Hence, $F_k(\vec{y}, t) \subset F_k^2(\vec{x}, t_{ij})$. Since this holds for any $(\vec{y}, t) \in F_k^2(\vec{x}, t_{ij})$, this proves $F_k^3(\vec{x}, t_{ij}) \subset F_k^2(\vec{x}, t_{ij})$. The result follows. \square

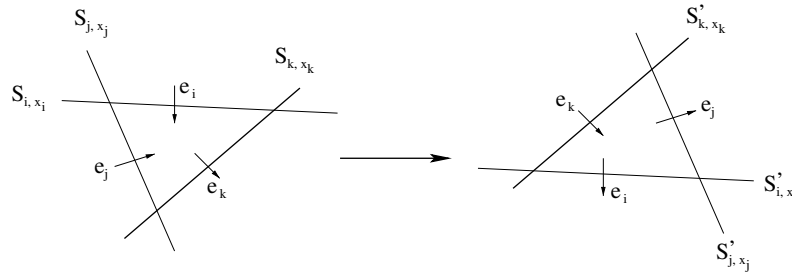


Fig. 8. If (\vec{x}, t_{ij}) must cross the section S_{k,x_k} to be transformed to (\vec{x}', t_{ij}) , then any unit face inside the triangle $T_{i,x_i}^+ \cap T_{j,x_j}^+ \cap T_{k,x_k}^-$ must also cross one of the sections S_{i,x_i} , S_{j,x_j} or S_{k,x_k} , hence is moved.

We are now in a position to prove that one can choose k_0 such that $F_{k_0}^*(\vec{x}, t_{ij})$ is finite and (\vec{x}, t_{ij}) should be translated by \vec{e}_{k_0} (the condition $k_0 \in D$ below). Lemma 1 then yields that (\vec{x}, t_{ij}) can be effectively translated by \vec{e}_{k_0} .

Lemma 3. Let $(\vec{x}', t_{ij}) \in S'$ and $(\vec{x}, t_{ij}) \in S$ such that $\pi_{ij}(\vec{x}', t_{ij}) = \pi_{ij}(\vec{x}, t_{ij})$. If $D = \{k | x'_k > x_k\} \neq \emptyset$, then there is $k_0 \in D$ such that $F_{k_0}^*(\vec{x}, t_{ij})$ is finite.

Proof. We first prove that $F_k(\vec{x}, t_{ij})$ is finite for any $k \in D$, and then that there is $k_0 \in D$ such that $F_{k_0}^*(\vec{x}, t_{ij}) = F_{k_0}^2(\vec{x}, t_{ij})$ is finite. Let $k \in D$. Note that $F_k(\vec{x}, t_{ij})$ is finite iff both S_{i,x_i} and S_{j,x_j} intersect S_{k,x_k} . Suppose that S_{i,x_i} does not intersect S_{k,x_k} . Thus, $S_{i,x_i} \subset T_{k,x_k}^-$. Then, since the shadows of S' are included in the shadows of S , there is $(\vec{z}, t) \in S$ such that $\pi_{ik}(\vec{x}') \in \pi_{ik}(\vec{z}, t)$. This yields $z_i = x'_i = x_i$ and $z_k = x'_k > x_k$. In particular, $\vec{z} \in S_{i,x_i} \cap T_{k,x_k}^+$. Since this contradicts $S_{i,x_i} \subset T_{k,x_k}^-$, we deduce that S_{i,x_i} intersects S_{k,x_k} . Similarly, S_{j,x_j} intersects S_{k,x_k} . The first result is proven.

Let us now choose $k_0 \in D$ being minimal in D for the following partial order:

$$n \leq m \Leftrightarrow T_{m,x_m}^+ \subset T_{n,x_n}^+.$$

In other words, k_0 is chosen such that there is no section S_{k,x_k} separating (\vec{x}, t_{ij}) from $S_{k_0,x_{k_0}}$, that is, such that $(\vec{x}, t_{ij}) \in T_{k,x_k}^-$ and $S_{k_0,x_{k_0}} \subset T_{k,x_k}^+$. This yields that a unit face (\vec{y}, t) of $F_{k_0}(\vec{x}, t_{ij})$ belongs to two de Bruijn sections both intersecting $S_{k_0,x_{k_0}}$. Thus, $F_k(\vec{y}, t)$ is finite. The second result follows. \square

Note that the previous lemma only proves that there is $k_0 \in D$ such that one can (and should) translate (\vec{x}, t_{ij}) by \vec{e}_{k_0} . However, this is not a problem for $d = 3$, since in this case D is reduced to a singleton; without going into details, let us just say that it is strongly connected with the fact that the set of $d \rightarrow 2$ rhombus tilings of a polygon forms a distributive lattice only for $d = 3$ (see, for example, [5,11]).

So, following the outline given at the beginning of this section, we can now, by performing flips, translate (\vec{x}, t_{ij}) by some \vec{e}_{k_0} such that $x'_{k_0} > x_{k_0}$. We can repeat this up to have $x'_k \leq x_k$ for any k . The way we can translate by $-\vec{e}_{k_0}$ a unit face (\vec{x}, t_{ij}) such that $x'_{k_0} < x_{k_0}$ is similar. So, we are able to move (\vec{x}, t_{ij}) to (\vec{x}', t_{ij}) . The end of the proof relies on the following lemma.

Lemma 4. Let $(\vec{x}', t_{ij}) \in S'$ and $(\vec{x}, t_{ij}) \in S$ such that $\pi_{ij}(\vec{x}', t_{ij}) = \pi_{ij}(\vec{x}, t_{ij})$. If $x'_k > x_k$, then $F_k^*(\vec{x}, t_{ij}) \cap S' = \emptyset$.

Proof. (sketch) Writing down a detailed proof is rather technical and obfuscating, but the underlying geometrical idea is quite easy. Indeed, $x'_k > x_k$ yields $(\vec{x}, t_{ij}) \in T_{k,x_k}^-$ and $(\vec{x}', t_{ij}) \in T_{k,x_k}^+$, as depicted on Fig. 8. So, suppose that there is a unit face $(\vec{y}, t) \in F_k(\vec{x}, t_{ij}) \cap S'$. Such a face thus should have the same position, in S and S' , relatively to any de Bruijn section. For example, if (\vec{y}, t) belongs to $T_{i,x_i}^+ \cap T_{j,x_j}^+ \cap T_{k,x_k}^-$ in S (as in the case of Fig. 8, left), then it should belong to $T_{i,x_i}^+ \cap T_{j,x_j}^+ \cap T_{k,x_k}^-$ in S' . However, this last set turns out to be empty (see Fig. 8, right). Thus, $F_k(\vec{x}, t_{ij}) \cap S' = \emptyset$. Suppose now that $(\vec{y}, t) \in F_k^2(\vec{x}, t_{ij}) \cap S'$. There is $(\vec{z}, t_z) \in F_k(\vec{x}, t_{ij})$ such that $(\vec{y}, t) \in F_k(\vec{z}, t_z)$. We prove $F_k(\vec{z}, t_z) \cap S' = \emptyset$ as above, with (\vec{z}, t_z) instead of (\vec{x}, t_{ij}) . \square

This lemma ensures that, once a unit face of S' is obtained, it is no more moved. We thus can get unit faces of S' over growing balls, and Theorem 1 follows. We end the paper by summing up the whole proof by the following algorithm:


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for  $r=0$  to  $\infty$ 
  while  $S_{B(\vec{0},r)} \neq S'_{B(\vec{0},r)}$ 
    choose  $(\vec{x}, t_{ij})$  in  $S_{B(\vec{0},r)} \setminus S'_{B(\vec{0},r)}$ 
     $(\vec{x}', t_{ij}) \leftarrow S'_{i,x_i} \cap S'_{j,x_j}$  (  $\pi_{ij}(S') \subset \pi_{ij}(S)$  )
    while  $\vec{x} \neq \vec{x}'$ 
      choose  $k$  s.t.  $x_k \neq x'_k$  and  $F_k^*(\vec{x}, t_{ij})$  is finite (Lemma 3)
       $x_k \leftarrow x_k \pm 1$  by performing flips over  $F_k^*(\vec{x}, t_{ij})$  (Lemma 1)
    endwhile
  endwhile (Lemma 4)
endfor

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