Discrete Mathematics 53 (1985) 35–53
North-Holland

ON BETTER QUASI-ORDERING COUNTABLE TREES

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The main result is that the class of countable trees is better-quasi-ordered under embeddability. R. Laver proved before that a certain class of well-founded trees is b.q.o. Actually our better-quasi-ordered class is larger than the countable class but does not contain Suslin like-trees nor certain Galvin trees of height $\omega+1$.

Introduction

Ce mémoire est une des parties des recherches de l'auteur sur le meilleur-préordre (b.q.o.) en rapport avec plusieurs structures mathématiques. Une première partie, déjà publiée, concerne le meilleur préordre des classes des p-groups abéliens non divisibles et dénombrables. On en déduit la construction inductive d'une classe plus grande que celle de ces p-groupes dénombrables et une généralisation du théorème d'Ulm.

Une troisième partie, non publiée, est consacrée aux classes b.q.o. de fonctions analytiques ordonnées par un plongement de type topologique. On en déduit, par exemple, que l'ordre circulaire des directions asymptotiques des fonctions entières est dispersé (scattered).

Ící nous étudions les algèbres ordinales des arbres dénombrables. Il existe une décomposition finie de tout arbre en arbres indécomposables. Ces 'atomes' sont indécomposables, soit par rapport à une chaîne, soit par rapport à une antichaîne soit leur indécomposabilité est du même type que celle de l'arbre dichotomique: ils se plongent en eux même en dessus de tout point. Ces derniers arbres, lorsqu'ils sont dénombrables, possèdent une chaîne, étiquetée par les nombres de branchements, qui est maximum; et leur comparaison est équivalente à celle de leurs chaînes maximum. Nous retrouvons ainsi beaucoup des résultats sur les chaînes dénombrables mais dans un contexte bien plus riche.

Nous utilisons une version de la construction d'une mauvaise application minimale qui redonne les constructions classiques. Cette construction évite l'emploi du 'forerun' de Nash-Williams bien qu'elle s'en inspire.

Nous utilisons les théorèmes de R. Laver sur les chaînes dispersées. La plupart de nos résultats sont donnés sous la forme de théorèmes sur les M-algèbres ordinales.

1. Indecomposable trees and embeddability

We call *tree* an ordered set T whose left segment $T^x = \{y \mid y \leqslant x\}$ are chains. A tree is *well branched* if all the infimums $x \wedge y$ exist. We call a *path* any set which is both a left segment and a chain.

An embedding of a well-branched tree T into another tree T' is any injective map $f: T \to T'$ such that $f(x \wedge y) = f(x) \wedge f(y)$ for every $x, y \in T$.

In order to define the embedding between trees which are not necessarily well branched we define the minimal branching of a tree.

For every tree T there is a well-branched tree $\hat{T}$ and an ordinal isomorphism i preserving existing infimums from T to $\hat{T}$ such that for every o.i.p.i $f: T \to T'$ (we write o.i.p.i instead of ordinal isomorphism preserving infimum) there is an embedding $\hat{f}: \hat{T} \to T'$ such that $f = \hat{f} \circ i$.

Using this we say that an o.i.p.i $f: T' \to T''$ is an embedding if there is an *embedding $\hat{f}: \hat{T}' \to \hat{T}''$ such that $i'' \circ f = \hat{f} \circ i'$*. I.e. we get the following commutative diagram:

$$\begin{array}{ccc} \hat{T}' & \longrightarrow & \hat{T}'' \\ {\scriptstyle i'}\big| & & \big|{\scriptstyle i''} \\ T' & \xrightarrow[f]{} & T'' \end{array}$$

An explicit definition of i and $\hat{T}$ is: $\hat{T} = \{T^x \cap T^y \mid x, y \in T\}$ ordered by inclusion and $i(x) = T^x$. Indeed, clearly $\hat{T}$ is a well-branched tree. Now let $f: T \to T'$ be an i.o.p.i into a well-branched tree T'. Let $P \in \hat{T}$, observing that $f(P)$ has a greatest element we put $\hat{f}(P) = \operatorname{Max} f(P)$.

The embedding of trees is a quasi-order relation, stronger than the order preserving embedding. We always will be concerned with the first relation, and we will denote it $T' \leqslant T'$ (we will denote $T' \leqslant_0 T''$ the order preserving embedding).

We look first at labelled trees. Usually a labelled tree (T, l) is a tree T and a map $l: T \to L$ whose range is a quasi-ordered set L. Our labelled trees are slightly different: a labelled tree is a pair (T, l) where l is a map defined on T, the range being a quasi-ordered set L. *We say that (T, l) is countable if T is countable and the domain of l is countable. In this section all the labelled trees are countable.*

The embedding of labelled trees is defined as follows: $(T', l') \leqslant (T'', l'')$ if there is a tree embedding $f: T' \to T''$ such that for every $P \in \operatorname{Dom} l'$, the path $\bar{f}(P) = (\leftarrow f(P)]$ belongs to $\operatorname{Dom} l''$ and $l'(P) \leqslant l''(\bar{f}(P))$. We call any intersection $P = A \cap B$ of two maximal different chains of T, *a junction path.* To a junction path is associated an important cardinal. Let K be any set of maximal chains C such that the intersection of two distinct members is P. If K' and K'' are two such sets maximal for inclusion, then for any $C' \in K'$ there exists only one $C'' \in K''$ such that $C' \cap C'' \neq P$ and conversely. Therefore the cardinals of K' and K'' are the same. We put $b(P) = |K|$, the cardinal of K when K is infinite and $b(P) = |K| - 1$ when K is finite. We extend the definition of b to all paths, putting $b(P) = 0$ if P is not a junction path. This defines a labelled tree (T, b) whose range is included into the set Card of cardinal numbers ordered by magnitude. To every labelled tree (T, l) one can associate a labelled tree (T, b, l), where the labelling b, l has as domain the domain of l and as range the direct product $\mathrm{Card} \times L$, and gives the value $(b(P), l(P))$ to every $P \in \mathrm{Dom}\, L$. If (T, l) is a labelled tree and T' is a subset of T, then we can consider the labelled tree (T', l') where l' is the induced labelling defined by $l'(P') = l(\leftarrow P])$. When there is no possible confusion we denote by l this labelling. For instance a *branching* chain of (T, l) is a labelled chain, the chain being a subchain C of T, the labelling induced by b and l. The tree above x is denoted by $T_x = \{y \in T \mid x \leqslant y\}$ we recall that T^x denotes the chain beneath the point x. Let (T, l) be a labelled tree; the left segment, in the class of (b, l)-labelled chains ordered by embedding, generated by the branching chains of (T, l) will be denoted $B(T)$.

It is well known that the class of countable chains labelled by countable b.q.o. sets is a b.q.o.-class, and that any labelled countable chain is a finite sum of indecomposable labelled chains. In the theory of chains it is well known too that any countable chain is the inductive outcome of the iteration of finite sums, and indecomposable ω-sums, ω^*-sums or η-sums; the process begins with the singletons and the empty chains.

In the case we consider, our chains are labelled in a slightly different way, e.g. there are different labelled singletons $(1, l)$ and different labelled empty chains $(\emptyset, l)$.

In the sequel our set L of labels is countable and b.q.o. We can define a ranking function r on the collection of countable labelled chains. The rank of a (labelled) singleton or a (labelled) empty chain is zero. If $\alpha = \sum_{i \in \mu} \alpha_i$ is a finite sum, then $r(\alpha) = \max r(\alpha_i)$. If $\mu = \eta, \omega$ or ω^* and the sum is indecomposable, then $r(\alpha) = \sup(r(\alpha_i) + 1)$. If there are different possible values of r we will take the least possible one. Because our class is b.q.o., thus w.q.o., this function is well defined.

We shall now give the definition of equimorphic classes. Two trees (T', l') and (T'', l'') belong to the same equimorphic class if $(T', l') \leqslant (T'', l'')$ and $(T'', l'') \leqslant (T', l')$.

Lemma 1. *The set of equimorphic classes of countable labelled chains (labelled by a countable b.q.o), whose rank is bounded by an ordinal $\delta' < \omega_1$, is countable.*

Proof. The set of singletons and empty chains $(1, l)$ and $(\emptyset, l)$ is countable by hypothesis. Hence the set of finite chains is countable too.

Let us suppose that the set of equimorphic classes of the set H of all chains of rank less than δ' is countable. We know that the whole class of countable labelled chains is b.q.o, thus well founded. Hence the set of left segments of H is b.q.o. Moreover a right segment F of H has finitely many minimal elements. Hence the set of right segments of H is countable. Clearly the set of left segments of H is countable too. Now, if $\alpha' = \sum_{i \in \mu'} \alpha'_i$ and $\alpha'' = \sum_{i \in \mu''} \alpha''_i$ are two indecomposable sums of rank δ' and if in addition $\mu' = \mu''$ and for any i', there is i'' such $\alpha'_{i'} \leqslant \alpha''_{i''}$, reciprocally, then α' and α'' are equimorphic. In other words if the left segment generated by $\{\alpha'_i\}_{i'}$ and by $\{\alpha''_i\}_{i''}$ in H are the same, then α' are α'' are equimorphic. It is obvious then that the set of equimorphic classes of indecomposable labelled chains of rank δ' is countable. The same is true for chains of rank δ', because any one of them is the finite sum of indecomposable chains. □

Definition. A labelled tree (T, l) is *strongly indecomposable* if $(T, l) \leqslant (T_x, l)$ for every $x \in T$. If the tree T is not labelled it is strongly indecomposable if $T \leqslant T_x$ for every $x \in T$.

It is obvious that (T_x, l) is strongly indecomposable whenever (T, l) is strongly indecomposable.

Lemma 2. *The set of equimorphic classes of branching paths of a countable tree is countable.*

Proof. We claim that the supremum γ of the ranks of all paths, $\gamma = \sup_P r(P)$, satisfies $\gamma \leqslant \sup_x r(T^x) + 1$.

Suppose $\sup_x r(T^x) + 1 < \gamma$. Then there exists a path P such that $\sup_x r(T^x) + 1 < r(P)$ and next there is $y \in P$ such that $\sup_x r(T^x) < r(P^y)$, according to the definition of P. This is a contradiction.

Now T is countable and the rank of a countable chain is countable. Hence $\sup_x r(T^x)$ is countable. Hence γ is countable too and $r(P)$ for every path, and the set of equimorphic classes of all P is countable. □

We consider here trees whose maximal chains are not labelled; other trees will be considered later on (e.g. in Theorem 2). Lemma 2 is the starting-point of the following theorem about strongly indecomposable trees.

Theorem 1. *A countable tree T, labelled or not, is strongly indecomposable if and only if $B(T_x) = B(T)$ for all $x \in T$. A countable strongly indecomposable tree contains branching chains which are maximum under embeddability. These chains are right indecomposable.*

If $B(T') \subset B(T'')$ and T' and T'' are countable trees and T'' is strongly indecomposable, then $T' \leqslant T''$.

Corollary 1. *The set of countable strongly indecomposable trees is, under embeddability, a b.q.o. set.*

Proof. Indeed Theorem 1 says that $T' \leqslant T''$ is equivalent to $B(T') \subset B(T'')$ and equivalent to $\max B(T') \leqslant \max B(T'')$. Because the class of (b, l)-labelled chains is a b.q.o. class the result follows. □

Corollary 1 bis. *In any countable tree T there is an element x for which T_x is strongly indecomposable.*

Proof. As a matter of fact $B(T_y)$ is a left segment of a class of (b, l)-labelled chains which is a b.q.o. class and therefore is well founded. Hence, there is a value x for which $B(T_y)$ reaches a minimum and $B(T_y)$ is constant for $y \geqslant x$. □

The binary tree of length ω is the minimal strongly indecomposable tree which is not a chain. Its maximum branching chain is (ω, l_1), where $l_1(x)=1$.

There is an universal countable branching chain. It is the chain η labelled by $b(x)=\aleph_0$. There is therefore an universal strongly indecomposable tree.

Let T be the set of elements $x=(n_1, r_1, n_2, r_2, \ldots, n_s, r_s)$ where $n_i \in \mathbb{N}$, $r_i \in Q$ ordered by the last possible difference, if there is any, or by the length: $x' \leqslant x''$ if $n'_i = n''_i$, $r'_i = r''_i$ for $i=1,2,\ldots,s'-1$ and $n'_{s'}=n''_{s'}$, $r'_s < r'_{s'}$ or $n'_{s'}=n''_{s'}$, $r'_{s'}=r''_{s'}$ and $s' \leqslant s''$. T is a tree. T_x where $x=(n_1, r_1, \ldots, n_s, r_s)$, contains the tree T', isomorphic to T, where elements are $y=(n_1, r_1, \ldots, n_s, r_s, n'_{s+1}, r'_{s+1}, \ldots, n'_{s'}, r'_{s'})$. Hence T is strongly indecomposable. T contains the chain $\{(0, r_1)\}_{r_1 \in Q}$ and the branching cardinal of its elements is $\aleph_0$.

Proof of Theorem 1. The set of paths with no last element is cofinal in $B(T)$. A maximal chain of a strongly indecomposable tree has no last point, except if the tree is a singleton.

If $(\tau_i)_{i\in\mathbb{N}}$ is a sequence of types of branching paths with no last element, we can write $\tau_i = \sum_{j\in\omega} \tau_{ij}$.

Let us suppose (inductive hypothesis) that the chain $C_{00}+C_{01}+C_{10}+C_{02}+C_{11}+C_{20}+\cdots+C_{p,q}+\cdots+C_{ij}+\{x_{ij}\}$ is in T, where C_{ij} are of type τ_{ij} and the order on the (p,q) is the lexicographical order of $(p+q, q)$.

The tree $T_{x_{ij}}$ contains the type of any branching chain of T (because $T \leqslant T_{x_{ij}}$), namely a chain of type $\tau_{i'}$ if (i', j') is the successor of (i, j). Hence, it contains $C_{i'j'}+\{x_{i',j'}\}$, and T contains $C_{00}+C_{01}+C_{10}+\cdots+C_{ij}+C_{i'j'}+x_{i',j'}$. Going up step by step we have proved that in T there is a chain $\sum C_{ij}$ whose type $\tau = \sum_{i,j} \tau_{ij}$ is clearly $\geqslant \tau_i$ for every i. Hence

(1) Any countable sequence of branching paths of T is bounded by another branching path of T; in particular any couple of branching paths τ' and τ'' is bounded by another path of T (indeed if $\tau_{2i}=\tau'$ and $\tau_{2i+1}=\tau''$, then $\tau \geqslant \tau', \tau''$).

Now the countability of the set of equimorphic classes of branching paths of T and (1) insure the existence of a maximum element of $B(T)$. There is only one maximum equimorphic class in $B(T)$. Moreover, if τ' is a maximum branching chain in $B(T)$ we can write $\tau' = \tau_i = \sum_j \tau_{ij}$ with τ_{ij} independent of i. Then, there exists a common bound $\tau = \sum_{i,j} \tau_{ij} \geqslant \tau_i = \tau'$. In this case τ is trivially right indecomposable, because every term occurs infinitely many times in the sum. Now $\tau' \leqslant \tau$ and $\tau \leqslant \tau'$ (because τ' is maximum). Therefore the maximum chain is right indecomposable.

We shall now prove that: $B(T')\subset B(T'')$ and $B(T'')\subset B(T''_x)$ for every $x\in T$ implies $T'\leqslant T''$.

Let us write $T'=\bigcup_{i\in\mathbb{N}} P'_i$ where the P'_i are maximal paths of T'. We know that any countable tree is the union of countably many maximal chains.

Let us suppose that the embedding f_n of $I'_n=\bigcup_{i<n} P'_i$ into I'' has been already defined.

If $P'=I'_n\cap P'_n$ is the junction path of the chains $P'_{i_1}, P'_{i_2}, \ldots, P'_{i_\delta}\subset I'_n$ with P'_n, then the branching cardinal $b(P')$ is at least δ and $f_n(P')$ is the junction path of the chains $f_n(P'_{i_1}), \ldots, f_n(P'_{i_\delta})$. Above $f_n(P')$ in $f_n(I_n)$ there are the δ chains $f_n(P'_{i_1})-f_n(P'), \ldots, f_n(P'_{i_\delta})-f_n(P')$. Now f_n is an embedding of (I_n, b', l) into (T'', b'', l'') and therefore $b''(f_n(P'))\geqslant b(P')\geqslant \delta$: therefore in T'' there is above $f_n(P')$ at least another chain. Let us pick an element x in $T''-f_n(I'_n)$ above $f_n(P')$. We know by hypothesis that $B(T'')\subset B(T_x)$, hence it is possible to embed (P'_n-P', b', l') in T''_x. Let us denote this embedding by φ. If we take $f_{n+1}\mid I'_n=f_n$ and $f_{n+1}\mid P'_n-P'=\varphi$ we get the embedding $f_{n+1}:(I_{n+1}, b', l')\rightarrow(T'', b'', l'')$ which extends f_n to the next I'_{n+1}. The general embedding f extends all f_n.

Let us suppose now that $B(T_x)=B(T)$ for every $x\in T$. It is obvious that the tree $S=T_x$ has the same property: $B(S_y)=B(T_y)=B(T_x)=B(S)$. Thus $B(T)\subset B(S)$ and $B(S_y)=B(S)$ for every $y\in S$, and hence $T\leqslant S$ or $T\leqslant T_x$. Hence $B(T_x)=B(T)$ for every $x\in T$ implies that T is strongly indecomposable. Finally, if $B(T')\subset B(T'')$ and T'' is strongly indecomposable and both are countable trees, then $T''\leqslant T''_x$. Hence $B(T'')\subset B(T''_x)$. Now $B(T')\subset B(T'')$ and $B(T'')\subset B(T''_x)$ for every $x\in T$. Hence $T'\leqslant T''$. □

A countable and direct union of trees $T=\coprod_{i\in I} T_i$ is called indecomposable if $T\leqslant\coprod_{i\in i-F} T_i$, for any finite subset F of I.

We will look next to another kind of indecomposability, related to the operation of tree sums.

Let (I, f) be a labelled and top labelled tree whose labels $f(i)$ are trees indexed by a set C of paths i of I. The maximal paths i of I are now allowed. Nevertheless C must still be countable for countable trees I. Then the set U, denoted by $\sum_{i\in I} f(i)$, is the union of I and $F=\sum_{i\in I} f(i)$, endowed with the order relation $x\leqslant_U y$, whose restrictions on I and F are its own orders. Moreover $x\leqslant_U y$ if $x\in i\subset I$, $i\in C$ and $y\in f(i)$. This tree sum consists in putting a tree $f(i)$ just above every path $i\in C$ of I.

The tree sum is an increasing operator:

$$(I', f')\leqslant(I'', f'') \Rightarrow \sum_{i'\in I'} f'(i')\leqslant \sum_{i''\in I''} f''(i'').$$

The left-hand side means that there exists an embedding g of (I', f') into (I'', f'') such that $f'(i')\leqslant f''(i'')$ for any i of C' and $g(C')\subset C''$.

Associated with any (I, f) there is a tree $U=\sum_I f(i)$ with a natural partition $U=I\cup F$ where $F=\sum_{i\in C} f(i)$. Conversely, let $U=I\cup F$ be a partition of a tree in a left segment I and right segment F. There are paths i of I such that the tree in F above i, $f(i)$, is not the same for a path $i' \supset i$ different of i. Then $f(i)$ is the tree in F just above the path i of I. Let the set of those paths be denoted by C. It is clear then that $U = \sum_{i \in I} f(i)$.

To check that the tree sum is an increasing operator: suppose $g : (I', f') \to (I'', f'')$ is an embedding and $\varphi(i') : f(i') \to f''(g(i''))$ is the embedding associated with the label $f'(i')$ of i'. If we look at the mapping defined by $\tilde{g} \mid I = g$ and $\tilde{g} \mid f'(i') = \varphi'(i')$ we get an embedding of $U' = \sum f'(i')$ into $U'' = \sum f''(i'')$. It is easy to see that the intersection i' of a maximal chain C of I' and a maximal chain of $i' \cup f(i')$ is mapped by g onto $g(i')$. Therefore the rule of the intersection of maximal chains is preserved by $\tilde{g}$.

A tree $U = \sum_I f(i)$, where I is a chain, is called right (resp. left, right-left) indecomposable if the tree sum is right (resp. left, right-left indecomposable). U is right (resp. left, right-left) indecomposable if for any non-trivial decomposition $I = I' + I''$, $U \leqslant \sum_{I''} f(i)$ (resp. $U \leqslant \sum_{I'} f(i)$, $U \leqslant \sum_{I'} f(i)$, $\sum_{i''} f(i)$).

The different types of trees we have talked about until now, are: the strongly indecomposable trees, the direct indecomposable trees, and the trees we can obtain as sums of a one side or two-side indecomposable labelled chain.

In addition of these blocks there is still another kind of block which is a composite block consisting of four units, related to each other in a decomposable way. Members consist of a tree sum $U = \sum_I f(i)$, such that $I = (I, f)$ is strongly indecomposable (i.e. $(I, f) \leqslant (I_x, f)$ for every $x \in I$ and I not a chain).

In a tree sum $U = \sum_I f(i)$, there are I and the right segment $F = \coprod_{i \in C} f(i)$. Our tree has three parts: L, S and T, related to three parts C_i of C. C_1 contains the paths of C which are not maximal in I, C_2, contains the paths of C which are maximal in (I, f^*) but which are not of maximum type, and finally C_3 contains the paths of C whose types restricted to (I, f^*) are all equal to the greatest type of (I, f^*) ((I, f^*) stands for (I, f) whose labels in the top are deleted). It is obvious that (I, f^*) is strongly indecomposable too and has therefore (Theorem 1) a branching chain of the greatest type.

Our new trees (I, f) are now labelled and top labelled. Their structure is slightly different from that of the labelled trees when they are strongly indecomposable. A strongly indecomposable labelled and top labelled tree has in general no branching chain of the greatest type. Nevertheless some of the statement of Theorem 1 remains true.

Theorem 2. *A countable labelled and top labelled tree $J = (I, f)$ is strongly indecomposable if and only if $B(J_x)$ is constant for every $x \in I$ and equal to $B(J)$. If J is strongly indecomposable and countable, $B(J)$ is generated by a countable set of branching chains, namely for any set of branching chain $\{(P_n, b, f)\}_{n \in \mathbb{N}}$ such that $I = \bigcup_{n \in \mathbb{N}} P_n$.*

Moreover if $B(J') \subset B(J'')$, and J', J'' are countable and J'' is strongly indecomposable, then $J' \leqslant J''$.

Corollary 2. *Let J' and J'' be strongly indecomposable and $I' = \bigcup_{n\in\mathbb{N}} P'_n$. If $J' \not\leqslant J''$ then there exist an $n \in \mathbb{N}$ such that the chain (P'_n, b, f') is not embeddable in J''.*

We recall that all the time the ranges of f' and f'' are supposed to be b.q.o. The corollary does not need a proof. It is easy to see that $L = \coprod_{C_1} f(i)$, $T = \coprod_{C_2} f(i)$ are direct countable indecomposable sums. The same is true for $S = \coprod_{C_2} f(i)$ if (I, f) is strongly indecomposable and $I \cup L \cup S \cap T$ is not equimorphic to $I \cup L \cup T$. We will not need these properties.

Proof of Theorem 2. Suppose J'' is strongly indecomposable and $I'' = \bigcup_n P''_n$. *Suppose too that any branching chain of J' is embeddable in one of the chains (P''_n, b, f'') for one $n \in \mathbb{N}$. We will prove then that if $J' = \bigcup_n P'_n$ there exists an embedding h from J' into J'' such that any one of the chains (P'_n, b, f') is sent into a chain (P''_n, b, f'').* And this is true for any decomposition of $J' = \bigcup_n P'_n$. Hence any branching chain of J' is embeddable in one of the chains (P''_n, b, f''). If we apply the above to $J' = J''$ and J'', it becomes clear that any branching chain of J'' is embeddable in one of the chains (P''_m, b'', f''). Hence $B(J'')$ is generated by a countable set of branching chains of J''.

Let $I' = \bigcup_{n\in\mathbb{N}} P'_n$ and $I'' = \bigcup_{n\in\mathbb{N}} P''_n$ be two decompositions in maximal chains. Suppose that any branching chain of J' is embeddable in one of the chains $\{(P''_n, b'', f'')\}_n$ and that J'' is strongly indecomposable. We put $I'_n = \bigcup_{i<n} P'_i$. Suppose that the embedding $h_n = (I'_n, b', f') \to (I'', b'', f'')$ has already been defined, and let P' be a junction path. We know that the branching cardinals $b(P; I'_n)$ and $b(h_n(P); h_n(I'_n))$, restricted to I'_n and to $h_n(I'_n)$, are finite and equal. Moreover $b'(P') \succ b'(P'; I'_n)$ and $b'(P') \leqslant b''(h_n(P'))$ (because h_n is an embedding of I'_n *into* I'').

Hence $b''(h_n(P')) \geqslant b'(P'_n) \succ b'(P'_n, I'_n) = b''(h_n(P'), h_n(I''))$ and therefore there is an x above $h_n(P')$ in $I'' - h_n(I'_n)$. Now $J'' \leqslant J''_x$ and $I'' = \bigcup_n P''_n$. By hypothesis any chain of J', i.e. $(P'_n - P', b', f')$, is embeddable in a chain (P''_n, b', f'') of J'' and hence in J''_x. Let φ be the embedding from $(P'_n - P', b', f')$ into J''_x. Finally $h_{n+1} \mid I'_n = h_n$ and $h_{n+1} \mid P'_n - P' = \varphi$ are an embedding from (I'_{n+1}, b', f') into (I'', b'', f''). Hence if $B(J') \subseteq B(J'')$ and J'' strongly indecomposable then $J' \leqslant J''$. Following the proof of Theorem 1 we get the converse: if $B(J_x)$ is constant and equal to $B(J)$, then J is strongly indecomposable. □

It should also be observed that in certain cases, when the labels are trees, the four units block $U = \sum_I f(i)$, with (I, f) strongly indecomposable, has as parts $L = \coprod_{i\in C_1} f(i)$, $S = \coprod_{i\in C_2} f(i)$ and $T = \coprod_{i\in C_3} f(i)$ which not only are indecomposable direct sums but also are built with strongly indecomposable trees $f(i)$.

In the following example U has this property.

Let U be a set of $x = (n_1, \nu_1, n_2, \nu_2, \ldots, n_s, \nu_s)$ with $\sum \nu_i < \omega^4$ and $n_i \leqslant 3$ ordered by the last and only the last difference or if there is any difference by the length, i.e.
$$n'_i = n''_i, \quad \nu'_i = \nu''_i, \quad i = 1, \ldots, s'-1, \quad n'_{s'} = n''_{s''} \quad \text{and} \quad \nu'_{s'} < \nu''_{s''}$$

or

$$(n'_1, \nu'_1, \ldots, n'_{s'}, \nu'_{s'}) = (n''_1, \ldots, n''_{s'}, \nu''_{s'}) \quad \text{and} \quad s' \leqslant s''.$$

Let $E = \{i \mid i \text{ even and } \nu_i \geqslant \omega^2 i\}$,

$$P = \{i \mid \nu_i \geqslant \omega^3\} \quad \text{and} \quad B = \{i \mid n_i \geqslant 1 \text{ and } i < \min(E \cup P \cup \{s+1\})\}.$$

The set U satisfies one of the following restrictions:

(i) $E = P = B = \emptyset$;

(l) $E = \emptyset$, $B = \emptyset$ and $n_i \leqslant 1$ for $i \geqslant \min B$;

(s) $P = B = \emptyset$, $E \neq \emptyset$ and $\nu_i < \omega^2$, $n_i \leqslant 2$ for $i > e = \min E$ and $\nu_e < \omega^2(e+1)$;

(t) $B = \emptyset$, P, $E \neq \emptyset$, $p = \min P < e$ and $\nu_i < \omega$ for $i > p$ and $\nu_p < \omega^3 + \omega$.

Then $U = I \cup S \cup L \cup T$ where the elements of I (resp. L, S, T) satisfy (i) ((l), (s), (t)).

The elements x of I satisfy $n_i = 0$, $\nu_i < \omega^3$ for $i = 1, 2, \ldots, s$. If $x' = (x, 1, \nu_{s+1})$ then $x < x' \in L$. If $x'' = (x, 0, 0, 0, \omega^2(s+2))$ with s even or $x'' = (x, 0, \omega^2(s+1))$ with s odd then $x < x'' \in S$. Finally if $x''' = (x, 0, \omega^3)$ we have $x < x''' \in T$.

An element x of I satisfies $n_i = 0$, $\nu_i < \omega^2$ for $i \leqslant s$. (I, b) is the binary tree of length ω^3. An element of L satisfies $\nu_i < \omega^4$ and $n_i \leqslant 1$. Any connected part of L is the ternary tree of length ω^4. Finally any connected part of s (resp. of T) is the 4-ary (5-ary) tree of length ω^2 (resp. ω).

A minimal element x of L satisfies $n_i = 0$, $\nu_i < \omega^2$ for $i = 1, 2, \ldots, b-1$ and $n_b = 1$, $\nu_b = 0$, hence $x = (0, \nu_1, \ldots, \nu_{b-1}, 1, 0)$. The last element before x is $x' = (0, \nu_1, \ldots, \nu_{b-1}) \in I$ and there is $x'' = (0, \nu_1, \ldots, \nu_{b-1}+1)$ still in I such that $x' < x''$ and incomparable to x. Hence the restriction on I of the maximal chains in $I \cup L$ is not maximal.

A minimal element x of S satisfies $n_i = 0$, $\nu_i < \omega^3$ and $\nu_i < \omega^2 i$ for i even and $i = 1, 2, \ldots, e-1$ and $n_e = 0$, $\nu_e = \omega^2 e$. Then the last element $x' < x$ such that $s' = e - 1$ satisfies $n'_i = n_i$, $\nu'_i = \nu_i$ for $i < s - 1$ and $n'_{e-1} = n_{e-1}$, $\nu'_{e-1} < \nu_e = \omega^2 e$. Thus the restriction of a maximal chain of $I \cup S$ is maximal in I and its length is $\omega^2 e$; also $\sup_e \omega^2 e = \omega^3$. It is clear that the maximal chains in $I \cup S$ are not the extension of chains of I of the greatest length ω^3.

On the contrary one minimal element x of T' satisfies $n_i = 0$ and $\nu_i < \omega^3$ for $i = 1, 2, \ldots, q-1$ where $q = \min\{p, e\}$, and $s = q$, and $p < e$, hence $n_s = 0$ and $\nu_q = \omega^3$. The elements $x' < x$ with $s' = s$ are such that $n'_i = n_i$, $\nu_i = \nu'_i$ for $i \leqslant s-1$ and $n'_s = n_3 = 0$, $\nu'_s < \omega^3$. The length of the restriction of a maximal chain in $I \cup T$ is ω^3. The greatest one possible.

In short, the indecomposable blocks are: the side indecomposable trees, the indecomposable countable direct sum of trees and finally the trees $U = \sum_I f(i)$, with (I, f) strongly indecomposable, which are the union of four indecomposables parts: $U = I \cup L \cup S \cup T$. The latter kind of trees possess a countable indecomposable set $\{(P_n, b, f)\}$ of branching chains generating $B(I, f)$.

We define a canonical tree $V = \bigcup_\Phi U_j$ as the finite union of indecomposable trees or blocks such that the blocks which are indecomposable under direct sums and the blocks $U_j = \sum_{I_j} f_j(i)$ with (I_j, f_j) strongly indecomposable are always in the top of the *finite tree Φ. If $j' \in \Phi$ is not a top element and $j'' > j'$, then the elements of $U_{j''}$ are above the chain $I_{j'}$ of $U_{j'} = \sum_{i \in I_{j'}} f_j(i)$.*

If $V' = \sum_{\Phi'} U'_{j'}$ and $V'' = \sum_{\Phi''} U''_{j''}$ and there is an embedding $h : \Phi' \to \Phi''$ such that there is an embedding

$$h_{j'} : \sum_{I'_{j'}} f_{j'}(i) \to \sum_{I''_{h(j')}} f_{h(j')}(i)$$

sending $I'_{j'}$ into $I''_{h(j')}$—we write in short $(\Phi', U') \leqslant'' (\Phi'', U'')$—then any g such that $g \mid U'_{j'} = h_{j'}$, is an embedding from V' into V''.

Hence,

Lemma 3. *If $V' = \sum_{\Phi'} U'$ and $V'' = \sum_{\Phi''} U''$ are canonical countable trees then*

$$(\Phi', U') \leqslant'' (\Phi'', U'') \Rightarrow V' \leqslant V''.$$

Our main idea is to find the good class—the canonical class—of countable trees to prove later that this class is better quasi ordered; and owing to this particular order, to prove finally that any countable tree is a canonical tree.

In the sections we study quasi ordered ordinal algebras. The reader might go directly to III.

2. Finite basis theorems about ordinal algebras

Let A be a class and M an operator domain with arity $a : M \to O$ (where O is the class of ordinals). An operator $m \in M$ with arity $\alpha = a(m)$, is a mapping $m : A^\alpha \to A$, i.e. to any α-sequence $(a_i)_{i<\alpha}$ in A, m associates an element $m((a_i)_{i<\alpha})$ in A.

The set A with this structure is called an M-algebra.

Suppose further that there exists a quasi-order on A and M in such a way that the operators are increasing and extensive ones and the values in A are increasing with the operator itself, i.e. $f((a_i)_{i<\alpha}) \leqslant (g(b_j)_{j<\beta})$ whenever $(a_i)_{i<\alpha} \leqslant (b_j)_{j<\beta}$ and $f \leqslant g$; $a_j \leqslant f((a_i)_{i<\alpha})$ for every $j < \alpha$.

The quasi-order of the sequences $(a_i)_{i<\alpha} \leqslant (b_j)_{j<\beta}$ means that there exists $\varphi : \alpha \to \beta$ increasing and injective, such that $a_i \leqslant b_{\varphi(i)}$ for $i < \alpha$. This is the quasi-order of the labelled chains (α, a) with $a(i) = a_i$.

The M-algebra A with this quasi-order, is called an M-ordinal algebra (it is known also as an M-algebra with a divisibility quasi-order).

We call, as usually, basis a subset that generates A.

Pouzet's basis theorem may be stated as follows:

Let A be an M-ordinal algebra. If a basis B of A and M is better quasi-ordered, then A is better-quasi-ordered.

If we restrict the arities of M to be finite arities (stronger condition) and replace the b.q.o. condition by a well-quasi-ordered one, (weaker condition), we get the well-known Higman's theorem.

We need here to look at infinitary algebras (with infinite arities). In this case the key-tool is Pouzet's theorem.

The investigation of the class of countable trees under embedding leads to wider classes of operators. In a first approach, M is the class of chains generated by the rational chain η and the scattered chains. In a second one, it is necessary to adjoin the operators associated with the strongly indecomposable countable trees.

We only adjoin now operators m, such that $a(m)=\eta$, and $m: A^{\eta} \to A$. The sequence-quasi-order is, as usually, the chain-labelled-order.

Theorem 4. *Let A be an M-ordinal algebra such that $a(M)=O\cup\{\eta\}$. If a basis B of A and M is b.q.o., A is also b.q.o.*

The proof of this theorem will use the next definitions and lemmas.

Definition. We call an α-sequence $(a_i)_{i\in\alpha}$ in A a regular sequence, if and only if

(1) α is a finite ordinal, or an infinite regular ordinal or η;

(2) If α is infinite regular (resp. $\alpha=\eta$), then $(a_i)_{i<\alpha} \leqslant (a_i)_{j\leqslant i<\alpha}$ for every $j<\alpha$ (resp. $(a_i)_{i\in\eta} \leqslant (a_i)_{j<i<k}$ for every $j, k\in\eta$).

Let us denote I_j (resp. I_{jk}) the left segment in A, $I_j=\{x \mid x\in A,\ x\leqslant a_i$ for some $i\geqslant j\}$ (resp. $I_{jk}=\{x \mid x\in A,\ x\leqslant a_i$ for some $i\in\eta$ such that $j<i<k\}$).

Trivially condition (2) implies $I_j=I_0$ *constant* (*resp.* I_{jk} *constant*) and conversely, $j\to I_j$ *constant* (*resp.* $(j,k)\to I_{jk}$ *constant*) implies 2.

Notations. If $s=(a_i)_{i\in\alpha}$, $u(s)=\alpha$ and $I(s)=\{x\in A \mid x\leqslant a_i$ for some $i\in\alpha\}$.

Lemma 4. *If s and t are infinite regular sequences in A, $I(s)\subset I(t)$ and $u(s)\leqslant u(t)$ imply $s\leqslant t$.*

Proof. *Case 1:* $u(t)\neq\eta$. If t is a regular sequence $t=(b_j)_{j<\beta}$, the subset of β, $\{k<\beta \mid k\geqslant b_j, b_k\geqslant b_j\}$, is cofinal in β for any $j<\beta$. If a_i is any element of s, $E(s)\subset E(t)$ implies that for every $j\geqslant i$ there is a k such that $b_k\geqslant a_i$, in any right segment of β. All this enable us to define $\varphi:\alpha\to\beta$ increasing injective such that $a_i\leqslant b_{\varphi(i)}$ (the cardinal of a proper left segment of α is $<|\beta|$).

Case 2: $u(s)=u(t)=\eta$. If t is a regular η-sequence $t=(b_j)_{j\in\eta}$, the subset of η, $\{p\in\eta \mid k<p<l,\ b_p\geqslant b_j\}$, is dense in $\eta\cap]k,l[$. If a_i is any element of $s=(a_i)_{i\in\eta}$, $E(s)\subset E(t)$ implies the existence of b_j such that $a_i\leqslant b_j$, therefore *the set of* $p\in\eta$ such that $a_i\leqslant b_p$ is dense in $\eta\cap]k,l[$ for any $k<l$ in η. We construct thus, as usually, the embedding of $(a_i)_{i\in\eta}$ in $(b_j)_{j\in\eta}$. □

Definitions. Let O_r be the set of all regular ordinals and S the union of O_r, N and $\{\eta\}$. Let us denote $S(A)$ the set generated by iterating ordinal α-sums $\alpha\in S$, of sequences in A and mappings of M and again transfinitely in the new obtained sets. Let us denote $\tilde{S}(A)$ the subset of $S(A)$ obtained by iterating only α-sums and m-mappings with regular sequences. The α-sum of $(s_j)_{j\in\alpha}$ is defined as usually by $s=\sum_{j\in\alpha} s_i=(t_{ji})_{(j,i)\in\beta}$ where $s_j=(t_{ji})_{i\in\beta_i}$ and $\beta=\sum_{j\in\alpha}\alpha_j$.

Lemma 5. *If* $\tilde{S}(A)$ *is better quasi-ordered then* $S(A)=\tilde{S}(A)$. *(We recall that* $S=O_r\cup N\cup\{\eta\}$.*)*

Proof. *Case 1:* $\alpha\in N$. There is nothing to prove, because every finite sequence is defined regular.

Case 2: $\alpha\in O_r$. We proceed inductively.

Suppose that $s=\sum_{i\in\alpha} s_i$, $s_i\in\tilde{S}(A)$ and $\nu<\alpha$, $\nu\in O_r$ implies $s\in\tilde{S}(A)$.

Let s be α-sum, in $\tilde{S}(A)$, i.e. $s=\sum_{i<\alpha} s_i$ where $s_i\in\tilde{S}(A)$. The set $\{I_j\}_{j<\alpha}$ where $I_j=\{x\in\tilde{S}(A)\mid x\leqslant s_i$ for some $i\geqslant j\}$ is a set of left segments in this b.q.o. set $S(A)$. Thus $\{I_j\}_{j<\alpha}$ is b.q.o. and a fortiori well founded. Let I_{j_0} be minimal, hence $I_j\supset I_{j_0}\supset I_j$ for every $j\geqslant j_0$ in α and I_j is constant (when $J_0\leqslant j<\alpha$). This amounts to say that the sequence $\{s_i\}_{j_0\leqslant i<\alpha}$ is regular. Therefore $\sum_{j_0\leqslant i<\alpha} s_i\in\tilde{S}(A)$ and $\sum_{i<j_0} s_i\in\tilde{S}(A)$ (consequence of the inductive hypothesis), hence $s\in\tilde{S}(A)$.

Case 3: $\alpha=\eta$. $s=\sum_{i\in\eta} s_i$, $s_i\in\tilde{S}(A)$.

Suppose $s\in\tilde{S}(A)$. Then $(s_i)_{i\in\eta}$ is not regular. Let us take left segments in $\tilde{S}(A)$.

$I_{jk}=\{x\in\tilde{S}(A)\mid x\leqslant s_i$ for some $i\in\eta$ such that $j<i<k\}$. For every $p<q$ in η there is j_0, k_0 such that $p<j_0<k_0<q$ such that I_{j_0,k_0} is minimal, i.e. $I_{j_0,k_0}\subset I_{j,k}$ for $p<j<k<q$. Thus $I_{j,k}$ is constant when $j,k\in]j_0,k_0[\cap\eta$. This amounts to say that the medium regular segments of s—and consequently their sum in $\tilde{S}(A)$—are dense in η. We remark now that every $\sigma_{j,k}=\sum_{j<i<k,i\in\eta} S_i$ such that $\sigma_{j,k}\in\tilde{S}(A)$ is contained in a $\sigma_{i',j'}$, maximal and in $\tilde{S}(A)$. Indeed the union of ω-sequence of medium segments whose sums are in $\tilde{S}(A)$ is also in $\tilde{S}(A)$. (It is an $(\omega^*+\omega)$-sum or ω^* or ω-sum.) Thus the set M of maximal medium segments in $\tilde{S}(A)$ is dense in η. It is also dense in itself. Suppose $\sigma_1,\sigma_2\in M$ are adjacent or either separated by only a point $\{s_i\}$. Both cases are impossible, because $\sigma_1,\sigma_2\in\tilde{S}(A)\Rightarrow\sigma_1+\sigma_2\in\tilde{S}(A)$ (first case) or $\sigma_1+\{s_i\}+\sigma_2\in\tilde{S}(A)$ in the second case. In both cases σ_1 and σ_2 would be not maximal. Thus the induced order in the set of medium segments M by η has just the same type $\tau(M)=\eta$. Hence $s=\sum_{\sigma\in M}\sigma$ with $\tau(M)=\eta$ and $\sigma\in\tilde{S}(A)$. We know again that there is a medium non-trivial segment M' of M such that $\sigma'=\sum_{\sigma\in M}\sigma$, $\sigma'\supsetneqq\sigma$, $\sigma'\in M$. This is impossible. This contradiction proves *ad absurdum* that $s\in\tilde{S}(A)$. □

We will denote $\tilde{S}_M(B)$ the set obtained by iterating the mappings of M on B and the α-regular-sums of sequences with $\alpha\in S$. Thus $\tilde{S}_M(B)$ is closed by the M-operators and the regular sum of sequences on $\tilde{S}_M(B)$, $\tilde{S}_M(B)\subset\tilde{S}(A)$. We will prove next that $\tilde{S}_M(B)=S(A)$.

We come now to the ranking of $S_M(B)$. It is a mapping $r:S_M(B)\rightarrow O\cdot O^*$ where $O\cdot O^*$ is the ordinal product of the class of ordinals by itself minus zero.

The purpose is that r must fulfil the condition $r(s_i) < r(\sum_{i<\alpha} s_i)$ (strictly extensibility, whenever $\alpha \in O_r \cup \{\eta\}$).

Definition. For every $b \in \tilde{S}_M(B)$, $r(b)$ is the least $r(b') \in O \cdot O$ for $b' \geqslant b$, $b' \in \tilde{S}_M(B)$ such that $r(b') \geqslant (0.1)$, $r(m(b')) \geqslant r(b')$ and if $(r_1(b_i))_{i\in\alpha}$ has a greatest element:

$$r\Big(\sum_{i\in\alpha} b_i\Big) \geqslant \Big(\max r_1(b_i), \sum_{i\in\alpha} r_2(b_i)\Big), \quad \alpha \in O_r,$$

$$r\Big(\sum_{i\in\eta} b_i\Big) \geqslant (\max r_1(b_i) + 1, 1);$$

otherwise $(r_1(b_i))_{i\in\alpha}$ without a greatest element)

$$r\Big(\sum_{i\in\alpha} b_i\Big) \geqslant \Big(\sup_{i\in\alpha} r_1(b_i), 1\Big).$$

There is always a least element in any subset of $O \cdot O^*$. Thus there is no ambiguity in the definition of $r(b)$. The last three conditions make sure that it is strictly extensive.

We will, hence, go forward to prove that $\tilde{S}_M(B)$ is b.q.o.

Suppose $S_M(B)$ is not b.q.o., then there exist bad sequences in $\tilde{S}_M(B)$. It is then known that there exist r-minimal bad sequences. That needs an explanation.

We call D an extended restriction of the barrier C, denoted $C < D$, if every element t of D is an extension of an element s of C, $s < t$ (that means that s is a left segment of t). There is only one s for each t, denoted $s = d(t)$ (because C is a barrier). It is possible that $s = t = d(t)$ for every $t \in D$, hence it is possible that $D \subset C$. Sometimes there is a $t \neq d(t) \gneqq s$ and then D is a proper extension of a restriction of C.

If $f: C \to H$, $g: D \to H$, we say that g is an subextended restriction of f if

(1) D is an extended restriction of C;

(2) $g(t) \leqslant f(d(t))$, $r(g(t)) \leqslant r(f(d(t))$;

(3) When $t = d(t)$, $r(g(t)) = r(f(d(t))) \Rightarrow g(t) = d(d(t))$;

(4) $d(t) < t \Rightarrow g(t) < f(d(t))$, $r(g(t)) < r(f(d(t)))$.

We say g is a proper subextended restriction of f, denoted $g < f$, when there exists $t \in D$ such that $r(g(t)) < r(f(dt))$. f is minimal bad when there is no bad proper subextended restriction of f.

To build a minimal bad mapping is a matter to get r minimal.

For instance, if $f: C \to H$ is not minimal bad, then there exists a bad $g: D \to H$ such that $g < f$. Let K and L be respectively the bases of C and D, and let $[\mathbb{N}]^{<\omega}$ be ordered by the last difference or by inclusion if there is no last difference. It is clear that with this order $[\mathbb{N}]^{<\omega}$, and also any barrier, is isomorphic to $\mathbb{N}$.

We choose the smallest $c_0 = d(d_0)$ such that there is a bad $g < f$ with $r(g(d_0)) < r(f(c_0))$. Let us write $K = K' + K''$, $L = L' + L''$ where the last element of K', L' and

c_0 is k_0. If C_0 is the set of elements of $C\cap[K'+L'']^{<\omega}$ which have no extension into an element of D, then $C_p\cap D\neq\emptyset$.

We change g into $h:E\to H$, putting $E=C_0\cup D$, $M=K'+L''$, $h\mid C_0=f\mid C_0$, $h\mid D=g\mid D$.

It is not difficult to check that E is a barrier whose base is M, that h is bad $<f$, and that f and h are equal on the left segment of C, $C'=C\cap[\leftarrow c_0[=E\cap[\leftarrow c_0[=E'$ and that the bases in $\mathbb{N}\cap[0,k_0]$ are the same. We will iterate next this construction, denoting $h=\lambda(f)$.

Then $f_0=f$ and $f_{n+1}=\lambda(f_n)$, with $c_n=d(d_{n+1})$ smallest such that $r(f_{n+1}(d_{n+1}))<r(f_n(C_n))$, $C_n'=C_n\cap[\leftarrow c_n[$ and $K_n'=K_n\cap[\leftarrow h_n]$, $k_n=\max c_n$.

If no f_n is minimal bad, we define $f_\omega:C_\omega\to H$ by

$$K_\omega=\bigcup K_n',\qquad C_\omega=\bigcup C_n'\quad\text{and}\quad f_\omega\mid C_n'=f_n\mid C_n'.$$

Trivially, C_n is increasing, like K_n' and C_n' are. Moreover c_n is not bounded. Indeed, if c_n would reach a bound c for $n\geqslant n_0$, the sequence $(r(f_n(c))_{n>n_0})$ would be strictly decreasing, and the range of r would be not well founded. Hence $k_n\in K_n'\subset K$ is not bounded either, and K is infinite. We check likewise that C_ω is a barrier on K_ω and f_ω is bad $<f_n$. Trivially f_ω is r-minimal bad.

Suppose now that $f:C\to\tilde{S}_M(B)$ is minimal.

We recall that all the α-sequence in $\tilde{S}_M(B)$ are regular sequences and $\alpha\in O_r\cup N\cup\{\eta\}$. Then $f(s)=(f_i(s))_{i\in\delta_s}$. Three cases are possible:

(1) $\delta_s=1$;

(2) $1<\delta_s<\omega$;

(3) $\delta_s\in O_r\cup\{\eta\}$.

There is a restriction of f such that for all s we always get the same case (Ramsey–Nash-William's theorem).

(2) $1<\delta_s<\infty$. Then there exist a decomposition and a restriction such that $\delta_s=\delta_s'+\delta_s''$ with δ_s', $\delta_s''>0$ and $s\to(f_i(s))_{i<\delta_s'}$, $s\to(f_i(s))_{i<\delta_s''}$ are good. This arises from the minimal character of f and the lesser rank. Then there is another restriction such that these sequences are perfect. Finally $s\to f(s)$ is good. We dismiss this case.

(3) $\delta_s\in O_r\cup\{\eta\}$. ($\delta_s\leqslant\delta_t$ in O_r or $\delta_s=\delta_t=\eta$.) The sequence $(f_i(s))_{i\in\delta_s}$ for every s is regular and $s\lhd t\Rightarrow(f_i(s))_{i\in\delta_s}\not\leqslant(f_j(f))_{j\in\delta_t}$. Then Lemma 1 enables us to claim that there exists $f_i(s)$ such that $f_i(s)\not\leqslant f_j(t)$ for every $j\in\delta_t$. We associate $u=s\cup t$ with $s,t\in C$, $s\lhd t$; we will denote $f_i(s)$ by $g(u)$. Then $g(u)=f_i(s)<f_j(t)$ for every $t\in C$. Hence we have $g(u)\leqslant g(w)$ (where $w=t\cup v$, $t\lhd v$), if $f_j(t)=g(w)$.

The sequence $g:C^2\to\tilde{S}(B)$ is bad ($g(u)\not\leqslant g(w)$ with $u\lhd w$). C^2 is a proper extension of C and the rank $r(g(u))=r(f_i(s))<r(\sum_{i\in\delta_s}f_i(s))$ is strictly lesser. Then f is not minimal bad. We dismiss this case also.

If $\delta_s=1$, then $f(s)=m_{1s}\circ m_{2s}\circ\cdots\circ m_{rs}((g_j(s))_{j\in\delta_s'})$, $r=k(s)$ where $m_{ks}\in M$ and $m_{r-1,s}$ is the last 1-ary operator. But M is b.q.o. and the m_{ks} are increasing extensive. There is a restriction such that $s\to m_{1s}\circ\cdots\circ m_{rs}$ is perfect. Consequently $s\to(g_j(s))_{j\in\delta_s'}$ is bad.

Now there are only one possibility left: $\delta_s'=1$ and $g(s)\in B$. This is not possible either because B is also b.q.o. This contradiction concludes the proof of theorem 4. □

Let $S[\eta]$ be the class of chains generated by the one element chain by iterating α-ordinal sums with $\alpha\in a(M)=O_r\cup O_r^*\cup\{\omega\backslash\{0\}\}\cup\{\eta\}$, where O_r^* denotes the set of dual regular ordinals.

Thus $S[\eta]$ is an M-algebra, whose arities belong to the set $a(M)$. Let us identify the operator and its arity, then $M=a(M)$. M is the union of three well-ordered chains, the basis B is the chain of one element. M and B are b.q.o., so is $S[\eta]$.

Let $S_r[\eta]$ be the class of chains labelled in $\{0, 1\}$ generated by M and let the basis be $(\{\emptyset\}, l')$, $(\{1\}, l_0)$, $(\{1\}, l_1)$ where $l'(\emptyset)=1$, $l_i(1)=i$. We do not allow chains such that before an element $c=(\{\emptyset\}, l')$ there is a last element, i.e. the path P before c is open and $l'(\{\emptyset\})$ is the label of P. There is no element $(\{\emptyset\}, l'')$, with $l''(\emptyset)=0$. $S_e[\eta]$ is clearly b.q.o.

Theorem 5. *Let a be an M-ordinal algebra whose arity set, $a(M)$, is $S_e[\eta]$. If M and a basis B are b.q.o. then A is b.q.o.*

Proof. Theorem 4 leads us to consider the set of sequences in A, $S(A)$ as an $(M\cup L)$-algebra. If $m\in M$ and $s=(s_i)_{i\in\alpha}$ have the same arity $\alpha\in S_e[\eta]$, then, if S is the iterating α-sums of sequences s_i, $S\rightarrow m(s)$ becomes an 1-any operator, when we associate $m(s)$ to S. Thus $S(A)$ is generated by $(\{\emptyset\}, l')$ and $(\{1\}, l_0)$ and $(\{1\}, l_1)$ with L-ordinal sums and the operators of M. We know that L and M are b.q.o., then $L\cup M$ is b.q.o. The basis has only three elements. Hence $S(A)$ (the set of α-sequences in A when $\alpha\in S_e[\eta]$) is b.q.o. and particularly, A is b.q.o.

If (T, t) is a labelled tree whose labels are trees (for every path P in T, $t(P)$ is a tree, empty or not), then $\tau=(t_P)_P$ (P a path in T) is a sequence of trees. Then the ordinal tree-sum $S(\tau)$ is defined as:

(1) A tree whose underlying set is $T\cup(\bigcup_P t_P)$ (P a path in T) and,

(2) Its order relation in T and in t_P (for all P) is the same plus the relations $x\leqslant y$ when $x\in P$ and $y\in t_P$. In fact $S(\tau)$ is the tree we obtain when we place the tree t_P above every path P in T. The arity of S and τ is T.

Two particular cases are important: (1) when the tree T is a chain; (2) when the tree T is an antichain.

Definition. *A tree T is slender if it does not embed (as a partially ordered set) the binary tree B of height ω, that we denote by $B\not\leqslant_o T$.*

If a tree T is such that for every x, T_x is not a chain (the empty tree is a chain) *then $B\leqslant_o T$.* Let T' be the subtree of T defined by $T'=\{x\in T \mid T_x$ is not a chain$\}$. Define inductively T^α by $T^{\alpha+1}=(T^\alpha)'$ and $T^\alpha=\bigcup_{\nu<\alpha}T^\nu$ when α is a limit ordinal. There is a first α, such that $T^{\alpha+1}=T^\alpha$. If $T^{\alpha+1}=T^\alpha\neq\emptyset$ then for every

$x \in T^\alpha$, T_x^α is not a chain and $B \leqslant_o T^\alpha \leqslant_o T$. Conversely, if $B \leqslant_o T$, we have by induction $B \leqslant_o T^\beta$ and $B \leqslant_o T^\alpha = T^{\alpha+1}$, $T^\alpha \neq \emptyset$. Hence

Proposition. *A tree T is slender iff* $T^\alpha = T^{\alpha+1} = \emptyset$.

Definition. If T is an slender tree, the degree $d(t)$ is the least ordinal α such that T^α is a chain or an antichain.

Let A be the class of all slender trees T such that the chain of $\hat{T}$ is in $S[\eta]$ (the class of η-completed scattered chains).

Let M be the set of all α-sums of tree sequences with $\alpha \in S_e[\eta]$ or either α an antichain. The arities of the antichains are in $N \cup O_r \cup S_e[\eta]$.

Theorem 6. *Let A be the class of slender trees T such that the chains of T are in* $S[\eta]$; *let M contain all the* α*-sums of* α*-tree sequences when* α *is a chain in* $S_e[\eta]$, *or either* α *is an antichain. Then the singleton tree is a basis of A, and A is b.q.o.*

Proof. The problem is now that (A, M) is not an ordinal M-algebra. Indeed the α-tree sums, or sums of trees following a α chain, are not increasing. There is a way to avoid this difficulty.

Let $\tilde{A}(1)$ be (1 denotes the singleton tree) the subalgebra generated by the singleton and the indecomposable operations plus the finite operations. In every step we get indecomposable trees of first or third class (sided indecomposable, or directed indecomposable). Let us endow $\tilde{A}(1)$, step by step with a new order denoted $\leqslant'$: if T' and T'' are first class indecomposable with the chains of indecomposability C' and C'' then $T' \leqslant' T''$, if there exists an embedding f from T' into T'' that maps C' into C'' (It is possible to get $T' \leqslant T''$ and $T' \leqslant' T''$.) This order is indeed strictly stronger than the order of trees under embedding.

$(\tilde{S}(1), M)$ endowed with the order $\leqslant'$ is a true M-ordinal algebra. Therefore Theorem 5 applies. Hence $(\tilde{A}(1), \leqslant')$ is b.q.o. and $(\tilde{A}(1), \leqslant)$ with a weaker order is a fortiori a b.q.o. set.

We know that every α-tree sum, of trees in $\tilde{A}(1)$, is a finite sum of indecomposable sums and thus $\tilde{A}(1) = A(1)$ (Theorem 3).

Suppose now that every $T \in A$ such that $d(T) < \alpha$ belongs to $A(1)$ (subalgebra generated by the singleton).

If $d(T) = \alpha$, then $T^\alpha \neq 1$ is a chain or antichain. Let i be a path in T and T_i the tree in $T - T^\alpha$ above i. Its clear that $d(T^i) < \alpha$, i.e. $T^i \in A(1)$. But if α is a chain or antichain and $T^i \in A(1)$, T is an α-sum, it is canonical and $T \in A(1)$. Hence $A(1) = A$. There is nothing left to prove. □

3. Better quasi ordering of the class of countable trees

We shall define a ranking and a new order in the class of countable canonical trees $\mathscr{C}$. The transfinite construction of $\mathscr{C}$ begins with the singleton tree. The operations are:

(a) The finite canonical sums;

(b) the direct indecomposable sums;

(c) the chain indecomposable sums;

(d) the strongly indecomposable labelled and top labelled sums.

All these operations are increasing except the canonical finite sums.

If by the embedding of side indecomposable trees the existence of an embedding $\varphi : \sum_{I'} f(i') \to \sum_{I''} f(i'')$ such that φ sends the chain I' into the chain I'' is meant, then Lemma 3 says that

$$(\Phi', U') \leqslant (\Phi'', U'') \Rightarrow \sum_{\Phi'} U'_{ji} \leqslant \sum_{\Phi''} U''_{ji}.$$

As a matter of fact there exists an embedding φ sending $\sum_{I'} f(i')$ into $\sum_{I''} f(i'')$, but not I' into I''. From now on we will be concerned with this stronger embedding relation, which will be carried on step by step. It will be denoted $(\mathscr{C}, \leqslant')$. It is now clear that $(\mathscr{C}, \leqslant')$ with the operations (a), (b), (c) and (d) is an ordinal algebra.

The range of $r = (r_1, r_2)$ is the ordinal product of the class of non zero ordinals by itself.

The inductive rules are the following:

$$r(u) = (1, 1)$$

where u is the singleton tree;

$$r\Big(\sum_{\Phi} U_j\Big) = (\max r_1(U_j), |\Phi|)$$

where $|\Phi|$ denotes the finite number of elements of Φ;

$$r\Big(\sum_{I} f(i)\Big) = \Big(\sup_{i} (r_1(f(i)) + 1), 1\Big)$$

where $I = \eta, \omega, \omega^*, \aleph_0$;

$$r\Big(\sum_{I} f(i)\Big) = \Big(\sup_{n} \Big(r_1\Big(\sum_{P_n} f(i)\Big) + 1\Big), 1\Big)$$

where (I, f) is strongly indecomposable labelled and top labelled and $\{(P_n, b, f)\}_{n \in \mathbb{N}}$ is a sequence of branching chains, generating $B((I, f))$.

If $(\mathscr{C}, \leqslant')$ is not b.q.o., there exists an r-bad minimal mapping $g : B \to (\mathscr{C}, \leqslant')$. The Ramsey–Nash-Williams theorem enables us to take a restriction $B' \subset B$, such that $g(s) = V_s$ is a sum of one of the types (a), (b), (c) or (d) and when the type is (a), to take $|\Phi_s|$ increasing and finally when the type is (b), to take $I_s = \aleph_0$, ω, ω^* or η.

If $|\Phi_s| > 1$, then there exists a partition $\Phi_s = \Phi'_s \cup \Phi''_s$ such that the sum is direct or either is a left-right partition such that $|\Phi'_s|$, $|\Phi''_s| \neq 0$. According to the lesser ranking and the definition of g, the mappings $s \to \sum_{\Phi'_s} U_j$, $s \to \sum_{\Phi''_s} U_j$ are good mappings. Hence $g': B' \to (\mathscr{C}, \leqslant')$ is good. The next possibility is $g'(s) = \sum_{I'_s} f_s(i)$, where I_s is a countable antichain or a chain of type ω, ω^* or η. Now, it is possible to choose i in I_s such that $f_s(i) \not\leqslant f_t(j)$ for every $j \in I_t$, where $s \lhd t$ in B' $(g(s) \leqslant g(t))$. The $f_s(i)$ chosen tree, depending upon s and t, will be denoted $h(s \cup t)$ (with $v = s \cup t$, $s \lhd t$, and s, t are elements of the barrier B'^2, the extended restriction of B). Therefore $h: B'^2 \to (\mathscr{C}, \leqslant')$ is a bad mapping. Indeed $v = s \cup t$, $w = t \cup u$, $v \lhd t \Rightarrow h(v) \not\leqslant h(w)$. Now, $r(h(v)) = r(f_s(i)) < r(\sum_{I'_s} f(i)) = r(g'(s))$, hence g' is good, and this is a contradiction. The only one possibility left is that $g'(s) = \sum_{I_s} f_s(i)$ is an strongly indecomposable sum.

In this case we have $s \lhd t$, $g'(s) \not\leqslant g'(t)$. But the corollary of Theorem 4 implies the existence of chains $(P_{s,n}, b, f_s)$ embeddable in none of the $(P_{t,m}, b, f_t)$, therefore $U_{s,n} = \sum_{P_{s,n}} f(i) \not\leqslant \sum_{P_{t,m}} f(i)$. If we define $h(s \cup t)$ by $h(s \cup t) = U_{s,n}$ we will get $h(v) \leqslant h(w)$ and will run again in a contradiction. Therefore $(\mathscr{C}, \leqslant')$ is b.q.o. class. If $g: B \to (\mathscr{C}, \leqslant)$ is a mapping into the class $\mathscr{C}$, ordered under embedding, $s \lhd t$ implies $g(s) \leqslant' g(t)$ and a fortiori we get the weaker relation $g(s) \leqslant g(t)$. The class $(\mathscr{C}, \leqslant)$ is also better quasi ordered.

3.1. *Reduction process on countable trees*

Let T be a countable tree and let $T = I^\nu \cup F^\nu$ be a left-right segment partition of T depending upon an ordinal ν. Suppose that the partitions are already defined for any ordinal $\nu < \alpha$ and $F^0 = \emptyset$, $I^0 = I$. Then, if α is a limit ordinal, $I^\alpha = \bigcap_{\nu < \alpha} I^\nu$, and if it is not, $x \in I^{\alpha+1}$, if and only if $x \in I^\alpha$ (I^α_x, f) is strongly indecomposable, and therefore $U = \sum_{I^\alpha_x} f(i)$ is a block with four subtrees $I^\alpha_x, L^\alpha_x, S^\alpha_x, T^\alpha_x$ (see Theorem 2 and its definitions) not all of them necessarily non empty. We shall call degree α, denoted $d(T)$, the first ordinal α—when it exists—such that $I^\alpha = \emptyset$.

Lemma 6. *If T is a countable tree such that T_x is a canonical tree for every $x \in T$, then T is a canonical tree.*

If T has a root x, then $T = T_x$ and there is nothing to prove. Any tree is the direct sum of its connected parts, and any connected tree V has a maximal chain C and then it is the sum $V = \sum_C f(i)$ where $f(i)$ is just the subtree of V above the junction path i (of the chain C). Hence $T = \bigcup_{j \in N} T_j$ (where N is $\mathbb{N}$ or a finite subset of $\mathbb{N}$) and $T_j = \sum_{I_j} f_j(i)$. Now (I_j, f_j) is a countable labelled chain whose labels $f_j(i)$ are trees T_x (or a right segment of a T_x). Hence, all the $f_j(i)$ are canonical trees. Therefore the chains (I_j, f_j) are b.q.o. labelled countable chains. Accordingly, the trees T_j are finite sums (under chains) of side indecomposable trees, i.e. they are canonical trees.

Finally T is a direct sum of trees of the b.q.o. class $(\mathscr{C}, \leqslant)$. T is therefore a finite set of indecomposable direct sums, and it is necessarily a canonical tree. We will see next that:

Any tree of countable degree is canonical.

Let us suppose this statement true for any degree $<\alpha$, and let T be of countable degree α. If α is an ordinal limit then for any x of T there is an ordinal ν such that $x \in F^{\nu}$, and T_x is then canonical (according to the reduction process). Lemma 6 allows us to claim that T is canonical. If α is not an ordinal limit, then T_x is canonical for any $x \in F^{\alpha-1}$ and if $x \in I^{\alpha-1}$, the hypothesis $I^{\alpha} = \emptyset$ implies that $T_x = \sum_{I_x^{\alpha-1}} f(i)$ is a block where $(I_x^{\alpha-1}, f)$ is strongly indecomposable and then also T_x is canonical. According again to Lemma 6 T is canonical. And now we will prove that: *Any countable tree has a countable degree and is a canonical tree.*

We will return to the reduction process. If we have reached the νth partition of $T' = T^{\nu} \cup P^{\nu}$, then for any $x \in I^{\nu}$, $T_x = \sum_{I_x^{\nu}} f(i)$, and (I_x^{α}, f) is a labelled and top labelled tree with trees $f(i)$ which are trees of degree $\leqslant \nu$. Therefore the trees $f(i)$ are canonical trees and $B((I_x^{\alpha}, f))$ is a left segment of a b.q.o. set of (b, f)-labelled chains. Hence the elements $B((I_x^{\alpha}, f))$ are elements of a well-founded set and if $B((I_{x_0}^{\alpha}, f))$ is a minimal element for $x = x_0$, $(I_{x_0}^{\alpha}, f)$ is strongly indecomposable and $x_0 \in I^{\nu}$. Therefore we have $I^{\nu} \supsetneq I^{\nu+1}$, where ν is any countable ordinal.

It is obvious then that for any countable $T = I^0$ there exists a countable ordinal α such that $I^{\alpha} = \emptyset$. We can now draw the following conclusion.

Theorem 8. *The class of countable trees is a better quasi ordered class under embedding and any countable tree is the finite canonical sum of indecomposable trees.*

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