On rank-one perturbations of normal operators

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Abstract

This paper is concerned with operators on Hilbert space of the form $T = D + u \otimes v$ where $D$ is a diagonalizable normal operator and $u \otimes v$ is a rank-one operator. It is shown that if $T \notin \mathbb{C}1$ and the vectors $u$ and $v$ have Fourier coefficients $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ with respect to an orthonormal basis that diagonalizes $D$ that satisfy $\sum_{n=1}^{\infty} \left( \frac{|\alpha_n|^2}{3} + \frac{|\beta_n|^2}{3} \right) < \infty$, then $T$ has a nontrivial hyperinvariant subspace. This partially answers an open question of at least 30 years duration.

Keywords: Invariant subspace; Hyperinvariant subspace; Normal operator; Rank-one perturbation

1. Introduction

Let $\mathcal{H}$ be a separable, infinite-dimensional, complex Hilbert space, and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. For $T$ in $\mathcal{L}(\mathcal{H})$, we write $\{T\}'$ for the commutant of $T$ (i.e., for the algebra of all $S \in \mathcal{L}(\mathcal{H})$ such that $TS = ST$) and $\{T\}'' = ((T)')'$ for the double commutant of $T$. As usual in what follows, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{T}$ will denote the sets of positive integers, real numbers, complex numbers, and complex numbers of modulus one, respectively. We now choose an ordered orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ for $\mathcal{H}$ which will remain fixed throughout.

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the paper. If $\Lambda = \{\lambda_n\}_{n \in \mathbb{N}}$ is any bounded sequence in $\mathbb{C}$, we write $D_\Lambda$ for the normal operator in $\mathcal{L}(\mathcal{H})$ determined by the equations

$$D_\Lambda(e_n) = \lambda_n e_n, \quad n \in \mathbb{N}. \quad (1)$$

This notation for $\Lambda = \{\lambda_n\}_{n \in \mathbb{N}}$ and $D_\Lambda$ will also remain fixed throughout, as well the notation $\Lambda'$ the derived set of $\Lambda$. By definition, we shall say that an operator $T$ in $\mathcal{L}(\mathcal{H})$ is a rank-one perturbation of a diagonal normal operator if there exist nonzero vectors $u = \sum_{n \in \mathbb{N}} \alpha_n e_n$ and $v = \sum_{n \in \mathbb{N}} \beta_n e_n$ in $\mathcal{H}$ and a bounded sequence $\Lambda = \{\lambda_n\}_{n \in \mathbb{N}}$ in $\mathbb{C}$ such that $T$ is unitarily equivalent to the operator $D_\Lambda + u \otimes v$, where, as usual, $u \otimes v$ is the operator of rank one defined by

$$(u \otimes v)(x) = \langle x, v \rangle u, \quad x \in \mathcal{H}. \quad (2)$$

The notation $\{\alpha_n\}_{n \in \mathbb{N}}$ and $\{\beta_n\}_{n \in \mathbb{N}}$ for the Fourier coefficients of $u$ and $v$, respectively, will also remain fixed throughout this paper. There is a vast literature devoted to the study of this class of operators and its various subclasses (cf., e.g., the bibliography of [4] or do a search on Math. Sci. Net), but almost all of these studies are concerned with the special case in which the sequence $\Lambda$ lies either on $\mathbb{R}$ or $\mathbb{T}$. In fact, very little is known about the structure of operators $T = D_\Lambda + u \otimes v$ when no restriction is placed on the location of the eigenvalues $\lambda_n$ of $D_\Lambda$, and one of the most annoying unsolved problems in operator theory (on Hilbert space) is the following.

(R) Does every rank-one perturbation $T = D_\Lambda + u \otimes v \in \mathcal{L}(\mathcal{H}) \setminus \mathbb{C}1_\mathcal{H}$ of a diagonal normal operator $D_\Lambda$ have a nontrivial invariant subspace (n.i.s.), or better yet, a nontrivial hyperinvariant subspace (n.h.s.)?

Despite the fact that Problem (R) is at least thirty years old (cf., for example, [5, Problem 8K] where it is explicitly posed, but probably not for the first time), it has remained stubbornly intractable, although E. Ionascu [4] addressed the problem. It is thus natural to regard this paper as a sequel to [4], some results from which we use below.

The purpose of this article is to provide a partial solution to Problem (R) by exhibiting a rather substantial subset of operators of the form $T = D_\Lambda + u \otimes v$ each of which has an n.h.s. More precisely, our main result is as follows.

**Theorem 1.1.** Let $T = D_\Lambda + u \otimes v$ be any rank-one perturbation of a diagonal normal operator such that $T \notin \mathbb{C}1_\mathcal{H}$ and $\sum_{n \in \mathbb{N}} (|\alpha_n|^{2/3} + |\beta_n|^{2/3}) < +\infty$. Then $T$ has an n.h.s.

To prove this theorem, we first treat some rather easy cases in Section 2, and thereby reduce the proof of Theorem 1.1 to the derivation of the following technical result.

**Theorem 1.2.** With the notation as introduced above, suppose $T = D_\Lambda + u \otimes v$ is such that

(i) the map $n \rightarrow \lambda_n$ of $\mathbb{N}$ onto $\Lambda$ is injective and $\Lambda'$ is not a singleton,

(ii) for every $n \in \mathbb{N}$, $\alpha_n \beta_n \neq 0$, and

(iii) $\sum_{n \in \mathbb{N}} (|\alpha_n|^{2/3} + |\beta_n|^{2/3}) < +\infty$ (the nontrivial assumption).

Then either
there exists an idempotent \( F \) with \( 0 \neq F \neq 1_H \) such that \( F \in \{ T \}' \), and consequently, \( T \) has a complemented n.h.s. (i.e., there exist n.h.s. \( M \) and \( N \) of \( T \) with \( M \cap N = (0) \) and \( M + N = \mathcal{H} \), or

(II) there exists an uncountable set \( \{ \mu : \mu \in P \} \) of eigenvalues of \( T \) and an associated family \( \{ u_{\mu} \}_{\mu \in P} \) of linearly independent eigenvectors (with \( Tu_{\mu} = \mu u_{\mu} \)) such that \( M = \bigvee_{\mu \in P} \{ u_{\mu} \} \) is an n.h.s. for \( T \) and \( \mathcal{H} \ominus M \) is infinite-dimensional.

The techniques and results herein also allow us to show, in a sequel [3] to this paper, that the operators \( T = D_A + u \otimes v \) satisfying (i)–(iii) above but not (II) are decomposable in the sense of [1].

2. Preliminaries

In this section we introduce some needed notation and set forth some known results from [4] bearing on Problem (R). The ideal of compact operators in \( \mathcal{L}(\mathcal{H}) \) will be denoted by \( \mathbf{K} \) and the Calkin map \( \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathbf{K} \) by \( \pi \). For \( T \) in \( \mathcal{L}(\mathcal{H}) \) we denote by \( \sigma(T) \) the spectrum of \( T \), by \( \sigma_{le}(T) \) [\( \sigma_{re}(T) \)] the left essential [right essential] spectrum of \( T \), and

\[
\sigma_e(T) = \sigma(T) \cap \sigma_{le}(T), \quad \sigma_{le}(T) = \sigma(T) \cap \sigma_{re}(T).
\]

Moreover, we write, as usual, \( \sigma_p(T) \) for the point spectrum of \( T \).

We first take note of some cases treated in [4].

**Proposition 2.1.** (See [4].) If \( T = D_A + u \otimes v \in \mathcal{L}(\mathcal{H}) \setminus \mathbb{C}1_H \) and there exists \( n_0 \in \mathbb{N} \) such that \( \alpha_{n_0} \beta_{n_0} = 0 \), then either \( \lambda_{n_0} \in \sigma_p(T) \) or \( \bar{\lambda}_{n_0} \in \sigma_p(T^*) \). Moreover, if there exist \( m_0, n_0 \in \mathbb{N} \) with \( m_0 \neq n_0 \) such that \( \lambda_{m_0} = \lambda_{n_0} \), then \( \lambda_{n_0} \in \sigma_p(T) \). Finally, if \( \Lambda' \) is a singleton, then \( \{ T \}' \) contains a nonzero compact operator. Consequently, in all cases \( T \) has an n.h.s.

Thus in what follows we restrict our attention to the class \( \mathcal{RO} \) consisting of all operators \( T = D_A + u \otimes v \in \mathcal{L}(\mathcal{H}) \) for which all coefficients \( \alpha_n \) and \( \beta_n \) are nonzero, \( \Lambda = \{ \lambda_n \}_{n \in \mathbb{N}} \) is a one-to-one map of \( \mathbb{N} \) into \( \mathbb{C} \), and \( \Lambda' \) is not a singleton. We remark that it follows easily that if \( T_1 = D_{A_1} + u_1 \otimes v_1 \) and \( T_2 = D_{A_2} + u_2 \otimes v_2 \) belong to \( \mathcal{RO} \) with \( T_1 = T_2 \), then the sequences \( A_1 \) and \( A_2 \) coincide and \( u_1 \otimes v_1 = u_2 \otimes v_2 \) [4, Proposition 1.1]. It is also clear that for all \( T = D_A + u \otimes v \in \mathcal{RO} \), we have \( \sigma_e(T) = \sigma_{le}(T) = \sigma_{re}(D_A) = \Lambda' \).

The following proposition gives very useful necessary and sufficient conditions that a number \( \lambda \in \mathbb{C} \) belong to \( \sigma_p(T) \).

**Proposition 2.2.** (See [4].) Let \( T = D_A + u \otimes v \in \mathcal{RO} \). Then a point \( \mu \in \mathbb{C} \) is an eigenvalue of \( T \) if and only if

\[
\begin{aligned}
(\text{a}) & \quad \mu \notin \Lambda, \\
(\text{b}) & \quad \sum_{n \in \mathbb{N}} \frac{|\alpha_n|^2}{|\mu - \lambda_n|^2} < +\infty \quad \text{(which implies by the Schwarz inequality that} \quad \sum_{n \in \mathbb{N}} \frac{|\alpha_n \beta_n|}{|\mu - \lambda_n|} < +\infty), \quad \text{and} \\
(\text{c}) & \quad \sum_{n \in \mathbb{N}} \frac{\alpha_n \beta_n}{\mu - \lambda_n} = +1.
\end{aligned}
\]

Moreover, if \( \mu \in \sigma_p(T) \) [respectively \( \bar{\mu} \in \sigma_p(T^*) \)], then the eigenspace associated with \( \mu \) [respectively \( \bar{\mu} \)] is spanned by the single vector \( \sum_{n \in \mathbb{N}} (\frac{\alpha_n}{\mu - \lambda_n}) e_n \) [respectively \( \sum_{n \in \mathbb{N}} (\frac{\beta_n}{\mu - \lambda_n}) e_n \)].
and so is one-dimensional. Finally, \((\Lambda \setminus \Lambda') \cap \sigma(T) = \emptyset\) (i.e., all isolated points \(\lambda_n\) of the set \(\Lambda\) lie outside of \(\sigma(T)\)).

We observe that the last statement of Proposition 2.2 can be proved in two lines by noting that if \(\lambda_n\) is isolated in \(\Lambda\), then \((D_{\Lambda} - \lambda_n)\) (and thus \((T - \lambda_n)\)) is a Fredholm operator of index zero, and hence necessarily either \(\lambda_n \in \sigma_p(T)\) (which is impossible by (a)) or \(\lambda_n \in \mathbb{C} \setminus \sigma(T)\).

One might expect that an arbitrary \(T\) in \((\mathcal{RO})\) would satisfy \(\sigma_p(T) \cup \sigma_p(T^*) \neq \emptyset\) (and thus trivially have an n.h.s.), but that this is false has been known (in the case \(D_{\Lambda} = D_{\Lambda}^*\)) for at least fifty years (cf., e.g., [2]). Perhaps the first example of an operator \(T \in (\mathcal{RO})\) such that \(\Lambda'\) has positive planar Lebesgue measure and \(\sigma_p(T) = \emptyset\) was given by Stampfli [6].

Before turning to more serious business, there is one more easy case to dispose of by using the Riesz–Dunford functional calculus and elementary Fredholm theory.

**Proposition 2.3.** If \(T = D_{\Lambda} + u \otimes v \in (\mathcal{RO})\) and either \(\sigma_e(T) = \sigma_{\text{irr}}(T) = \Lambda'\) is not connected or \(\sigma(T) \neq \sigma_e(T)\), then either conclusion (I) or (II) of Theorem 1.2 obtains.

**Proof.** Suppose first that \(\sigma_e(T)\) is not connected. Then, either (1) \(\sigma(T)\) is not connected, in which case the well-known argument consisting of integrating the resolvent of \(T\) about a curve surrounding a separated part of \(\sigma(T)\) produces an idempotent \(0 \neq E \neq 1\) in \(\{T\}'\), or (2) \(\sigma(T)\) is connected, from which one deduces, since \(\sigma_e(T)\) is not a singleton, that \(\sigma(T)\) must fill at least one hole \(H\) in \(\sigma_e(T)\), and (via the normality of \(D_{\Lambda}\)) \(H\) necessarily has associated Fredholm index zero. Thus every point \(\mu \in H\) lies in \(\sigma_p(T)\) and \(\bar{\mu} \in \sigma_p(T^*)\). It follows easily (see Proposition 3.5, where the needed notation is available) that conclusion (II) of Theorem 1.2 holds.

Now suppose that \(\sigma_e(T)\) is connected but \(\sigma(T) \neq \sigma_e(T)\). Then clearly either \(\sigma(T)\) contains an isolated point, in which case \(\{T\}'\) contains a nonzero idempotent as above, or \(\sigma(T)\) is connected but fills at least one hole in \(\sigma_e(T)\), in which case (II) of Theorem 1.2 holds (again via Proposition 3.5).

3. Some new results

Our first order of business is to delineate a class of operators of the form \(T = D_{\Lambda} + u \otimes v\) with which we shall be concerned in the remainder of the paper. In view of Proposition 2.3, to establish Theorems 1.1 and 1.2 (whose proof will be completed in Section 4), it suffices to deal with those \(T\) in the subset \((\mathcal{RO})_1\) defined as follows.

**Definition 3.1.** Suppose \(T = D_{\Lambda} + u \otimes v \in (\mathcal{RO}) \subset \mathcal{L}(\mathcal{H})\). If \(\sigma(T) = \sigma_e(T) = \Lambda'\), \(\sigma(T)\) is a (perfect) connected subset of \(\mathbb{C}\), and the sequences \(\{\alpha_n\}_{n \in \mathbb{N}}\) and \(\{\beta_n\}_{n \in \mathbb{N}}\) satisfy

\[
\sum_{n \in \mathbb{N}} |\alpha_n|^{2/3} < +\infty, \quad \sum_{n \in \mathbb{N}} |\beta_n|^{2/3} < +\infty, \tag{3}
\]

then \(T\) will be said to belong to the class \((\mathcal{RO})_1\). Note that for \(T \in (\mathcal{RO})_1\), \(\sigma_p(T) \subset \sigma(T) = \Lambda'\).

The development of the techniques and results that will eventually yield the remainder of the proof of Theorems 1.1 and 1.2 now begins.
Definition 3.2. For $T = D_A + u \otimes v$ in $(\mathcal{RO})_1$, we define $\gamma_n = \max\{|\alpha_n|, |\beta_n|\}$, $n \in \mathbb{N}$, and set

$$c_1^2 = \sum_{n \in \mathbb{N}} \gamma_n^{2/3} \quad (< +\infty).$$

Moreover, for $\zeta \in \mathbb{C}$ and $s > 0$, we define the open disc $\mathcal{D}(\zeta, s)$ by

$$\mathcal{D}(\zeta, s) := \{\lambda \in \mathbb{C}: |\lambda - \zeta| < s\},$$

and, in particular, we set, for every $r > 0$,

$$A_r := \bigcup_{n \in \mathbb{N}} \mathcal{D}(\lambda_n, \gamma_n^{2/3} r), \quad \Delta_r := \mathbb{C} \setminus A_r,$$

and

$$\Delta_0 := \bigcup_{r > 0} \Delta_r.$$

Denoting planar Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$ by $m_2$, we obtain that

$$m_2(\Lambda_r) \leq \sum_{n \in \mathbb{N}} \pi \gamma_n^{4/3} r^2 = \pi r^2 \sum_{n \in \mathbb{N}} \gamma_n^{4/3}.$$

Remark 3.3. If $T = D_A + u \otimes v$ in $(\mathcal{RO})_1$ and $\Lambda$ consists—say—of the rational points in (the open unit disc) $\mathbb{D} = \mathcal{D}(0, 1)$ in $\mathbb{C}$, then $\sigma(T) = \sigma_{\text{re}}(T) = \mathbb{D}^-$ and $m_2(\sigma(T)) = \pi$, so if $r$ is chosen so small that $\pi r^2 \sum_{n \in \mathbb{N}} \gamma_n^{4/3}$ is very near 0, then $\Lambda_r$ will still be an open covering of $\Lambda$, but the subset $\sigma(T) \cap \Delta_r$ will have $m_2$-measure almost $\pi$.

Remark 3.4. The underlying idea that enables the basic constructions of this paper to be carried out is that even though $\sigma(T) \cap \Delta_r$ may be quite large, we are able to define the appropriate integrations over various simple closed Jordan curves that lie in $\Delta_r$ (for suitable $r > 0$), even though entire arcs on such curves may be contained in $\sigma(T)$.

We must now return briefly to the case in which $T \in (\mathcal{RO})$ and $\sigma(T)$ fills at least one hole in $\sigma_e(T)$.

Proposition 3.5. Suppose $T \in (\mathcal{RO})$ has the property that $\sigma_p(T) \cap \Delta_0$ is uncountable (which, of course, is true if $\sigma(T)$ fills a hole in $\sigma_e(T)$). Then $T$ satisfies conclusion (II) of Theorem 1.2.

Proof. Since $\sigma_p(T) \cap \Delta_0$ is uncountable, there exists $r_0 > 0$ such that $\sigma_p(T) \cap \Delta r_0$ is also uncountable, and thus contains a perfect set $P$. For $\mu \in P$, $u_\mu$ spans the eigenspace of $T$ corresponding to $\mu$ (by Proposition 2.2), and since $\langle u_\mu, v \rangle = -1$, by taking complex conjugates we get $\langle \bar{v}_\mu, u \rangle = -1$. Thus by another application of Proposition 2.2, we see that $\bar{\mu} \in \sigma_p(T^*)$ and $\bar{v}_\mu$ spans the associated eigenspace. Partition $P$ as $P = P_1 \cup P_2$, where $P_1$ is countably
infinite and \( P_2 \) is uncountable, and set \( \mathcal{M} = \bigvee_{\mu \in P_2} \{ u_\mu \} \). Note that since each one-dimensional space \( C u_\mu \) is an n.h.s. for \( T \), so is \( \mathcal{M} \). Moreover, the computation

\[
\mu_1 \langle u_{\mu_1}, \bar{v}_{\mu_2} \rangle = (T u_{\mu_1}, T^* \bar{v}_{\mu_2}) = \mu_2 \langle u_{\mu_1}, \bar{v}_{\mu_2} \rangle,
\]

valid for all \( \mu_1 \in P_1, \mu_2 \in P_2 \), shows that \( u_{\mu_1} \perp \bar{v}_{\mu_2} \) for all such \( \mu_1, \mu_2 \).

Thus, for \( \mu_1 \in P_1, \bar{v}_\mu \in H \ominus \mathcal{M} \), and since these \( \bar{v}_\mu \) with \( \mu \in P_1 \) are linearly independent, we see that \( H \ominus \mathcal{M} \) is infinite-dimensional, and thus \( T \) does, indeed, satisfy (II) of Theorem 1.2. \( \Box \)

Note that this result also completes the proof of Proposition 2.3. Because of the frequency with which notation such as \((D \Lambda - \lambda H_1 \Lambda)\) or \((D \Lambda - \lambda_1 H)\) occurs below, we shall henceforth simply use the slightly simplified notation \((D \Lambda - \lambda)\) or \((D \Lambda - \lambda_1)\), etc., where the inverse maps make sense (as possibly unbounded, densely defined, linear transformations) whenever the respective maps are injective.

**Lemma 3.6.** Suppose \( T = D \Lambda + u \otimes v \in (RO)_1 \) and \( r > 0 \) is fixed. Then for every \( \lambda \in \Delta_r \), we have \( u, v \in \text{ran}(D \Lambda - \lambda) \cap \text{ran}(D^* \Lambda - \bar{\lambda}) \), the vectors

\[
u_\lambda := (D \Lambda - \lambda)^{-1} u, \quad v_\lambda := (D \Lambda - \lambda)^{-1} v,
\]

\[
\bar{u}_\lambda := (D^* \Lambda - \bar{\lambda})^{-1} u, \quad \bar{v}_\lambda := (D^* \Lambda - \bar{\lambda})^{-1} v,
\]

are nonzero and satisfy

\[
\max \{ \| u_\lambda \|, \| v_\lambda \|, \| \bar{u}_\lambda \|, \| \bar{v}_\lambda \| \} \leq c_1 / r, \quad \lambda \in \Delta_r.
\]

**Proof.** Calculations show that, providing the two series converge, we have

\[
\| u_\lambda \|^2 = \| \bar{u}_\lambda \|^2 = \sum_{n \in \mathbb{N}} \frac{|\alpha_n|^2}{|\lambda - \lambda_n|^2} > 0, \quad \| v_\lambda \|^2 = \| \bar{v}_\lambda \|^2 = \sum_{n \in \mathbb{N}} \frac{|\beta_n|^2}{|\lambda - \lambda_n|^2} > 0,
\]

and the result thus follows immediately from the inequality

\[
\sum_{n \in \mathbb{N}} \frac{\max \{ |\alpha_n|^2, |\beta_n|^2 \}}{|\lambda - \lambda_n|^2} \leq \sum_{n \in \mathbb{N}} \frac{\gamma_n^2}{r^2} = c_1^2 / r^2, \quad \lambda \in \Delta_r. \tag{7}
\]

**Lemma 3.7.** With \( T = D \Lambda + u \otimes v \in (RO)_1 \), \( r > 0 \) fixed, and \( u_\lambda, v_\lambda, \bar{u}_\lambda, \bar{v}_\lambda \) as in Lemma 3.5, each of these four functions (of \( \lambda \)) is strongly continuous on \( \Delta_r \). Consequently, functions of the form \( \lambda \rightarrow \langle u_\lambda, \bar{v}_\lambda \rangle \) are also continuous on \( \Delta_r \).

**Proof.** The equality

\[
\sum_{n \geq N} \frac{|\alpha_n|^2}{|\lambda - \lambda_n|^2} \leq \frac{1}{r^2} \sum_{n \geq N} \gamma_n^{2/3}, \quad N \in \mathbb{N},
\]
shows that the partial sums \( \sum_{n=1}^{N} \left( \frac{a_n}{\lambda - \lambda_n} \right) e_n \) (which are clearly strongly continuous functions of \( \lambda \) on \( \Delta_r \)) converge uniformly there to \( u_\lambda \). This establishes the strong continuity of \( u_\lambda \), and the arguments for the other functions are similar. \( \square \)

**Definition 3.8.** We write \( (RO)_2 \) for the set of all \( T \in (RO)_1 \) such that \( \sigma_p(T) \cap \Delta_0 \) is a countable set, and note that to complete the proofs of Theorems 1.1 and 1.2, it suffices to show that each \( T \in (RO)_2 \) has the appropriate properties.

**Proposition 3.9.** Suppose \( T = D_\Lambda + u \otimes v \in (RO)_2 \), \( r > 0 \) is fixed, \( u_\lambda, v_\lambda, \tilde{u}_\lambda, \) and \( \tilde{v}_\lambda \) are as in Lemma 3.5, and we define

\[
\varphi_\lambda := 1 + \langle u_\lambda, v \rangle = 1 + \langle (D_\Lambda - \lambda)^{-1}u, v \rangle, \quad \lambda \in \Delta_r.
\]

Then for every compact subset \( K \subset \Delta_r \) such that \( \varphi_\lambda \) does not vanish on \( K \),

\[
u, v \in \text{ran}(T - \lambda) \cap \text{ran}(T^* - \tilde{\lambda}), \quad \lambda \in K,
\]

each of the four functions

\[
u^T_\lambda := (T - \lambda)^{-1}u, \quad \tilde{\nu}^T_\lambda := (T^* - \tilde{\lambda})^{-1}v,
\]

is strongly continuous on \( K \) (where here again, the linear transformations \( (T - \lambda)^{-1} \) and \( (T^* - \tilde{\lambda})^{-1} \) are possibly unbounded but densely defined), and there exists \( \epsilon_{K,r} > 0 \) such that

\[
|\varphi_\lambda| \geq \epsilon_{K,r}, \quad \|\nu^T_\lambda\|, \|\tilde{\nu}^T_\lambda\|, \|\tilde{\nu}^T_\lambda\| \leq c_1/r\epsilon_{K,r}, \quad \lambda \in K.
\]

**Proof.** We treat only the case of \( \nu^T_\lambda \); the arguments for the other three functions are similar. Clearly

\[
(T - \lambda)u_\lambda = (D_\Lambda - \lambda)u_\lambda + \langle u_\lambda, v \rangle u = \varphi_\lambda u, \quad \lambda \in \Delta_r,
\]

and we know from Proposition 3.6 that \( \varphi_\lambda \) is continuous on \( \Delta_r \). Since \( \varphi_\lambda \) does not vanish on \( K \), there exist \( 0 < \epsilon_{K,r} < M_{K,r} < \infty \) such that \( \epsilon_{K,r} \leq |\varphi_\lambda| \leq M_{K,r} \) on \( K \). Moreover, (10) yields

\[
u^T_\lambda = (T - \lambda)^{-1}u = (1/\varphi_\lambda)u_\lambda, \quad \lambda \in K,
\]

which shows, via the continuity of \( \varphi_\lambda^{-1} \) and strong continuity of \( u_\lambda \) on \( K \), that \( \nu^T_\lambda \) is strongly continuous there and also, via (7), that \( \|\nu^T_\lambda\| \leq c_1/r\epsilon_{K,r} \) for all \( \lambda \in K \). \( \square \)

The following result is established by some calculations closely resembling those in Lemmas 3.6, 3.7, and Proposition 3.9, so we only sketch the proof. Nevertheless, the linear manifold \( \mathcal{L} \) introduced therein plays a central role in Section 4.
Definition 3.10. For \( T = D_A + u \otimes v \in (RO)_2 \) and \( r > 0 \) fixed, we write \( D_A = \int \lambda dE \) (so \( E \) is the spectral measure of \( D_A \)), and for every \( x \in \mathcal{H} \), we define the extended real number \( c_x \in [0, +\infty] \), by

\[
c^2_x := \sum_{n \in \mathbb{N}} \left( |\langle x, e_n \rangle|^2 / \gamma_n^{4/3} \right)
\]

and the set \( \mathcal{L} \subset \mathcal{H} \) as

\[
\mathcal{L} := \{ x \in \mathcal{H} : c_x < +\infty \}.
\]

Theorem 3.11. For \( T \in (RO)_2 \) and \( r > 0 \) fixed, the set \( \mathcal{L} \) in (12) is a dense linear manifold in \( \mathcal{H} \), invariant under \( D_A, D^*_A, T, T^* \), \( (D_A - \lambda)^{-1} \), and \( (D^*_A - \bar{\lambda})^{-1} \) for every \( \lambda \in \Delta_r \). Moreover \( \mathcal{L} \) contains \( u, v \), the basis vectors \( \{ e_n \}_{n \in \mathbb{N}} \), and is invariant under every value of \( E \). Furthermore, for every compact subset \( K \subset \Delta_r \) on which \( \varphi_\lambda \) does not vanish,

\[
\mathcal{L} \subset \bigcap_{\lambda \in K} (\text{ran}(D_A - \lambda) \cap \text{ran}(T - \lambda) \cap \text{ran}(D^*_A - \bar{\lambda}) \cap \text{ran}(T^* - \bar{\lambda})),
\]

and, upon defining, for each \( x \in \mathcal{L} \) and \( \lambda \in \Delta_r \),

\[
x_\lambda := (D_A - \lambda)^{-1} x, \quad \bar{x}_\lambda := (D^*_A - \bar{\lambda})^{-1} x,
\]

\[
x^T_\lambda := (T - \lambda)^{-1} x, \quad \bar{x}^T_\lambda := (T^* - \bar{\lambda})^{-1} x,
\]

we obtain, for all \( x \in \mathcal{L} \) and \( \lambda, \bar{\lambda} \in K \), that the four functions in (13) take values in \( \mathcal{L} \), that

\[
x^T_\lambda = x_\lambda - \langle x, \bar{v}_\lambda \rangle u^T_\lambda = x_\lambda - \varphi^{-1}_\lambda \langle x, \bar{v}_\lambda \rangle u_\lambda, \quad \lambda \in K,
\]

where \( \varphi_\lambda \) is as in (8), that

\[
\| x_\lambda \|, \| \bar{x}_\lambda \| \leq c_x / r, \quad \lambda \in \Delta_r,
\]

and that

\[
\| x^T_\lambda \|, \| \bar{x}^T_\lambda \| \leq (c_x / r) + (c_1 / r)^2 \left( \| x \| / \varepsilon_{K,r} \right), \quad \lambda \in K,
\]

where \( \varepsilon_{K,r} \) is a lower bound on \( |\varphi_\lambda| \) on \( K \). Finally, for every \( x \in \mathcal{L} \), each of the functions appearing in (13) is bounded and weakly continuous on \( K \).

Sketch of proof. It is clear that \( \mathcal{L} \) is a linear manifold invariant under every value of \( E \), and that \( \mathcal{L}^\perp = \mathcal{H} \) follows because every \( x \in \mathcal{H} \) with only finitely many nonzero Fourier coefficients \( \langle x, e_n \rangle \) belongs to \( \mathcal{L} \). Thus \( \{ e_n \}_{n \in \mathbb{N}} \subset \mathcal{L} \) and earlier calculations showed that \( u, v \in \mathcal{L} \). For each \( x \in \mathcal{L} \), we calculate

\[
\| x_\lambda \|^2 = \| (D_A - \lambda)^{-1} x \|^2 = \| \bar{x}_\lambda \|^2 = \| (D^*_A - \bar{\lambda})^{-1} x \|^2
\]

\[
= \sum_{n \in \mathbb{N}} |\langle x, e_n \rangle|^2 / |\lambda - \lambda_n|^2 \leq \sum_{n \in \mathbb{N}} |\langle x, e_n \rangle|^2 / r^2 \gamma_n^{4/3} = c^2_x / r^2 < +\infty, \quad \lambda \in \Delta_r.
\]
so \( x \in \text{dom}(D_A - \lambda)^{-1}, \) \( \mathcal{L} \) is invariant under \((D_A - \lambda)^{-1}\) and \((D_A^* - \bar{\lambda})^{-1}\) (for \( \lambda \in \Delta_r \)), \( x \in \text{ran}(D_A - \lambda) \cap \text{ran}(D_A^* - \bar{\lambda}) \), and \( \|x\|, \|\bar{x}\| \) are bounded by \( c_{x/r} \), as desired. Moreover, for \( \lambda \in \mathcal{K} \) almost the same calculation shows that \( \mathcal{L} \) is invariant under \( D_A, D_A^*, T, \) and \( T^* \), and the weak continuity on \( \mathcal{K} \) of the four functions in (13) is established by an argument like that in Lemma 3.6. Next, (14) is verified by a calculation similar to (11). Then, by (14) and what has already been shown, for \( \lambda \in \mathcal{K}, \mathcal{L} \) is also invariant under \((T - \lambda)^{-1}\) and \((T^* - \bar{\lambda})^{-1}\). Finally, (16) follows from (14), (17), (7), and (9), where \( \varepsilon_{K,r} \) is as in (9). \( \square \)

We shall need one additional easy lemma.

**Lemma 3.12.** Let \( T = D_A + u \otimes v \in (\mathcal{RO})_2, \) \( A \in (T)' \), \( r > 0 \) be fixed, and let \( \emptyset \neq \mathcal{K} \subset \Delta_r \) be a compact subset on which \( \varphi \) does not vanish. Then for every \( x \in \mathcal{L} \) and \( \lambda \in \mathcal{K}, (T - \lambda)^{-1}Ax \) is well-defined and \((T - \lambda)^{-1}Ax = \Lambda(T - \lambda)^{-1}x \). Consequently, \((T - \lambda)^{-1}Ax \) is bounded and weakly continuous on \( \mathcal{K} \).

**Proof.** We know from Theorem 3.11 that for \( x \in \mathcal{L} \), \( \lambda \in \mathcal{K}, x^T_\lambda = (T - \lambda)^{-1}x \) is well-defined, bounded, and weakly continuous on \( \mathcal{K} \), and therefore so is \( A(x^T_\lambda) = (T - \lambda)^{-1}x \). Moreover, since \( A(\text{ran}(T - \lambda)) \subset \text{ran}(T - \lambda) \) for every \( \lambda \in \mathcal{K} \), \((T - \lambda)^{-1}Ax \) is also well-defined, and a trivial calculation shows that

\[
(Ax)^T_\lambda := (T - \lambda)^{-1}Ax = A(T - \lambda)^{-1}x = A(x^T_\lambda), \quad x \in \mathcal{L}, \lambda \in \mathcal{K}.
\] (18)

**Remark 3.13.** One must exercise great care in making calculations with functions like those in (13), to keep in mind always that, while for \( x \in \mathcal{L}, \) the function \( x^T_\lambda = (T - \lambda)^{-1}x \) is well-defined and weakly continuous on certain compact subsets of \( \Delta_r \), the transformation \((T - \lambda)^{-1} \) appearing, while certainly linear, is not (for many \( \lambda \in \Delta_r \)) continuous (i.e., in \( \mathcal{L}(\mathcal{H}) \)) but is unbounded and only densely defined.

### 4. Some useful integrals

We are now almost prepared to write down some integrals that will be needed to complete the proof of Theorems 1.1 and 1.2 (for an arbitrary \( T \) in \( (\mathcal{RO})_2 \)). We will use without further comment the notation, definitions, and results of Section 3, and we shall need some additional notation and a standing convention. Recall that if \( \Gamma \subset \mathbb{C} \) is a simple, closed Jordan curve in \( \mathbb{C} \), then according to the Jordan curve theorem, \( \mathbb{C} \setminus \Gamma \) consists of exactly two disjoint open connected sets which we shall denote by \( \text{Int}(\Gamma) \) and \( \text{Ext}(\Gamma) \), with \( \text{Ext}(\Gamma) \) being unbounded.

**Standing Conventions 4.1.** Thus far, for \( T = D_A + u \otimes v \in (\mathcal{RO})_2, \) no assumption has been made concerning the size of \( \|T\| \) or the location of \( \sigma(T), \) and the significance of this work is that none is needed. Nevertheless, to simplify the notation in the plane geometry to be undertaken below, it will be convenient in what follows to, first, recall that \( \sigma(T) = \Lambda' \) is a perfect connected set, and thus has diameter \( d > 0, \) and then to replace \( D_A \) and \( T \) by \( \zeta D_A \) and \( \zeta T \) for a suitable \( \zeta \in \mathbb{C} \setminus \{0\} \) (which will have no effect on the validity of Theorems 1.1 and 1.2 or any other result to follow), so that every \( T \in (\mathcal{RO})_2 \) under consideration satisfies the following *standing conventions*: \( \|D_A\|, \|T\| < 1, \)

\[-1 < a := \min\{\text{Re}(\lambda): \lambda \in \sigma(T)\} < b := \max\{\text{Re}(\lambda): \lambda \in \sigma(T)\} < 1, \]
and all $r > 0$ under consideration satisfy $r \in (0, \min\{1 - \|T\|, (b - a)/(4c_1^2)\})$. Note that these conventions ensure that $\Lambda_r \subset \mathbb{D}$. Moreover, we write $\rho(T) := \mathbb{C} \setminus \sigma(T)$, the resolvent set of $T$, and since

$$\lim_{|\lambda| \to \infty} \| (T - \lambda)^{-1} \| = 0,$$

one knows that the function $\lambda \mapsto (T - \lambda)^{-1}$ is analytic and norm-continuous on $\rho(T)$ and bounded in $\mathbb{C} \setminus \{0\}$. Also we shall denote by $P$ the projection of $\mathbb{C} = \mathbb{R}^2$ onto $\mathbb{R} \subset \mathbb{C}$. For $r > 0$ fixed, it follows immediately from the definition of the set $\Lambda_r$ in (5), the connectedness of $\sigma(T) = \Lambda'$, and the standing conventions, that $P(\sigma(T)) = [a, b]$ and that $P(\Lambda_r)$ is a union of open subintervals of $\mathbb{R}$ of total length at most $2r \sum_{n \in \mathbb{N}} y_n^{2/3} (= 2rc_1^2)$. Therefore

$$\Pi_r := (a, b) \setminus \{ P(\Lambda_r) \cup (\sigma_p(T) \cap \Delta_0) \}$$

and $\Pi_r$ has (linear, Lebesgue) measure larger than $(b - a)(1 - rc_1^2) > (b - a)/2$ (since $\sigma_p(T) \cap \Delta_0$ is a countable (perhaps void) set). We note that an important and needed property of $\Pi_r$ is that for every $s \in \Pi_r$, the vertical line $x = s$ lies entirely in $\Delta_r$. We also will use the fact that the subset $\Pi'_r$ consisting of all points of $\Pi_r$ with Lebesgue density 1 has the same linear measure as does $\Pi_r$. Consequently, $\Pi'_r$ is dense in $\Pi_r$, and for each $s \in \Pi'_r$, there exist monotone sequences $\{s_n^-\}_{n \in \mathbb{N}}$ and $\{s_n^+\}_{n \in \mathbb{N}}$ in $\Pi'_r$, with $a < s_n^- < s < s_n^+ < b$, such that $s_n^- \nearrow s$ and $s_n^+ \searrow s$.

The following result, whose proof is long and is, in particular, given in a sequence of five steps, implies (what remains to be proved to establish) Theorems 1.1 and 1.2.

**Theorem 4.2.** Let $T = D_A + u \otimes v \in (\mathcal{RO})_2$. Then, with $T$ and $r > 0$ as in the standing conventions, for every $s \in \Pi'_r$, there exist two nonzero idempotents $F_j^s \in \{T\}^n$, $j = 1, 2$, such that $F_1^s + F_2^s = 1_{\mathcal{H}}$. Furthermore, for all $s, s' \in \Pi'_r$ with $s \neq s'$, and for $j = 1, 2$, $F_j^s \neq F_j^{s'}$.

**Proof.** The proof will be given in several steps.

**Step I.** Since $T$ satisfies the standing conventions, we have $\sigma(T) \cup \sigma(D_A) = \Lambda' \cup \Lambda \subset \mathbb{D}$. We fix an arbitrary $s \in \Pi'_r \subset (a, b)$, so the vertical line segment $I_s \subset \mathbb{D}^-$ defined by

$$I_s = \{ s + it: - (1 - s^2)^{1/2} \leq t \leq (1 - s^2)^{1/2} \}$$

lies entirely in $\Delta_r \cap \mathbb{D}^-$ and has endpoints on $\mathbb{T}$.

We next construct two positively oriented, piecewise smooth, simple closed, Jordan curves $\Gamma_1^s, \Gamma_2^s \subset \mathbb{T} \cup I_s$ as follows. Let $\Gamma_j^s$, $j = 1, 2$, consist of the line segment $I_s$ together with an arc $a_j^s$ of $\mathbb{T}$ (each properly oriented), where

$$a_1^s = \{ e^{i\theta} \in \mathbb{T}: \Re(e^{i\theta}) \geq s \}, \quad a_2^s = \{ e^{i\theta} \in \mathbb{T}: s \leq \Re(e^{i\theta}) \}.$$ 

Note that both $\Gamma_1^s$ and $\Gamma_2^s$ contain $I_s$ (with opposite orientations) as a subarc and are compact sets. Thus $\mathbb{T} = a_1^s \cup a_2^s \subset \rho(T) \cap \rho(D_A)$, so the resolvents $R_\lambda(T) = (\lambda - T)^{-1}$ and $R_\lambda(D_A)$ are analytic in a neighborhood of $\mathbb{T} = a_1^s \cup a_2^s$. Since $I_s \cup \mathbb{T} (= \Gamma_1^s \cup \Gamma_2^s)$ is a compact set on which $\varphi_\lambda$ does not vanish, we see that for every $x \in \mathcal{L}$ (the dense linear manifold of Theorem 3.11),
the functions \(x_\lambda, \bar{x}_\lambda, x_\lambda^T, \bar{x}_\lambda^T\) from Theorem 3.11, as well as all functions \((Ax)_\lambda^T\) as in (18) where \(A \in \{T\}'\), are bounded and weakly continuous on \(l_s \cup T\). Therefore, these functions are weakly measurable and (since \(\mathcal{H}\) is separable) strongly measurable on \(\Gamma_1^s \cup \Gamma_2^s\). Consequently, the vector-valued integrals

\[
E_j^s x := \frac{1}{2\pi i} \int_{\Gamma_j^s} (\lambda - D\Lambda)^{-1} x \, d\lambda \quad \left(= - \frac{1}{2\pi i} \int_{\Gamma_j^s} x_\lambda \, d\lambda\right), \quad x \in \mathcal{L}, \quad j = 1, 2, \tag{19}
\]

and

\[
F_j^s x := \frac{1}{2\pi i} \int_{\Gamma_j^s} (\lambda - T)^{-1} x \, d\lambda \quad \left(= - \frac{1}{2\pi i} \int_{\Gamma_j^s} x_\lambda^T \, d\lambda\right), \quad x \in \mathcal{L}, \quad j = 1, 2, \tag{20}
\]

exist in the strong topology on \(\mathcal{H}\), and from (19), (20), and (14) we get

\[
F_j^s x = E_j^s x + \frac{1}{2\pi i} \int_{\Gamma_j^s} \langle x, \bar{v}_\lambda \rangle u_\lambda^T \, d\lambda
= E_j^s x + \frac{1}{2\pi i} \int_{\Gamma_j^s} \varphi_\lambda^{-1}(x, \bar{v}_\lambda) \, d\lambda, \quad x \in \mathcal{L}, \quad j = 1, 2. \tag{21}
\]

Moreover, with \(D\Lambda = \int \lambda \, dE\), as in Definition 3.10, so \(E\) is the (purely atomic) spectral measure of \(D\Lambda\), it is easy to check (for example, by computing the weak integrals \(\langle E_j^s x, e_n \rangle\) for \(x \in \mathcal{L}\)) that

\[
E_j^s x = E(\text{Int}(\Gamma_j^s)) x, \quad \|E_j^s x\| \leq \|x\|, \quad x \in \mathcal{L}, \quad j = 1, 2, \tag{22}
\]

and hence from (21), (22), (7), and (9), we obtain that

\[
\|F_j^s x\| \leq (1 + \frac{c_1^2}{r_2^2 \varepsilon_{K,r}}) \|x\|, \quad x \in \mathcal{L}, \quad j = 1, 2, \tag{23}
\]

where \(|\varphi_\lambda| \geq \varepsilon_{K,r}\) on \(K = \Gamma_1^s \cup \Gamma_2^s\) as in Proposition 3.9. Since it is now obvious from (19)–(23) that \(E_j^s\) and \(F_j^s\), \(j = 1, 2\), are bounded linear transformations defined on \(\mathcal{L}\), we may extend them by continuity (without changing the notation) to be elements of \(\mathcal{L}(\mathcal{H})\) (but the equalities (19)–(21) obtain only for \(x \in \mathcal{L}\)), so

\[
E_j^s = E(\text{Int}(\Gamma_j^s)), \quad j = 1, 2,
\]

and since \(A \subset \text{Int}(\Gamma_1^s) \cup \text{Int}(\Gamma_2^s)\) and \(\text{Int}(\Gamma_1^s) \cap \text{Int}(\Gamma_2^s) = \emptyset\),

\[
E_1^s + E_2^s = 1_{\mathcal{H}}, \quad E_1^s \cdot E_2^s = 0. \tag{24}
\]

Since, by Theorem 3.11, \(E_j^s \mathcal{L} \subset \mathcal{L}\), we also get from (24) that

\[
\mathcal{L} = E_1^s \mathcal{L} + E_2^s \mathcal{L}, \quad E_1^s \mathcal{L} \perp E_2^s \mathcal{L}, \tag{25}
\]
the direct sum of the indicated mutually orthogonal linear manifolds. Moreover, since for \( x \in \mathcal{L} \), in the integral \((F_1^s + F_2^s)x\) the integrations along \( l_s \) cancel one another, we get immediately that

\[
(F_1^s + F_2^s)x = \frac{1}{2\pi i} \int_{\mathbb{C}} (\lambda - T)^{-1} x d\lambda, \quad x \in \mathcal{L},
\]

and since \( \sigma(T) \subset \mathbb{D} \), we see that also, by the Riesz–Dunford functional calculus,

\[
F_1^s + F_2^s = 1_{\mathcal{H}}. \tag{26}
\]

Therefore to show that \( F_1^s \) and \( F_2^s \) are idempotents, it clearly suffices to show that

\[
F_1^s \cdot F_2^s = 0.
\]

**Step II.** We expand the set of \( x \in \mathcal{H} \) for which (20) is valid, as follows.

**Lemma 4.3.** With \( T \in (\mathcal{R} \mathcal{O})_2 \) and \( r \) and \( s \) fixed as in Theorem 4.2, let \( \mathcal{L}' \supset \mathcal{L} \) denote the set of all \( x \) in \( \mathcal{H} \) for which the function \( x_j^s = (T - \lambda)^{-1} x \) is well-defined, bounded, and weakly continuous on \( \Gamma_j^s \cup \Gamma_j^s \). Then \( A(x_j^s) \in \mathcal{L}' \) for every \( x \in \mathcal{L} \) and every \( A \in \{T\}' \). Moreover,

\[
F_j^s x = \frac{1}{2\pi i} \int_{\Gamma_j^s} (\lambda - T)^{-1} x d\lambda, \quad x \in \mathcal{L}', \ j = 1, 2, \tag{27}
\]

and

\[
F_j^s A = AF_j^s, \quad A \in \{T\}', \ j = 1, 2.
\]

**Proof.** Obviously the hypotheses guarantee that the integral in (27) exists, so we fix \( x_0 \in \mathcal{L}' \) and, via the density of \( \mathcal{L} \) in \( \mathcal{H} \), let \( \{x_n\}_{n \in \mathbb{N}} \subset \mathcal{L} \) be such that \( \|x_n - x_0\| \to 0 \). Then from (20), we have

\[
F_j^s x_n = \frac{1}{2\pi i} \int_{\Gamma_j^s} (\lambda - T)^{-1} x_n d\lambda, \quad n \in \mathbb{N}, \ j = 1, 2,
\]

and since \( F_j^s \in \mathcal{L}(\mathcal{H}) \), clearly \( \|F_j^s x_n - F_j^s x_0\| \to 0 \) for \( j = 1, 2 \) (so the sequence \( \{\|F_j^s x_n\|\}_{n \in \mathbb{N}} \) is bounded). Thus it suffices to show that

\[
\langle F_j^s x_n, y \rangle \to \left\langle \frac{1}{2\pi i} \int_{\Gamma_j^s} (\lambda - T)^{-1} x_0 d\lambda, y \right\rangle, \quad y \in \mathcal{L}, \ j = 1, 2.
\]

But, for \( j = 1, 2 \), and \( y \in \mathcal{L} \),
\[ \langle F_j^* x, y \rangle = \left( \frac{1}{2\pi i} \int_{\Gamma_j^s} (\lambda - T)^{-1} x_{\lambda} d\lambda, y \right) \]

\[ = \frac{1}{2\pi i} \int_{\Gamma_j^s} \langle (\lambda - T)^{-1} x_n, y \rangle d\lambda, \quad n \in [0] \cup \mathbb{N}, \]

since the integrals in question are limits of finite (Riemann–Stieltjes) sums, and moreover, the convergence

\[ \int_{\Gamma_j^s} \langle (\lambda - T)^{-1} x_n, y \rangle d\lambda = -\int_{\Gamma_j^s} \langle x_n, \bar{y}^T_{\lambda} \rangle d\lambda \rightarrow -\int_{\Gamma_j^s} \langle x_0, \bar{y}^T_{\lambda} \rangle d\lambda \]

now follows from the fact that the sequence of continuous functions \( \{\langle x_n, \bar{y}^T_{\lambda} \rangle\}_{n \in \mathbb{N}} \) (on \( \Gamma_j^s \)) converges uniformly on \( \Gamma_j^s \) to \( \langle x_0, \bar{y}^T_{\lambda} \rangle \). Next, note that by Lemma 3.12, \( A(x_\lambda^T) (= (Ax_\lambda^T) \in \mathcal{L}' \),

and thus from (27) we obtain that

\[ F_j^s Ax = \frac{1}{2\pi i} \int_{\Gamma_j^s} (\lambda - T)^{-1} Ax d\lambda = \frac{1}{2\pi i} \int_{\Gamma_j^s} A(\lambda - T)^{-1} x d\lambda \]

\[ = AF_j^s x, \quad x \in \mathcal{L}, \quad A \in (\mathcal{T}')', \quad j = 1, 2, \]

so \( F_j^s \) commutes with \( \{T\}' \) as desired. \( \square \)

**Step III.** We formulate the penultimate step of the proof as follows.

**Lemma 4.4.** With \( T \in (\mathcal{RO})_2 \) and \( r, s \) fixed as in Theorem 4.2, we have that for \( j = 1, 2 \), and each fixed \( \xi \in \text{Ext}(\Gamma_j^s) \), there exist operators \( B_j^s(\xi), A_j^s(\xi) \) in \( \mathcal{L}(\mathcal{H}) \) with \( B_j^s(\xi) \in \{D_A\}' \), \( A_j^s(\xi) \in (\mathcal{T})' \) such that

\[ B_j^s(\xi)(D_A - \xi) = E_j^s, \quad (\xi \in \text{Int}(\Gamma_j^s)), \quad A_j^s(\xi)(T - \xi) = F_j^s, \quad \xi \in \text{Ext}(\Gamma_j^s), \quad j = 1, 2. \]

Moreover, for each \( x \in \mathcal{L} \) and \( j = 1, 2 \),

\[ B_j^s(\xi)x = \frac{1}{2\pi i} \int_{\Gamma_j^s} (\xi - \lambda)^{-1} x_{\lambda} d\lambda, \quad (29) \]

\[ A_j^s(\xi)x = \frac{1}{2\pi i} \int_{\Gamma_j^s} (\xi - \lambda)^{-1} x_{\lambda}^T d\lambda, \quad (30) \]
and $B_j^s(\cdot)x$, $A_j^s(\cdot)x : \text{Ext}(\Gamma_j^s) \to \mathcal{H}$ are analytic (vector-valued) functions. Furthermore, $F_j^s$ is an idempotent different from 0 and $1_{\mathcal{H}}$, $M_j^s := \text{ran}(F_j^s)$ is a nontrivial hyperinvariant subspace for $T$, and

$$
\sigma(T|_{M_j^s}) \subset \text{Int}(\Gamma_j^s) \cup \Gamma_j^s \quad (= \mathbb{C}\setminus \text{Ext}(\Gamma_j^s)), \quad j = 1, 2.
$$

(31)

**Proof.** We give the argument for $j = 1$; the other argument is essentially the same. Fix $\zeta \in \text{Ext}(\Gamma_1^s)$. It is clear that the functions $\lambda \to (\zeta - \lambda)^{-1}x_{\lambda}$ and $\lambda \to (\zeta - \lambda)^{-1}x_{\lambda}^s$ are bounded and weakly continuous on $\Gamma_1^s$, so the integrals in (29) and (30) are well-defined for each $x \in \mathcal{L}$, and thus we define $B_j^1(\zeta)$ and $A_j^1(\zeta)$ on $\mathcal{L}$ by (29) and (30). (We note here that since $D_A = \int \lambda \, dE$, one could also define $B_j^1(\zeta)$ by using the functional calculus for the normal operator $D_A$, but we need both $B_j^1(\zeta)$ and $A_j^1(\zeta)$ to be written as line integrals so we can compare them later in the proof.) We shall first show that $B_j^1(\zeta)$ is bounded on $\mathcal{L}$, and thus extends to an element of $\mathcal{L}(\mathcal{H})$, and then use this fact to show that $A_j^1(\zeta)$ is also bounded on $\mathcal{L}$. First, since $\mathcal{L} = E_1^s\mathcal{L} + E_2^s\mathcal{L}$ and $(D_A - \zeta)E_j^s\mathcal{L} \subset E_j^s\mathcal{L} \subset \mathcal{L}$, $j = 1, 2$ (via Theorem 3.11 and (22)), we compute, with $x = x_1 + x_2 \in E_1^s\mathcal{L} + E_2^s\mathcal{L}$ arbitrary in $\mathcal{L}$,

$$
B_j^1(\zeta)(D_A - \zeta)x_k = \frac{1}{2\pi i} \int_{\Gamma_1^s} (\zeta - \lambda)^{-1}(D_A - \lambda)^{-1}(D_A - \zeta)x_k \, d\lambda
$$

$$
= \frac{1}{2\pi i} \int_{\Gamma_1^s} (\zeta - \lambda)^{-1}(D_A - \lambda + \lambda - \zeta)(D_A - \lambda)^{-1}x_k \, d\lambda
$$

$$
= E_1^s x_k - \frac{1}{2\pi i} \int_{\Gamma_1^s} (\lambda - \zeta)^{-1}x_k \, d\lambda
$$

$$
= E_1^s x_k - 0, \quad k = 1, 2,
$$

$$
= \begin{cases} 
  x_1, & \text{if } k = 1, \\
  0, & \text{if } k = 2,
\end{cases}
$$

(32)

where the next-to-last equality results because the function $\lambda \to (\lambda - \zeta)^{-1}x$ is analytic on a neighborhood of the simply connected region $\Gamma_1^s \cup \text{Int}(\Gamma_1^s)$. Since $(D_A - \zeta)|_{E_1^s\mathcal{H}}$ is clearly invertible,

$$(D_A - \zeta)[B_1^1(\zeta) - (D_A|_{E_1^s} - \zeta)^{-1}E_1^s]x = 0, \quad x \in \mathcal{L}.$$ 

Therefore for all $\zeta \in \text{Ext}(\Gamma_1^s) \setminus \Lambda$, we have

$$
B_1^1(\zeta)x = (D_A|_{E_1^s} - \zeta)^{-1}E_1^sx, \quad x \in \mathcal{L}.
$$

But from (29) the left-hand side of this last equality is analytic (in $\zeta$) on $\text{Ext}(\Gamma_1^s)$, and obviously so is the right-hand side. Therefore that equality holds for all $\zeta \in \text{Ext}(\Gamma_1^s)$. In particular, $B_1^1(\zeta)$ extends to a bounded operator in $\mathcal{L}(\mathcal{H})$ satisfying the same equation for each $\zeta$ in $\text{Ext}(\Gamma_1^s)$ and every $x$ in $\mathcal{H}$.

We now show that $A_1^1(\zeta)|_{\mathcal{L}}$ is bounded on $\mathcal{L}$, first by computing, using (14), (29), and (30):
\[ A_1^s(\zeta)x = \frac{1}{2\pi i} \int_{\Gamma_1^s} (\zeta - \lambda)^{-1} x_\lambda \, d\lambda - \frac{1}{2\pi i} \int_{\Gamma_1^s} (\zeta - \lambda)^{-1} \varphi_\lambda^{-1}(x, \bar{v}_\lambda) u_\lambda \, d\lambda, \]
\[ = B_1^s(\zeta)x - \frac{1}{2\pi i} \int_{\Gamma_1^s} (\zeta - \lambda)^{-1} \varphi_\lambda^{-1}(x, \bar{v}_\lambda) u_\lambda \, d\lambda, \quad x \in \mathcal{L}. \]

Then, using (16), we obtain (with \( K = \Gamma_1^s \))
\[ \| A_1^s(\zeta)x \| \leq \left\{ \| B_1^s(\zeta) \| + c_1^2/(r^2 \varepsilon_{K,r} \text{dist}(\zeta, \Gamma_1^s)) \right\} \| x \|, \quad x \in \mathcal{L}. \]
Thus \( A_1^s(\zeta) \) is bounded on \( \mathcal{L} \), and extends by continuity to an operator in \( \mathcal{L}(\mathcal{H}) \). Recall that from (29) and (30) we also obtain that for \( x \in \mathcal{L} \), the functions \( A_1^s, B_1^s : \text{Ext}(\Gamma_1^s) \to \mathcal{H} \) are analytic on \( \text{Ext}(\Gamma_1^s) \). Moreover, that \( A_1^s(\zeta) \in \{ T \}' \) is immediate from the computation
\[ A_1^s(\zeta)T x = \frac{1}{2\pi i} \int_{\Gamma_1^s} (\zeta - \lambda)^{-1} (T - \lambda)^{-1} T x \, d\lambda \]
\[ = \frac{1}{2\pi i} \int_{\Gamma_1^s} T (\zeta - \lambda)^{-1} x_\lambda^T \, d\lambda \]
\[ = T A_1^s(\zeta)x, \quad x \in \mathcal{L}, \; \zeta \in \text{Ext}(\Gamma_1^s), \quad (33) \]
which is valid since \( T \mathcal{L} \subset \mathcal{L} \subset \bigcap_{\lambda \in K} \text{ran}(T - \lambda) \). Next, we calculate
\[ A_1^s(\zeta)(T - \zeta)x = \frac{1}{2\pi i} \int_{\Gamma_1^s} (\zeta - \lambda)^{-1} (T - \zeta)(T - \lambda)^{-1} x \, d\lambda \]
\[ = \frac{1}{2\pi i} \int_{\Gamma_1^s} (\zeta - \lambda)^{-1} (T - \lambda + \lambda - \zeta)(T - \lambda)^{-1} x \, d\lambda \]
\[ = F_1^s x - \frac{1}{2\pi i} \int_{\Gamma_1^s} (\lambda - \zeta)^{-1} x \, d\lambda \]
\[ = F_1^s x, \quad x \in \mathcal{L}, \; \zeta \in \text{Ext}(\Gamma_1^s), \]
since the function \( \lambda \to (\lambda - \zeta)^{-1} x \) is analytic in a neighborhood of the simply connected region \( \Gamma_1^s \cup \text{Int}(\Gamma_1^s) \). Hence
\[ (T - \zeta)A_1^s(\zeta) = A_1^s(\zeta)(T - \zeta) = F_1^s, \quad \zeta \in \text{Ext}(\Gamma_1^s), \quad (34) \]
and we observe that this (together with its counterpart for \( j = 2 \)) shows that both \( F_1^s \) and \( F_2^s \) are nonzero. For instance, if \( F_2^s = 0 \), then \( F_1^s = 1_{\mathcal{H}} \) and (33) gives that \( \sigma(T) \cap \text{Ext}(\Gamma_1^s) = \emptyset \), which we know to be false since \( s \in \Pi_r' \) and there exists a point \( \lambda_0 \in \sigma(T) \subset \mathbb{D} \) such that \( s < P(\lambda_0) < b \).
**Step IV.** In this step we show that $F_s^1 F_s^2$ (which equals $F_s^2 F_s^1$ by Lemma 4.3) is the zero operator, which simultaneously shows that $F_s^1$ and $F_s^2$ are idempotents. To accomplish this, however, and also for use in the sequel [3] to obtain the decomposability of the operators in $(RO)_2$, we must introduce some additional machinery.

Since $T \in (RO)_2$, $T$ has the single-valued-extension property (SVEP); i.e., if $\emptyset \neq G \subset \mathbb{C}$ is a connected open set and $w : G \to \mathcal{H}$ is an analytic (vector-valued) function such that $(T - \lambda)w(\lambda) \equiv 0$ on $G$, then $w \equiv 0$ on $G$. (Indeed, if $G \cap \sigma(T) = \emptyset$, this is trivial. Otherwise, let $l$ be a vertical line with $l \cap G \neq \emptyset$ and $P(l) \in \Pi_s^j$. Since $\sigma_p(T) \cap \Pi_s^j = \emptyset$, $w \equiv 0$ on $l \cap G$, which contains an open interval, and thus $w \equiv 0$ on $G$ via the analyticity of $w$.) This makes it possible to define for every $x$ in $\mathcal{H}$, the local spectrum $\sigma_T(x) \subset \sigma(T)$ of $T$ at $x$ to be the (compact) set $\mathbb{C}\setminus \rho_T(x)$, where $\rho_T(x)$, the local resolvent of $T$ at $x$, is defined as the (open) set consisting of all $\lambda_0 \in \mathbb{C}$ such that there exists an open neighborhood $\mathcal{N}_{\lambda_0}(x)$ of $\lambda_0$ and an analytic function $x_{\lambda_0} : \mathcal{N}_{\lambda_0}(x) \to \mathcal{H}$ satisfying $(T - \lambda)x_{\lambda_0}(\lambda) \equiv x$ on $\mathcal{N}_{\lambda_0}(x)$. The SVEP guarantees the uniqueness of $x_{\lambda_0}$ and therefore one has an analytic function $x_T(\lambda)$ defined on $\rho_T(x)$ such that $(T - \lambda)x_T(\lambda) \equiv x$ on $\rho_T(x)$. It is well known (cf. [1, p. 1]) that $\sigma_T(x) = \emptyset$ if and only if $x = 0$ and also that $\sigma_T(Ax) \subset \sigma_T(x)$ for every $A \in \{T\}'$. In particular,

$$
\sigma_T(F_j^x x) \subset \sigma_T(x) \subset \sigma(T), \quad x \in \mathcal{H}, \quad j = 1, 2,
$$

and using Lemma 4.4 (see also (34)), we obtain

$$
A_j^x(\xi)(T - \xi)x = (T - \xi)A_j^x(\xi)x = F_j^x x, \quad \xi \in \text{Ext}(\Gamma_j^s), \quad x \in \mathcal{H}, \quad j = 1, 2.
$$

The analyticity of $A_j^x(\cdot)x$ on Ext($\Gamma_j^s$), together with the definition of local spectrum, gives

$$
\sigma_T(F_j^x x) \subset I_s \cup \text{Int}(\Gamma_j^s), \quad x \in \mathcal{H}, \quad j = 1, 2,
$$

and putting (35) and (36) together, we get

$$
\sigma_T(F_j^x x) \subset \sigma_T(x) \cap (I_s \cup \text{Int}(\Gamma_j^s)), \quad x \in \mathcal{H}, \quad j = 1, 2.
$$

(37)

To complete the argument that $F_s^1$ and $F_s^2$ are idempotents (for each $s \in \Pi_s^j$), it is convenient now to fix $s \in \Pi_s^j$, and introduce monotone sequences $\{s_n^-\}_{n \in \mathbb{N}}$ and $\{s_n^+\}_{n \in \mathbb{N}}$ in $\Pi_s^j$ such that $s_n^- \not\rightarrow s$ and $s_n^+ \not\rightarrow s$. Since $s$ was completely arbitrary in $\Pi_s^j$, all of the preceding results are valid for each $s_n^\pm$ as well as for $s$. Thus we obtain from (37) that

$$
\sigma_T(F_s^1 F_s^2 x) \subset \sigma_T(F_s^2 x) \cap (I_s \cup \text{Int}(\Gamma_1^s))
\subset (I_{s_n^+} \cup \text{Int}(\Gamma_2^s)) \cap (I_s \cup \text{Int}(\Gamma_1^s))
= \emptyset, \quad x \in \mathcal{H}, \quad n \in \mathbb{N}.
$$

Hence, by what was said above, $F_s^1 F_s^2 = 0$ for each $n \in \mathbb{N}$, and to complete the argument that $F_s^1 F_s^2 = 0$, we shall show that the sequence $\{F_s^2 n\}_{n \in \mathbb{N}}$ converges to $F_s^2$ in the weak operator topology (WOT). For this purpose we note that $\widetilde{K} = \Gamma_2^s \cup (\bigcup_{n \in \mathbb{N}} \Gamma_n^s)$ is a compact set, and
thus (23) with $K$ replaced by $\tilde{K}$ gives that the sequence $F_2^{s_n^+}$ is uniformly bounded. Thus it suffices to show that

\[ \langle (F_2^s - F_2^{s_n^+}) e_m, e_k \rangle \to 0, \quad k, m \in \mathbb{N}. \]

Next we use (21) and (22) to write

\[ F_2^s e_m = E \left( \text{Int}(\Gamma_2^s) \right) e_m + G_2^s e_m, \quad m \in \mathbb{N}, \]

where

\[ G_2^s e_m = \frac{1}{2\pi i} \int_{\Gamma_2^s} \varphi^{-1}_\lambda \{(D\Lambda - \lambda)^{-1} e_m, v\} u_\lambda d\lambda = \frac{\bar{\beta}_m}{2\pi i} \int_{\Gamma_2^s} \varphi^{-1}_\lambda (\lambda_m - \lambda)^{-1} u_\lambda d\lambda, \]

and similarly for $F_2^{s_n^+} e_m, n \in \mathbb{N}$. Since it is obvious from the definitions of the Jordan loops $\Gamma_2^{s_n^+}$ that

\[ \bigcup_{n \in \mathbb{N}} \text{Int}(\Gamma_2^{s_n^+}) = \text{Int}(\Gamma_2^s), \]

the regularity of the spectral measure $E$ gives us that $E_2^{s_n^+} \to E_2^s$ in the strong operator topology, and thus what remains is to show that

\[ \frac{1}{2\pi i} \left( \int_{a_2^s} - \int_{a_2^{s_n^+}} \right) (\alpha_k \bar{\beta}_m \varphi^{-1}_\lambda (\lambda - \lambda_m)^{-1} (\lambda - \lambda_k)^{-1}) d\lambda \\
+ \frac{1}{2\pi i} \left( \int_{l_s} - \int_{l_{s_n}^+} \right) (\alpha_k \bar{\beta}_m \varphi^{-1}_\lambda (\lambda - \lambda_m)^{-1} (\lambda - \lambda_k)^{-1}) d\lambda \\
= \langle (G_2^s - G_2^{s_n^+}) e_m, e_k \rangle \to 0, \quad k, m \in \mathbb{N}, \tag{38} \]

where the arcs $a_2^s, a_2^{s_n^+}, l_s,$ and $l_{s_n}^+$ are all properly oriented to agree with their definitions at the beginning of Section 4. Moreover, since $s_n^+ \to s, s_n^+ \to s$, and

\[ |\alpha_k \bar{\beta}_m \varphi^{-1}_\lambda (\lambda - \lambda_m)^{-1} (\lambda - \lambda_k)^{-1}| \leq \left( |\alpha_k \bar{\beta}_m|/\epsilon \tilde{K} \right) (1/ \min \{ \text{dist}(\lambda_k, \Gamma_2^s), \text{dist}(\lambda_m, \Gamma_2^s) \})^2, \quad \lambda \in a_2^s, k, m \in \mathbb{N}, \tag{39} \]

it is obvious that the first term on the left side of (38) tends to zero as $s_n^+ \to s$. On the other hand, if the line segments $l_s$ and $l_{s_n}^+$ are parametrized as at the beginning of the proof of Theorem 4.2, the second term on the left-hand side of (38) becomes
\[
\frac{\alpha_k \beta_m}{2\pi} \left[ \frac{\sqrt{1 - s^2}}{-1}\int_{-\sqrt{1 - s^2}}^{\sqrt{1 - s^2}} \varphi_{s+i\lambda}^{-1}(s + i\lambda_m)^{-1}(s + i \lambda_k)^{-1} \, d\lambda \\
- \frac{\sqrt{1 - s_n^2}}{-1}\int_{-\sqrt{1 - s_n^2}}^{\sqrt{1 - s_n^2}} \varphi_{s_n+i\lambda}^{-1}(s_n^+ + i\lambda_m)^{-1}(s_n^+ + i \lambda_k)^{-1} \, d\lambda \right] \]
\[
= \frac{\alpha_k \beta_m}{2\pi} \int_{-\sqrt{1 - s^2}}^{\sqrt{1 - s^2}} \left( \psi(t) - \chi_{[-\sqrt{1 - s_n^2}, \sqrt{1 - s_n^2}]}(t)\psi_n(t) \right) \, dt,
\]
where
\[
\psi(t) = \varphi_{s+i\lambda}^{-1}(s + i\lambda_m)^{-1}(s + i \lambda_k)^{-1},
\]
and the functions \(\psi_n(t)\) are defined analogously. Since \(\psi\) and \(\psi_n\) (for \(n\) large enough) are uniformly bounded as in (39), and \(\{\psi_n\}_{n\in\mathbb{N}}\) converges pointwise on \([-\sqrt{1 - s^2}, \sqrt{1 - s^2}]\) to \(\psi\), the convergence in the WOT of \(\{F_2^{s_n}\}_{n\in\mathbb{N}}\) to \(F_2^s\) follows, for example, from the Lebesgue bounded convergence theorem.

**Step V.** To complete the proof of Theorem 4.2, we first notice that from Lemmas 4.3, 4.4, and Step IV, we know that for each \(s \in \Pi', F_1^s\) and \(F_2^s\) are nonzero idempotents in \(\{T\}''\), and therefore that for all such \(s\), ran\((F_1^s)\) and ran\((F_2^s)\) are n.h.s. for \(T\). Thus it only remains to show that for \(s, s' \in \Pi', F_1^s \neq F_1^{s'}\). Thus suppose that \(F_1^s = F_1^{s'}\) with \(s < s'\). Then \(1_{\mathcal{H}} = F_1^s + F_2^s\), and therefore for every \(x \in \mathcal{H}, \sigma_T(x) \subset \sigma_T(F_1^s x) \cup \sigma_T(F_2^s x) \subset \text{Int}(\Gamma_1^s)^{-} \cup \text{Int}(\Gamma_2^s)^{-}\). Hence for every \(\lambda \in \mathbb{C}\) such that \(s < \text{Re}(\lambda) < s'\), and every \(x \in \mathcal{H}\), we have \((T - \lambda)x_T(\lambda) = x\). Thus \((T - \lambda)\mathcal{H} = \mathcal{H}\), and it follows that \(\sigma(T) \subset \text{Int}(\Gamma_1^s)^{-} \cup \text{Int}(\Gamma_2^s)^{-}\), which contradicts the fact that \(\sigma(T)\) is a connected set. \(\square\)

**Remark 4.5.** The alert reader will no doubt have noted that in Standing Conventions 4.1, we could have defined \(P_L\) to be the projection of \(\sigma(T)\) onto any given line \(L\) through the origin in place of the \(x\)-axis, and then we could have proved the analog of Theorem 4.2 with the \(x\)-axis in \(\mathbb{C}\) replaced by \(L\) and the arguments would be (except for notation) exactly the same. This fact will be important in the sequel [3] in proving the decomposability of the operators in the class \((RO)\).

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**References**