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Doubly resonant semilinear elliptic problems via nonsmooth critical point theory

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ABSTRACT

We consider the existence of weak solutions for classical doubly resonant semilinear elliptic problems. We show how the main technical assumptions can be used to define appropriate metrics on the underlying function space, so that extensions of the results already known in the literature can be obtained using only basic facts from critical point theory for continuous functionals on complete metric spaces.

1. Introduction

This paper deals with the existence of (weak) solutions of the semilinear elliptic problem

$$(P) \qquad -\Delta u = g(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a (smooth) bounded open subset of $\mathbb{R}^n$ ($n \geqslant 3$ for simplicity), and g is a Carathéodory function with subcritical growth, under so-called *double resonance* conditions. Namely, letting (λ_j) denote the nondecreasing sequence of (positive) eigenvalues of $-\Delta$ with 0-Dirichlet boundary condition, we say that the above problem is *strongly doubly resonant* if for some $i \in \mathbb{N}$ and almost every $x \in \Omega$, we have

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$$\lambda_i \leqslant \liminf_{|s|\to\infty} \frac{g(x,s)}{s} \leqslant \limsup_{|s|\to\infty} \frac{g(x,s)}{s} \leqslant \lambda_{i+1},$$

and we call it *weakly doubly resonant* if the weaker condition

$$\lambda_i \leqslant \liminf_{|s|\to\infty} \frac{2G(x,s)}{s^2} \leqslant \limsup_{|s|\to\infty} \frac{2G(x,s)}{s^2} \leqslant \lambda_{i+1},$$

where $G(x,s) := \int_0^s g(x,t)\,dt$, is satisfied. Such problems were first considered by Berestycki and Figueiredo [1] through a degree theoretic approach. Here, we use a variational approach and, amidst the vast literature on the subject, we parallel our study and the works (in decreasing order of generality) of Furtado and Silva [10], Costa and Magalhães [7], and Costa and Oliveira [8], which appear as key steps in this line (and to which we refer for further references). As well known, the functional f associated with problem (P) is of class C^1 on $H_0^1(\Omega)$; accordingly, the results of the quoted papers are based on critical point theory for C^1 functionals on Banach spaces. This leads to identify appropriate compactness conditions of (weak) Palais–Smale type satisfied by f, which are shown, in an abstract part, to suffice in order to obtain a suitable deformation property, whence an existence of critical point result.

The main purpose of this paper is to show how the main technical assumptions of the problem (namely, variants of the so-called *nonquadraticity at infinity* conditions, introduced in [7] and refined in [10]) can be used to define a metric d, equivalent to the norm metric, in such a way that, using only a basic form of the deformation principle from critical point theory for continuous functions on complete metric spaces [5,9], and a simple (new, however) linking principle, the functional f is shown to possess a Palais–Smale type sequence in $(H_0^1(\Omega), d)$ — which is ultimately shown to be norm-bounded, and we are done. For the definition of the metric d, we use a particular case of the general change-of-metric principle from [2], where the main motivation indeed was to unify various abstract results of Schechter in smooth critical point theory (see [16]), thus avoiding the repetitive construction of *ad hoc* deformations. We believe that our approach, in separating the specific technical features of the abstract theory and of its applications to partial differential equations, provides a clearer and more systematic view of the problems dealt with, what can thus be helpful for further studies.

In Section 2, we provide the necessary background in nonsmooth critical point theory, as well as the abstract results (Theorems 2.3 and 2.4) on existence of Palais–Smale type sequences in appropriately defined equivalent metrics, for continuous functions on Hilbert spaces. Though no min–max procedure is involved, these results can be seen as variants of the saddle-point theorem of Rabinowitz [15]. In Section 3, we introduce some nonquadraticity conditions (in the spirit of [10]), the connection of which with one of the afore-mentioned metrics is established. These conditions are used, together with Theorem 2.3, in Section 4, dealing with weak double resonance for problem (P), to obtain refinements of the corresponding results in [10]; and in Section 5, dealing with strong double resonance, in the case when the nonlinearity g has sublinear growth. Still in the sublinear case, we consider another set of nonquadraticity conditions in Section 6, from which existence results for problem (P) are deduced, under both weak and strong double resonance, as an application of Theorem 2.4.

2. Abstract results

Let X be a metric space endowed with the metric d, and let $f : X \to \mathbb{R}$ be continuous. If A is a subset of X, a *deformation* of A (in X) is a continuous map $\eta : A \times [0,1] \to X$ such that $\eta(u,0) = u$ for every $u \in A$. For $\rho > 0$, we denote by

$$B_\rho(A) := \{u \in X \colon d(u,A) < \rho\}$$

the open ρ-neighborhood of A, where $d(u, A) := \inf\{d(u, v)\colon v \in A\}$, with the convention $d(u, \emptyset) = +\infty$. If $A := \{u\}$, we simply write $B_\rho(u)$. If B is another subset of X, we set

$$d(A, B) := \inf\{d(u, v)\colon u \in A,\ v \in B\}$$

(with the convention $d(A, B) = +\infty$, if either A or B is empty). For $a \in \mathbb{R}$, we let

$$[f < a] := \{u \in X\colon f(u) < a\}.$$

We recall the notion of weak slope from [9].

Definition 2.1. Let (X, d) be a metric space, and let $f : X \to \mathbb{R}$ be continuous. For $u \in X$, we denote by $|df|(u)$ the supremum of the set of nonnegative reals σ such that there exist $\delta > 0$ and $\mathcal{H}\colon B_\delta(u) \times [0, \delta] \to X$ continuous with

$$d(\mathcal{H}(v, t), v) \leqslant t, \qquad f(\mathcal{H}(v, t)) \leqslant f(v) - \sigma t$$

for every $(v, t) \in B_\delta(u) \times [0, \delta]$. The extended real number $|df|(u)$ is called the *weak slope* of f at u.

Recall that if X is a C^1 Finsler manifold and if f is a C^1 function on X, then $|df|(u) = \|f'(u)\|$ for $u \in X$, where $f'(u)$ is the differential of f at u (see [9]).

The weak slope yields the following deformation theorem, which is a slight variant of [6, Theorem 2.1] (itself a direct extension of [3, Theorem 2], these results relying on the basic deformation theorem [5, Theorem (2.8)]).

Theorem 2.1. *Let (X, d) be a metric space, let $f : X \to \mathbb{R}$ be continuous, let C be a nonempty subset of X, and let $c \in \mathbb{R}$, $\rho > 0$, and $\sigma > 0$. Assume that $f^{-1}([c, b])$ is complete for every $b \in]c, c + \sigma\rho[$ and that*

$$|df|(u) > \sigma \quad \text{for every } u \in B_\rho(C) \text{ with } c < f(u) < c + \sigma\rho.$$

Then, there exist a continuous function $\tau : C \cap [f < c+\sigma\rho] \to [0, +\infty[$ and a deformation η of $C \cap [f < c+\sigma\rho]$ such that

(a) $\tau(u) \leqslant \max\{(f(u) - c)/\sigma, 0\} < \rho$;
(b) $d(\eta(u, t), u) \leqslant \tau(u)t$;
(c) $f(\eta(u, t)) \leqslant f(u) - \sigma\tau(u)t$;
(d) $f(u) \geqslant c \Rightarrow f(\eta(u, 1)) = c$.

Proof. Follow the arguments in the proof of [6, Theorem 2.1], replacing $B_{\rho_1+\rho_2}(C)$, $\overline{B}_{\rho_1}(C)$, and ρ_2 therein by $B_\rho(C)$, C, and ρ, respectively. □

Definition 2.2. Let (X, d) be a metric space, and let $S_1 \subset D_1$, $S_2 \subset D_2$ be four subsets of X, with at least three of them nonempty, and such that $d(S_1, D_2) > 0$ and $d(S_2, D_1) > 0$. We say that the pair (D_1, S_1) *links* the pair (D_2, S_2) if for every deformation η of D_1 satisfying

$$d(\eta(u, t), u) < \min\{d(S_1, D_2), d(S_2, D_1)\} \tag{2.1}$$

for every $(u, t) \in D_1 \times [0, 1]$, we have

$$\eta(D_1, 1) \cap D_2 \neq \emptyset. \tag{2.2}$$

Clearly, if (D_1, S_1) links (D_2, S_2) then $D_1 \cap D_2 \neq \emptyset$, so that for any function $f : X \to \mathbb{R}$ we have

$$\inf_{D_2} f \leqslant \sup_{D_1} f.$$

Theorem 2.2. *Let (X, d) be a complete metric space, let $f : X \to \mathbb{R}$ be continuous, let (D_1, S_1), (D_2, S_2) be two pairs in X such that (D_1, S_1) links (D_2, S_2), and set $\rho := \min\{d(D_2, S_1), d(D_1, S_2)\} > 0$. Assume that f is bounded from above on D_1 and is bounded from below on D_2. Then, for any reals β_1, β_2 with*

$$\beta_2 < \inf_{D_2} f \leqslant \sup_{D_1} f < \beta_1,$$

there exists $u \in B_\rho(D_1)$ such that

$$\beta_2 < f(u) < \beta_1 \quad \textit{and} \quad |df|(u) \leqslant \frac{\beta_1 - \beta_2}{\rho}.$$

Proof. We argue by contradiction. Since $A := \{u \in B_\rho(D_1)\colon\ \beta_2 < f(u) < \beta_1\} \neq \emptyset$, we thus assume that

$$|df|(u) > \frac{\beta_1 - \beta_2}{\rho} \quad \text{for every } u \in A.$$

Apply Theorem 2.1 with $C := D_1$, $c := \beta_2$, the given $\rho > 0$, and $\sigma := \frac{\beta_1 - \beta_2}{\rho}$. Noting that $D_1 \cap [f < \beta_1] = D_1$, we find a continuous function $\tau : D_1 \to [0, \rho[$ and a deformation η of D_1 satisfying properties (a), (b) and (d) of Theorem 2.1. Using (a) and (b), we see that η is a deformation of D_1 satisfying (2.1), and that $\eta(u,t) = u$ if $f(u) \leqslant \beta_2$. Taking also (d) into account, we thus have

$$f\big(\eta(u,1)\big) \leqslant \beta_2 < \inf_{D_2} f \quad \text{for every } u \in D_1,$$

so that $\eta(D_1, 1) \cap D_2 = \emptyset$, contrary to (2.2). This contradicts the fact that (D_1, S_1) links (D_2, S_2), whence the conclusion. □

In the remainder of this section, X is a real Hilbert space, and we denote by $\|\cdot\|$ the norm associated with the scalar product of X. We consider a proper linear subspace X_1 of X with $0 < \dim X_1 < +\infty$, and we let $X_2 := (X_1)^\perp$. For $i = 1, 2$ and $\rho > 0$, we set

$$D_{i,\rho} := \big\{u \in X_i\colon\ \|u\| \leqslant \rho\big\}, \qquad S_{i,\rho} := \big\{u \in X_i\colon\ \|u\| = \rho\big\}.$$

The following proposition, asserting that the pair $(D_{1,\rho_1}, S_{1,\rho_1})$ "topologically links" the pair $(D_{2,\rho_2}, S_{2,\rho_2})$, is a well-known fact (see, e.g., the proof of [12, Theorem (8.1)]). We sketch the proof for completeness.

Proposition 2.1. *Let $\rho_1, \rho_2 > 0$, and let η be a deformation of D_{1,ρ_1} in $(X, \|\cdot\|)$ such that*

$$\eta\big(S_{1,\rho_1} \times [0,1]\big) \cap D_{2,\rho_2} = \emptyset \quad \textit{and} \quad \eta\big(D_{1,\rho_1} \times [0,1]\big) \cap S_{2,\rho_2} = \emptyset.$$

Then, $\eta(D_{1,\rho_1}, 1) \cap D_{2,\rho_2} \neq \emptyset$.

Proof. Denote by π_i the orthogonal projection on X_i, $i = 1, 2$, set $Y := [-1, 1] \times D_{1,\rho_1}$, and define $\phi : Y \times [0,1] \to \mathbb{R} \times X_1$ by

$$\phi\big((s,u),t\big) := \left(\frac{\|\pi_2(\eta(u,t))\|}{\rho_2} + s, \pi_1\big(\eta(u,t)\big)\right).$$

Then, ϕ is continuous and $\phi((s,u),0) = (s,u)$ for every $(s,u) \in Y$. Moreover, $\phi((s,u),t) \neq (0,0)$ for every (s,u) on the boundary of Y and every $t \in [0,1]$. Indeed, let $(u,t) \in D_{1,\rho_1} \times [0,1]$ be such that $\pi_1(\eta(u,t)) = 0$. If $(s,u) \in [-1,1] \times S_{1,\rho_1}$, then

$$\frac{\|\pi_2(\eta(u,t))\|}{\rho_2} > 1, \quad \text{so that} \quad \frac{\|\pi_2(\eta(u,t))\|}{\rho_2} + s > 0,$$

while if $(s,u) \in \{-1,1\} \times D_{1,\rho_1}$, then

$$\frac{\|\pi_2(\eta(u,t))\|}{\rho_2} \neq 1, \quad \text{so that} \quad \frac{\|\pi_2(\eta(u,t))\|}{\rho_2} + s \neq 0.$$

Thus, the Brouwer degree of $\phi(\cdot,1)$ with respect to $(0,0)$ is 1 (that of the identity of Y), so that there exists $(s,u) \in Y$ with $\phi((s,u),1) = (0,0)$, and it is easy to see that this means that $\eta(u,1) \in D_{2,\rho_2}$. $\square$

The following proposition is a particular case of the change-of-metric principle [2, Theorem 4.1], refined in [3,6]. In particular, the formula in (b), refining the corresponding one in [2], is in [6, Theorem 2.2].

Proposition 2.2. *Let $\beta : [0,+\infty[\to]0,+\infty[$ be continuous. For $u, v \in X$, let $\Gamma_{u,v}$ denote the set of (piecewise) C^1 paths $\gamma : [0,1] \to X$ with $\gamma(0) = u$ and $\gamma(1) = v$, and set*

$$\tilde{d}(u,v) := \inf\left\{ \int_0^1 \beta\big(\|\gamma(t)\|\big)\,\|\gamma'(t)\|\,dt\colon\ \gamma \in \Gamma_{u,v} \right\}. \tag{2.3}$$

Then, $\tilde{d}$ is a metric on X which is topologically equivalent to the metric induced by the norm, and the following properties hold.

(a) *If $\int_0^{+\infty} \beta(s)\,ds = +\infty$, then $(X,\tilde{d})$ is complete.*
(b) *For every $u \in X$ we have*

$$\tilde{d}(u,0) = \int_0^{\|u\|} \beta(s)\,ds.$$

(c) *If $f : X \to \mathbb{R}$ is continuous, then for every $u \in X$ we have*

$$|\tilde{d}f|(u) = \frac{|df|(u)}{\beta(\|u\|)},$$

where $|df|$ and $|\tilde{d}f|$ denote the weak slope of f with respect to the norm metric and to the metric $\tilde{d}$, respectively.

We describe in the following a general procedure in order to use the change-of-metric principle.

Proposition 2.3. *Let $\beta : [0,+\infty[\to]0,+\infty[$ be continuous and such that $s\beta(s) \to +\infty$ as $s \to +\infty$, and let $\tilde{d}$ be the metric defined by (2.3) through the function β. For $i, j = 1,2$, $i \neq j$, we have*

$$\tilde{d}(S_{i,\rho}, D_{j,\rho}) \to +\infty \quad \text{as } \rho \to +\infty.$$

Proof. Let $i, j \in \{1, 2\}$ with $i \neq j$, let $\rho > 0$, $u \in S_{i,2\rho}$, $v \in D_{j,2\rho}$, and let $\gamma \in \Gamma_{u,v}$. Define

$$t_\gamma := \sup\bigl\{t \in [0,1]:\ \bigl\|\gamma(s)\bigr\| \in [\rho, 3\rho] \text{ for all } s \in [0,t]\bigr\}.$$

Note that t_γ is well defined (since $\|\gamma(0)\| = \|u\| = 2\rho$), and that $\|\gamma(t_\gamma) - \gamma(0)\| \geqslant \rho$ (if $t_\gamma < 1$, then either $\|\gamma(t_\gamma)\| = \rho$, or $\|\gamma(t_\gamma)\| = 3\rho$, while if $t_\gamma = 1$, then $\gamma(t_\gamma) = v$). We have

$$\begin{aligned}\int_0^1 \beta\bigl(\bigl\|\gamma(t)\bigr\|\bigr)\bigl\|\gamma'(t)\bigr\|\,dt &\geqslant \int_0^{t_\gamma} \beta\bigl(\bigl\|\gamma(t)\bigr\|\bigr)\bigl\|\gamma'(t)\bigr\|\,dt \\ &\geqslant \Bigl(\min_{s\in[\rho,3\rho]} \beta(s)\Bigr)\bigl\|\gamma(t_\gamma) - \gamma(0)\bigr\| \\ &\geqslant \rho \min_{s\in[\rho,3\rho]} \beta(s) \\ &\geqslant \frac{1}{3} \min_{s\in[\rho,3\rho]} s\beta(s).\end{aligned}$$

Since γ is arbitrary in $\Gamma_{u,v}$, and u, v are arbitrary in $S_{i,2\rho}$, $D_{j,2\rho}$, respectively, we thus obtain that

$$\tilde{d}(S_{i,2\rho}, D_{j,2\rho}) \geqslant \frac{1}{3} \min_{s\in[\rho,3\rho]} s\beta(s),$$

whence the conclusion. □

Combining the above propositions, the following is a corollary of Theorem 2.2.

Theorem 2.3. *Let $f : X \to \mathbb{R}$ be continuous, and let $\beta : [0, +\infty[\ \to\ ]0, +\infty[$ be continuous with $s\beta(s) \to +\infty$ as $s \to +\infty$. Assume that*

$$-\infty < a := \inf_{X_2} f \leqslant \sup_{X_1} f =: b < +\infty. \tag{2.4}$$

Then, there exist $c \in [a, b]$ and a sequence $(u_h) \subset X$ such that

$$f(u_h) \to c \quad \text{and} \quad \frac{|df|(u_h)}{\beta(\|u_h\|)} \to 0 \quad \text{as } h \to +\infty.$$

Proof. Recall that X is a Hilbert space with $X = X_1 \oplus X_2$, where $0 < \dim X_1 < \infty$ and $\dim X_2 > 0$. Let $\tilde{d}$ be the metric defined by (2.3) through the function β. Because $\beta(s) \geqslant \frac{1}{s}$ for large s, $(X, \tilde{d})$ is complete according to Proposition 2.2(a). Since the topology of $(X, \tilde{d})$ agrees with the norm topology, f is continuous on $(X, \tilde{d})$, while it follows from Proposition 2.1 that $(D_{1,h}, S_{1,h})$ links $(D_{2,h}, S_{2,h})$ in $(X, \tilde{d})$ for every $h \in \mathbb{N}$ (recall Definition 2.2, and note that $S_{1,h}$, $D_{1,h}$ are compact).

For each $h \in \mathbb{N}$, we may thus apply Theorem 2.2 in $(X, \tilde{d})$, to the function f, to the pairs $(D_{1,h}, S_{1,h})$, $(D_{2,h}, S_{2,h})$, and with $\beta_1 := b + \frac{1}{h}$, $\beta_2 := a - \frac{1}{h}$. Taking Proposition 2.2(c) into account, we find $(u_h) \subset X$ such that $a - \frac{1}{h} < f(u_h) < b + \frac{1}{h}$ and

$$\frac{|df|(u_h)}{\beta(\|u_h\|)} \leqslant \frac{\beta_1 - \beta_2}{\min\{\tilde{d}(S_{1,h}, D_{2,h}), \tilde{d}(S_{2,h}, D_{1,h})\}} \to 0 \quad \text{as } h \to +\infty,$$

according to Proposition 2.3. Up to a subsequence, we further have that $(f(u_h))$ converges to some $c \in [a, b]$. □

Remark 2.1. In critical point theory for continuous functions on metric spaces, the *Palais–Smale condition* for $f : X \to \mathbb{R}$ at the level $c \in \mathbb{R}$ (condition $(\mathrm{PS})_c$, for short) reads:

> If $(u_h) \subset X$ is a sequence such that $f(u_h) \to c$ and $|df|(u_h) \to 0$, then (u_h) has a convergent subsequence.

We thus see that if, in Theorem 2.3, we further assume that the function f satisfies condition $(\mathrm{PS})_c$ for every $c \in [a, b]$, *in the metric space* $(X, \tilde{d})$, then f *has a critical point at level* c (for some $c \in [a, b]$), that is, there exists $u \in X$ with $|df|(u) = 0$ and $f(u) = c$ (using the fact that the weak slope is (clearly) lower semicontinuous, and taking Proposition 2.2(c) into account). In [17], using the so-called *strong Cerami–Palais–Smale condition* at level $c \in \mathbb{R}$ (denoted by $(\mathrm{SCe})_c$), Silva obtains a result of this type through a classical approach in critical point theory for C^1 functionals, namely, by showing that condition $(\mathrm{SCe})_c$ allows to obtain a suitable deformation lemma for the function f. We believe it enlightening to rather rely on a general, basic deformation principle (Theorem 2.1), and on an elementary consequence of it (Theorem 2.2), while postponing, at the stage of the applications to variational problems, the choice (through the function β) of the appropriate metric in which to apply the abstract principles — a choice depending on the technical assumptions made in the problem.

As is well known, in applications to variational nonlinear elliptic problems, if the nonlinearity has *subcritical growth* then a bounded Palais–Smale sequence for the associated functional has a convergent subsequence. Arguing as above, and rather than verify a Palais–Smale condition in a generalized sense, we shall thus directly show, in the applications given in Sections 4 and 5, that the sequence (u_h) provided by Theorem 2.3 indeed is bounded.

A simple construction, to be used in applications, of a function β as in Theorem 2.3 is given below, through another function ω.

Proposition 2.4. *Let* $\omega :]0, +\infty[\to \mathbb{R}$ *be nondecreasing, with* $\omega(s) \to +\infty$ *as* $s \to +\infty$. *Then there exists a bounded, continuous function* $\beta_\omega : [0, +\infty[\to]0, +\infty[$ *such that*:

(a) $s^2 \beta_\omega(s)^2 \leqslant \omega(s)$ *for large* s;
(b) $s\beta_\omega(s) \to +\infty$ *as* $s \to +\infty$.

Proof. Let $s_0 > 0$ be such that $\omega(s) > 0$ for all $s \geqslant s_0$. Define, for $s \geqslant s_0$:

$$\tilde{\omega}(s) := \inf\bigl\{\bigl(\omega(t) + |t - s|\bigr)^{1/2}\colon t \geqslant s_0\bigr\} = \inf\bigl\{\bigl(\omega(t) + s - t\bigr)^{1/2}\colon t \in [s_0, s]\bigr\}.$$

Then, $\tilde{\omega} : [s_0, +\infty[\to]0, +\infty[$ is continuous, with $\tilde{\omega}(s)^2 \leqslant \omega(s)$ for $s \geqslant s_0$. Moreover, $\tilde{\omega}$ is nondecreasing and $\tilde{\omega}(s) \to +\infty$ as $s \to +\infty$. Then the function $\beta_\omega : [0, +\infty[\to]0, +\infty[$ defined by

$$\beta_\omega(s) := \min\left\{1, \frac{\tilde{\omega}(\check{s})}{\check{s}}\right\}, \quad \text{where } \check{s} := \max\{s, s_0\},$$

has the desired properties. □

We now provide a variant of Theorem 2.3 for a specific function β, but where we relax assumption (2.4): in this case, there is no related Palais–Smale condition, while from the point of view of applications (see Section 6), the approach is the same.

Proposition 2.5. *For $\mu > 0$, consider the metric*

$$d_\mu(u,v) := \inf\left\{ \int_0^1 \bigl(1 + \|\gamma(t)\|\bigr)^{\mu-1} \|\gamma'(t)\|\, dt\colon \gamma \in \Gamma_{u,v} \right\}$$

(*corresponding to the function $\beta_\mu(s) := (1+s)^{\mu-1}$ in* (2.3)). *For $i, j = 1, 2$, $i \neq j$, we have*:

(a) *if $\mu \geqslant 1$, then $d_\mu(S_{i,\rho}, D_{j,\rho}) = \frac{1}{\mu}(1+\rho)^\mu$;*
(b) *if $0 < \mu < 1$ and $\rho \geqslant 1$, then $d_\mu(S_{i,\rho}, D_{j,\rho}) \geqslant \rho^\mu\, d_\mu(S_{i,1}, D_{j,1})$.*

Proof. (a) Let π_1 denote the orthogonal projection on X_1, let $u \in X_1$, $v \in X_2$, and let $\gamma \in \Gamma_{u,v}$. Then, $\gamma_1 := \pi_1 \circ \gamma \in \Gamma_{u,0}$, and since $s \mapsto (1+s)^{\mu-1}$ is nondecreasing, we have

$$\int_0^1 \bigl(1 + \|\gamma(t)\|\bigr)^{\mu-1} \|\gamma'(t)\|\, dt \geqslant \int_0^1 \bigl(1 + \|\gamma_1(t)\|\bigr)^{\mu-1} \|\gamma_1'(t)\|\, dt \geqslant d_\mu(u,0),$$

so that $d_\mu(u,v) \geqslant d_\mu(u,0)$, from which we deduce, taking also Proposition 2.2(b) into account, that

$$d_\mu(S_{1,\rho}, D_{2,\rho}) = d_\mu(S_{1,\rho}, 0) = \frac{1}{\mu}(1+\rho)^\mu.$$

Similarly, through projecting γ on X_2, we obtain the other equality.

(b) Let $u \in S_{1,\rho}$, $v \in D_{2,\rho}$, and let $\gamma \in \Gamma_{u,v}$. Then, $u/\rho \in S_{1,1}$, $v/\rho \in D_{2,1}$, $\gamma/\rho \in \Gamma_{u/\rho, v/\rho}$, and we have

$$\begin{aligned}\int_0^1 \bigl(1 + \|\gamma(t)\|\bigr)^{\mu-1} \|\gamma'(t)\|\, dt &\geqslant \int_0^1 \bigl(\rho + \|\gamma(t)\|\bigr)^{\mu-1} \|\gamma'(t)\|\, dt \\ &= \rho^\mu \int_0^1 \bigl(1 + \|(\gamma/\rho)(t)\|\bigr)^{\mu-1} \|(\gamma/\rho)'(t)\|\, dt \\ &\geqslant \rho^\mu\, d_\mu(u/\rho, v/\rho) \geqslant \rho^\mu\, d_\mu(S_{1,1}, D_{2,1}),\end{aligned}$$

so that $d_\mu(S_{1,\rho}, D_{2,\rho}) \geqslant \rho^\mu\, d_\mu(S_{1,1}, D_{2,1})$. The other inequality is proved in a similar way. □

Theorem 2.4. *Let $f : X \to \mathbb{R}$ be continuous and bounded from below on the bounded subsets of X_2, and let $\mu > 0$. Assume that*

$$\lim_{h\to\infty} \frac{1}{h^\mu}\Bigl(\sup_{D_{1,h}} f - \inf_{D_{2,h}} f\Bigr) = 0.$$

Then, there exists a sequence $(u_h) \subset X$ such that

$$\inf_{X_2} f \leqslant \liminf_{h\to\infty} f(u_h) \leqslant \limsup_{h\to\infty} f(u_h) \leqslant \sup_{X_1} f$$

and

$$\lim_{h\to\infty} \frac{|df|(u_h)}{(1+\|u_h\|)^{\mu-1}} = 0.$$

Proof. Let d_μ be the metric defined in Proposition 2.5, so that, according to this result we have

$$\min\{d_\mu(S_{1,h}, D_{2,h}), d_\mu(S_{2,h}, D_{1,h})\} \geqslant \frac{h^\mu}{\alpha} \quad \text{for all } h \in \mathbb{N},$$

where $\alpha := \mu$ if $\mu \geqslant 1$, and $\alpha := (\min\{d_\mu(S_{1,1}, D_{2,1}), d_\mu(S_{2,1}, D_{1,1})\})^{-1}$ if $\mu < 1$. Similarly as in the proof of Theorem 2.3 (to which we refer for details), taking Proposition 2.2 into account, we may apply Theorem 2.2 in (X, d_μ): we find a sequence $(u_h) \subset X$ such that

$$\inf_{D_{2,h}} f - \frac{1}{h} \leqslant f(u_h) \leqslant \sup_{D_{1,h}} f + \frac{1}{h}$$

and

$$\frac{|df|(u_h)}{(1+\|u_h\|)^{\mu-1}} \leqslant \frac{\alpha}{h^\mu}\left(\sup_{D_{1,h}} f - \inf_{D_{2,h}} f + \frac{2}{h}\right) \to 0 \quad \text{as } h \to +\infty,$$

according to the assumption made. □

Remark 2.2. Note that the case of the *Cerami metric*, namely, the metric d_0 corresponding to the function $\beta_0(s) := 1/(1+s)$, is not covered by Theorems 2.3 and 2.4. Indeed, though (X, d_0) is not bounded – as is seen from Proposition 2.2(b) – we have that for $i, j = 1, 2$, $i \neq j$, $d_0(S_{i,\rho}, D_{j,\rho})$ is bounded with respect to ρ. Indeed, considering $u \in S_{i,\rho}$, $v \in S_{j,\rho}$, and $\gamma(t) := \rho(u + t(v-u))/\|u + t(v-u)\|$, $t \in [0, 1]$, an easy computation shows that

$$d_0(S_{i,\rho}, D_{j,\rho}) \leqslant d_0(S_{i,\rho}, S_{j,\rho}) \leqslant \int_0^1 \frac{\|\gamma'(t)\|}{1+\|\gamma(t)\|}\, dt = \frac{\pi}{2}\, \frac{\rho}{\rho+1}.$$

Note also that the sequence (u_h) provided by Theorem 2.4 is not truly a Palais–Smale sequence (see Remark 2.1), since $(f(u_h))$ may not be bounded. Theorem 2.4 is a variant of some (more involved) results in [2], see also Schechter [16] in a smooth setting.

3. The semilinear elliptic problem and nonquadraticity conditions

Let Ω be a nonempty, bounded domain in $\mathbb{R}^n$, $n \geqslant 3$, with a smooth boundary $\partial\Omega$. We consider the Hilbert space $H_0^1(\Omega)$ endowed with the scalar product

$$(u, v) := \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx,$$

and we denote by $\|\cdot\|$ its induced norm. For $1 \leqslant p < +\infty$, we denote by $\|\cdot\|_p$ the norm of $L^p(\Omega)$, by $p' = \frac{p}{p-1}$ the conjugate of p (for $1 < p < +\infty$), and we let $2^* = \frac{2n}{n-2}$ denote the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ in $L^p(\Omega)$.

We consider the problem (P) of the existence of a weak solution $u \in H_0^1(\Omega)$ of the semilinear equation

$$-\Delta u = g(\cdot, u) \quad \text{in } \mathcal{D}'(\Omega),$$

where (and throughout this section) $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for almost every $x \in \Omega$ and all $s \in \mathbb{R}$,

$$(g) \qquad \left|g(x,s)\right| \leqslant a(x) + b|s|^{p-1},$$

where $a \in L^{2^{*\prime}}(\Omega)$, $b \geqslant 0$, and $1 < p \leqslant 2^*$.

Under this assumption, and letting $G(x,s) := \int_0^s g(x,t)\,dt$, the functional f defined by

$$f(u) = \frac{1}{2}\int_\Omega \left|\nabla u(x)\right|^2 dx - \int_\Omega G\big(x, u(x)\big)\,dx \tag{3.1}$$

is (well defined and) of class C^1 on $(H_0^1(\Omega), \|\cdot\|)$, with

$$f'(u)(v) = \int_\Omega \nabla u \cdot \nabla v - \int_\Omega g(x,u)v$$

(from now on, we drop the "dx"s) so that for every $u \in H_0^1(\Omega)$ we have $|df|(u) = \|f'(u)\|_{H^{-1}(\Omega)}$, while

$$2f(u) - f'(u)(u) = \int_\Omega \big(g(x,u)u - 2G(x,u)\big).$$

As is well known, existence results for problem (P) heavily depend on the behavior of the integrand in the latter equality. Accordingly, a basic condition we shall use here is the following:

There exists $a_0 \in L^1(\Omega)$ such that for almost every $x \in \Omega$ and all $s \in \mathbb{R}$,

$$(G.g)^{\pm} \qquad \pm\big(g(x,s)s - 2G(x,s)\big) \geqslant -a_0(x).$$

This indeed contains two (dual) conditions: $(G.g)^+$ and $(G.g)^-$, which imply that

$$H_\pm(x) := \liminf_{|s|\to\infty}\big[\pm\big(g(x,s)s - 2G(x,s)\big)\big] > -\infty \quad \text{for a.e. } x \in \Omega. \tag{3.2}$$

We set

$$\Omega_\pm := \big\{x \in \Omega:\ H_\pm(x) = +\infty\big\}.$$

In the remainder of this section, we discuss some conditions on the "size" of the sets Ω_+ and Ω_-, that allow to make appropriate choices of a metric on $H_0^1(\Omega)$, in order to apply the abstract results of the previous section to obtain the existence of a weak solution for (P).

Let $0 < \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_h \to +\infty$ denote the eigenvalues of $-\Delta$ on $H_0^1(\Omega)$, counted according to their multiplicity, and let $(e_j)_{j\in\mathbb{N}}$ denote corresponding eigenvectors forming a Hilbert basis of $H_0^1(\Omega)$ ($e_1 > 0$ on Ω). For $j \in \mathbb{N}$, we denote by E_j the (finite dimensional) λ_j-eigenspace.

Let S denote the best constant for the embedding $H_0^1(\Omega) \subset L^{2^*}(\Omega)$, that is

$$S := \inf\left\{ \frac{\|u\|^2}{\|u\|_{2^*}^2} \colon u \in H_0^1(\Omega),\ u \neq 0 \right\}.$$

We set $\alpha_1 := 0$, and for $j \in \mathbb{N}$, $j \geqslant 2$:

$$\alpha_j := |\Omega| - \left(\frac{S}{\lambda_j}\right)^{n/2} > 0,$$

where $|\cdot|$ denotes the Lebesgue measure. (Note that $n/2 = (2^*/2)'$.)

Lemma 3.1. *Let $j \in \mathbb{N}$, and let $\Omega_0 \subset \Omega$ with $|\Omega_0| > \alpha_j$. There exists $\varepsilon_j > 0$ such that for every $u \in H_0^1(\Omega)$ with $\|u\| \leqslant 1$ and $\lambda_j \|u\|_2^2 \geqslant 1 - \varepsilon_j$, we have*

$$\left| \left\{ x \in \Omega_0 \colon u(x) \neq 0 \right\} \right| > 0.$$

Proof. Assume, for a contradiction, that there is a sequence (u_h) in $H_0^1(\Omega)$ such that for each h,

$$\|u_h\| \leqslant 1, \qquad \lambda_j \|u_h\|_2^2 \geqslant 1 - \frac{1}{h}, \quad \text{and} \quad u_h(x) = 0 \quad \text{for a.e. } x \in \Omega_0.$$

Up to a subsequence, we may assume that (u_h) converges to some function u, weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$, and almost everywhere in Ω, so that $\|u\| \leqslant 1$, $\lambda_j \|u\|_2^2 \geqslant 1$, and $u(x) = 0$ for almost every $x \in \Omega_0$. If $j = 1$, we thus have $u \in E_1 \setminus \{0\}$, so that $u(x) \neq 0$ for every $x \in \Omega$, which contradicts the fact that $|\Omega_0| > 0$. If $j \geqslant 2$, we have

$$1 \leqslant \lambda_j \left(\int\limits_{\Omega \setminus \Omega_0} |u|^{2^*} \right)^{2/2^*} |\Omega \setminus \Omega_0|^{2/n} \leqslant \frac{\lambda_j}{S} |\Omega \setminus \Omega_0|^{2/n},$$

which contradicts the fact that $|\Omega_0| > \alpha_j$. $\square$

According to the previous lemma, for $j \in \mathbb{N}$ we consider the two conditions:

$(NQ_j)^{\pm}$ $$|\Omega_{\pm}| > \alpha_j.$$

Lemma 3.2. *Let $j \in \mathbb{N}$, and assume that conditions $(G.g)^+$ and $(NQ_j)^+$ hold. Let $\varepsilon_j > 0$ be given by Lemma 3.1, corresponding to $\Omega_0 := \Omega_+$. Set*

$$E^{\varepsilon_j} := \left\{ u \in H_0^1(\Omega) \colon \|u\| \leqslant 1,\ \lambda_j \|u\|_2^2 \geqslant 1 - \varepsilon_j \right\},$$

and define a nondecreasing $\omega_{j,+} :]0, +\infty[\to \mathbb{R}$ by:

$$\omega_{j,+}(s) := \inf\left\{ \int\limits_{\Omega} \big(g(x, \rho u)\rho u - 2G(x, \rho u) \big) \colon \rho \geqslant s,\ u \in E^{\varepsilon_j} \right\}.$$

Then,

$$\lim_{s \to +\infty} \omega_{j,+}(s) = +\infty.$$

Proof. Let (u_h) be a sequence in E^{ε_j} and $0 < \rho_h \to +\infty$. Up to a subsequence, (u_h) converges to some function u, weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$, and almost everywhere in Ω. Thus, $u \in E^{\varepsilon_j}$ and, from $(NQ_j)^+$ and Lemma 3.1 we find $\tilde{\Omega} \subset \Omega_+$ with $|\tilde{\Omega}| > 0$ such that $u(x) \neq 0$ for every $x \in \tilde{\Omega}$. According to $(G.g)^+$ (recall (3.2)), we may apply Fatou's lemma to obtain

$$\lim_{h\to\infty} \int_{\Omega} \big(g(x, \rho_h u_h)\rho_h u_h - 2G(x, \rho_h u_h)\big) \geqslant \int_{\tilde{\Omega}} H_+ = +\infty.$$

The conclusion follows from the arbitrariness of the sequences (u_h) and (ρ_h). □

Remark 3.1. In a dual way, if conditions $(G.g)^-$ and $(NQ_j)^-$ hold, then the function $\omega_{j,-} :]0, +\infty[\to \mathbb{R}$ defined by:

$$\omega_{j,-}(s) := \inf\left\{ \int_{\Omega} \big(2G(x, \rho u) - g(x, \rho u)\rho u\big):\ \rho \geqslant s,\ u \in E^{\varepsilon_j} \right\},$$

also satisfies $\omega_{j,-}(s) \to +\infty$ as $s \to +\infty$.

Lemma 3.3. *Let $j \in \mathbb{N}$, and assume that $|\Omega_+| > 0$ (i.e., condition $(NQ_1)^+$ holds). There exists $\varepsilon_j > 0$ such that for every $u \in H_0^1(\Omega)$ with $\|u\| \leqslant 1$, $\lambda_j \|u\|_2^2 \geqslant 1 - \varepsilon_j$, and $d(u, E_j) \leqslant \varepsilon_j$, we have*

$$\big|\big\{x \in \Omega_+:\ u(x) \neq 0\big\}\big| > 0.$$

Proof. Assume, for a contradiction, that there is a sequence (u_h) in $H_0^1(\Omega)$ such that for each h: $\|u_h\| \leqslant 1$, $\lambda_j \|u_h\|_2^2 \geqslant 1 - \frac{1}{h}$, $d(u_h, E_j) \leqslant \frac{1}{h}$, and

$$\big|\big\{x \in \Omega_+:\ u_h(x) \neq 0\big\}\big| = 0.$$

Then, up to a subsequence, (u_h) converges weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$, and almost everywhere in Ω, to some function $u \in E_j$ with $\|u\| = 1$, so that $u(x) \neq 0$ for almost every $x \in \Omega$, and $|\{x \in \Omega_+:\ u(x) \neq 0\}| = 0$, so that $|\Omega_+| = 0$, contradicting our assumption. □

Arguing in a similar way as in Lemma 3.2 yields the following.

Lemma 3.4. *Let $j \in \mathbb{N}$, assume that conditions $(G.g)^+$ and $(NQ_1)^+$ hold, and let $\varepsilon_j > 0$ be given by Lemma* 3.3. *Set*

$$E_j^{\varepsilon_j} := \big\{u \in H_0^1(\Omega):\ \|u\| \leqslant 1,\ \lambda_j \|u\|_2^2 \geqslant 1 - \varepsilon_j,\ d(u, E_j) \leqslant \varepsilon_j\big\},$$

and define a nondecreasing $\hat{\omega}_{j,+} :]0, +\infty[\to \mathbb{R}$ by

$$\hat{\omega}_{j,+}(s) := \inf\left\{ \int_{\Omega} \big(g(x, \rho u)\rho u - 2G(x, \rho u)\big):\ \rho \geqslant s,\ u \in E_j^{\varepsilon_j} \right\}.$$

Then,

$$\lim_{s\to+\infty} \hat{\omega}_{j,+}(s) = +\infty.$$

Remark 3.2. (a) Set

$$\underline{H}_{\pm}(x) := \liminf_{s\to-\infty}\big[\pm\big(g(x,s)s - 2G(x,s)\big)\big],$$

$$\overline{H}_{\pm}(x) := \liminf_{s\to+\infty}\big[\pm\big(g(x,s)s - 2G(x,s)\big)\big].$$

Arguing similarly as in Lemma 3.3 and in Lemma 3.4, it is easily seen that the latter holds under the following condition, weaker than $(NQ_1)^+$:

$$(NQ_j)_0^+ \qquad \forall u \in E_j \setminus \{0\}\colon \quad \big|\{\overline{H}_+ = +\infty,\ u > 0\}\big| + \big|\{\underline{H}_+ = +\infty,\ u < 0\}\big| > 0.$$

In a dual way, results similar to the two previous lemmas hold under conditions $(G.g)^-$ and $(NQ_1)^-$, or, more generally, under conditions $(G.g)^-$ and

$$(NQ_j)_0^- \qquad \forall u \in E_j \setminus \{0\}\colon \quad \big|\{\overline{H}_- = +\infty,\ u > 0\}\big| + \big|\{\underline{H}_- = +\infty,\ u < 0\}\big| > 0.$$

(b) The conditions $(NQ_j)^{\pm}$ originate from the work of Furtado and Silva [10]. Precisely, conditions $(NQ)_{\pm}$ in [10] read: "$|\Omega_{\pm}| > 0$ and $(G.g)^{\pm}$ holds"; while the assumption that $|\Omega_{\pm}| > \alpha_j$ is delayed there until the verification of the compactness condition for the functional f mentioned in Remark 2.1 — which points up the difference of approach evoked in that remark. Also, our singling out conditions $(G.g)^{\pm}$ is justified by the specific role these are to play in the coming section. The stronger conditions

$$(H)^{\pm} \qquad \lim_{|s|\to\infty}\big[\pm\big(g(x,s)s - 2G(x,s)\big)\big] = +\infty \quad \text{uniformly for a.e. } x \in \Omega$$

(implying $\Omega_{\pm} = \Omega$, and $(G.g)^{\pm}$, due to uniformity) were considered by Costa and Magalhães [7], who introduced such *nonquadraticity conditions*. See Remark 4.3 below for further comments.

4. A weakly doubly resonant problem

In the remainder of the paper, $f : H_0^1(\Omega) \to \mathbb{R}$ is the functional defined by (3.1). In this section, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying (g). Using standard notations, for $x \in \Omega$ we let

$$L_{\pm}(x) := \liminf_{s\to\pm\infty}\frac{2G(x,s)}{s^2}, \qquad K_{\pm}(x) := \limsup_{s\to\pm\infty}\frac{2G(x,s)}{s^2},$$

and set $L := \min\{L_+, L_-\}$, $K := \max\{K_+, K_-\}$. We first point out the following consequence of conditions $(G.g)^{\pm}$, building on a technical device from [7, Lemma 4].

Lemma 4.1.

(a) *Assume* $(G.g)^+$. *Then,*

$$2G(x,s) \leqslant K(x)s^2 + a_0(x) \quad \textit{for a.e. } x \in \Omega \textit{ and for all } s \in \mathbb{R}.$$

(b) *Assume* $(G.g)^-$. *Then,*

$$2G(x,s) \geqslant L(x)s^2 - a_0(x) \quad \textit{for a.e. } x \in \Omega \textit{ and for all } s \in \mathbb{R}.$$

Proof. We show (a), the proof of (b) is similar. For almost every $x \in \Omega$ and every $s \neq 0$ we have

$$s^3\frac{d}{ds}\left(\frac{G(x,s)}{s^2}\right) = g(x,s)s - 2G(x,s) \geqslant -a_0(x),$$

so that for every t with $ts > 0$ and $|t| > |s|$ we obtain

$$\frac{G(x,s)}{s^2} \leqslant \frac{G(x,t)}{t^2} + \frac{a_0(x)}{2}\left(\frac{1}{s^2} - \frac{1}{t^2}\right) \tag{4.1}$$

which, taking the upper limits on the right-hand side as $t \to \pm\infty$, and the definition of K into account, yields the result. $\square$

Remark 4.1. Observe that, taking the lower limits on the right-hand side of (4.1), as $t \to \pm\infty$, then the upper limits on the left-hand side, as $s \to \pm\infty$, we indeed obtain $K_+ = L_+$ and $K_- = L_-$. That is, under condition $(G.g)^+$ (or under condition $(G.g)^-$ as well), there exist

$$\ell_+(x) := \lim_{s\to+\infty} \frac{2G(x,s)}{s^2} \quad \text{and} \quad \ell_-(x) := \lim_{s\to-\infty} \frac{2G(x,s)}{s^2}$$

(and, of course, $L = \min\{\ell_+, \ell_-\}$, $K = \max\{\ell_+, \ell_-\}$).

For $j \in \mathbb{N}$, we set

$$X_j^- := \operatorname{span}\{e_1, \ldots, e_j\} \quad \text{and} \quad X_j^+ := \overline{\operatorname{span}\{e_j, e_{j+1}, \ldots\}},$$

so that

$$\|u\|^2 \leqslant \lambda_j \|u\|_2^2 \quad \text{for } u \in X_j^- \quad \text{and} \quad \|u\|^2 \geqslant \lambda_j \|u\|_2^2 \quad \text{for } u \in X_j^+, \tag{4.2}$$

and we consider the following one-sided growth conditions on G:

There exists $\delta_j \in L^1(\Omega)$ such that for almost every $x \in \Omega$ and all $s \in \mathbb{R}$,

$(G_j)^\pm$
$$\pm 2G(x,s) \leqslant \pm\lambda_j s^2 + \delta_j(x).$$

Taking (4.2) into account, we see that $(G_j)^\pm$ (respectively) imply

$$\pm 2f(u) = \pm\|u\|^2 \mp \int_\Omega 2G(x,u) \geqslant -\|\delta_j\|_1 \quad \text{for every } u \in X_j^\pm. \tag{4.3}$$

We now consider the following conditions, weaker than $(G_j)^\pm$:

There exist $a_1 \in L^1(\Omega)$ and $b_1 \in L^{n/2}(\Omega)$ such that for almost every $x \in \Omega$ and all $s \in \mathbb{R}$,

$(G)^\pm$
$$\pm 2G(x,s) \leqslant a_1(x) + b_1(x)s^2.$$

Notation. For a function u defined on Ω, we set as usual: $u^+ := \max\{u, 0\}$, $u^- := \max\{-u, 0\}$. Moreover, equalities or large inequalities between functions defined on Ω are hereafter meant to hold in *almost everywhere* sense.

Lemma 4.2. *Let $(\hat{u}_h)$ be a sequence in $H_0^1(\Omega)$ with $\|\hat{u}_h\| = 1$ for every h, and weakly convergent to a function u, let $0 < \rho_h \to +\infty$, and let $j \in \mathbb{N}$. We have:*

(a) *If $(G)^+$ holds, then*

$$\limsup_{h\to\infty} \int_\Omega \frac{2G(x, \rho_h \hat{u}_h)}{\rho_h^2} \leqslant \int_\Omega \left(K_+\left(u^+\right)^2 + K_-\left(u^-\right)^2\right). \tag{4.4}$$

If, moreover, $K \leqslant \lambda_j$, $u \in X_j^+$, *and*

$$\liminf_{h\to\infty} \frac{f(\rho_h \hat{u}_h)}{\rho_h^2} \leqslant 0, \tag{4.5}$$

then $u \in E_j \setminus \{0\}$ *and* $(\lambda_j - K_+)u^+ + (\lambda_j - K_-)u^- = 0$ *(whence* $K = \lambda_j$*).*

(b) *If* $(G)^-$ *holds, then*

$$\liminf_{h\to\infty} \int_\Omega \frac{2G(x, \rho_h \hat{u}_h)}{\rho_h^2} \geqslant \int_\Omega \left(L_+\left(u^+\right)^2 + L_-\left(u^-\right)^2\right).$$

If, moreover, $L \geqslant \lambda_j$, $(\hat{u}_h)$ *converges strongly to* $u \in X_j^-$, *and*

$$\limsup_{h\to\infty} \frac{f(\rho_h \hat{u}_h)}{\rho_h^2} \geqslant 0,$$

then $u \in E_j \setminus \{0\}$ *and* $(L_+ - \lambda_j)u^+ + (L_- - \lambda_j)u^- = 0$ *(whence* $L = \lambda_j$*).*

Proof. (a) Up to a subsequence, we may assume that $\hat{u}_h \to u$ almost everywhere in Ω. Then, according to $(G)^+$,

$$\frac{2G(\cdot, \rho_h \hat{u}_h)}{\rho_h^2} \leqslant \frac{a_1}{\rho_h^2} + b_1 \hat{u}_h^2 \to b_1 u^2 \quad \text{strongly in } L^1(\Omega),$$

while

$$\limsup_{h\to\infty} \frac{2G(x, \rho_h \hat{u}_h(x))}{\rho_h^2} \leqslant K_+(x)\left(u^+(x)\right)^2 + K_-(x)\left(u^-(x)\right)^2$$

for almost every $x \in \Omega$, and (4.4) follows from Fatou's lemma. If (4.5) holds, we thus have

$$0 \geqslant \liminf_{h\to\infty} \frac{2f(\rho_h \hat{u}_h)}{\rho_h^2} = 1 - \limsup_{h\to\infty} \int_\Omega \frac{2G(x, \rho_h \hat{u}_h)}{\rho_h^2} \geqslant 1 - \int_\Omega \left(K_+\left(u^+\right)^2 + K_-\left(u^-\right)^2\right),$$

so that, if $K \leqslant \lambda_j$, if $u \in X_j^+$, and taking (4.2) into account, we have

$$\|u\|^2 \leqslant 1 \leqslant \int_\Omega \left(K_+\left(u^+\right)^2 + K_-\left(u^-\right)^2\right) \leqslant \lambda_j \|u\|_2^2 \leqslant \|u\|^2.$$

It follows that $u \in E_j$ with $\|u\| = 1$, and that $\int_\Omega ((\lambda_j - K_+)(u^+)^2 + (\lambda_j - K_-)(u^-)^2) = 0$, so that $(\lambda_j - K_+)u^+ + (\lambda_j - K_-)u^- = 0$. Since $u(x) \neq 0$ for almost every $x \in \Omega$, we obtain in particular that $K = \lambda_j$.

(b) The first conclusion is obtained similarly as in (a). Using it, under the further assumptions made, and taking (4.2) into account, we obtain

$$\lambda_j \|u\|_2^2 \leqslant \int_\Omega \left(L_+\left(u^+\right)^2 + L_-\left(u^-\right)^2\right) \leqslant 1 = \|u\|^2 \leqslant \lambda_j \|u\|_2^2.$$

It follows that $u \in E_j$, and then, that $(L_+ - \lambda_j)u^+ + (L_- - \lambda_j)u^- = 0$. □

Corollary 4.1. *Let $j \in \mathbb{N}$.*

(a) *If $(G)^+$ holds, if $K \leqslant \lambda_j$, and if $(\lambda_j - K_+)u^+ + (\lambda_j - K_-)u^- \neq 0$ for every $u \in E_j \setminus \{0\}$, then $f(u) \to +\infty$ as $\|u\| \to \infty$, $u \in X_j^+$.*

(b) *If $(G)^-$ holds, if $L \geqslant \lambda_j$, and if $(L_+ - \lambda_j)u^+ + (L_- - \lambda_j)u^- \neq 0$ for every $u \in E_j \setminus \{0\}$, then $f(u) \to -\infty$ as $\|u\| \to \infty$, $u \in X_j^-$.*

Proof. Assume, for a contradiction, that there is a sequence (u_h) in X_j^+ (resp., in X_j^-) with $0 < \rho_h := \|u_h\| \to \infty$ and $\sup(f(u_h)) < +\infty$ (resp., $\inf(f(u_h)) > -\infty$). Setting $\hat{u}_h := \frac{u_h}{\rho_h}$, so that, up to a subsequence, $(\hat{u}_h)$ converges weakly in X_j^+ (resp., strongly in (the finite dimensional) X_j^-), Lemma 4.2 yields a contradiction. $\square$

Remark 4.2. Lemma 4.2 and Corollary 4.1 are adaptations of classical arguments, see, e.g., [14, Lemma 1].

In view of the previous result and of Remark 4.1, we are led to consider the following condition, for $j \in \mathbb{N}$:

$$(E_j) \qquad (\lambda_j - \ell_+)u^+ + (\lambda_j - \ell_-)u^- \neq 0 \quad \text{for every } u \in E_j \setminus \{0\}.$$

(Note that (E_1) just reads: $\ell_+ \neq \lambda_1 \neq \ell_-$.)

The following is the main result of this section.

Theorem 4.1. *Assume (g) with $p < 2^*$. Let further $i \in \mathbb{N}$, assume that*

$$(WR)_{i,i+1} \qquad \lambda_i \leqslant L \leqslant K \leqslant \lambda_{i+1},$$

and that conditions $(G.g)^+$, $(NQ_{i+1})^+$, and either $(G_i)^-$ or (E_i) *and $(G)^-$, hold. Then, problem (P) has a weak solution.*

Proof. Considering the decomposition

$$H_0^1(\Omega) = X_1 \oplus X_2 \quad \text{with } X_1 := X_i^- \text{ and } X_2 := X_1^\perp \left(= X_{i+1}^+\right), \tag{4.6}$$

we first show that

$$-\infty < \inf_{X_2} f \leqslant \sup_{X_1} f < +\infty. \tag{4.7}$$

Indeed, $K \leqslant \lambda_{i+1}$ and $(G.g)^+$ imply that $(G_{i+1})^+$ holds (with $\delta_{i+1} := a_0$), according to Lemma 4.1(a), so that f is bounded below on X_2 according to (4.3). Likewise, (4.3) yields that f is bounded above on X_1 if $(G_i)^-$ holds. Finally, if (E_i) and $(G)^-$ hold, and since $L \geqslant \lambda_i$, f is anticoercive on X_1 according to Corollary 4.1(b), so that again $\sup_{X_1} f < +\infty$.

Thanks to $(G.g)^+$ and $(NQ_{i+1})^+$, letting $E^{\varepsilon_{i+1}}$ and $\omega_{i+1,+} =: \omega$ be defined as in Lemma 3.2, consider $\beta_\omega : [0, +\infty[\to]0, +\infty[$ given by Proposition 2.4. Applying Theorem 2.3, we obtain a sequence $(u_h) \subset H_0^1(\Omega)$ such that

$$\big(f(u_h)\big) \text{ is bounded} \quad \text{and} \quad \frac{f'(u_h)}{\beta_\omega(\|u_h\|)} \to 0. \tag{4.8}$$

We show that (u_h) is bounded. Arguing by contradiction, assume that, up to a subsequence, $0 < \rho_h := \|u_h\| \to \infty$ and $\hat{u}_h := \frac{u_h}{\rho_h}$ strongly converges in $L^2(\Omega)$. Since $(G_{i+1})^+$ holds (as already said), we have

$$1=\lim_{h\to\infty}\left(1-\frac{2f(u_h)}{\rho_h^2}\right)=\lim_{h\to\infty}\int\limits_{\Omega}\frac{2G(x,\rho_h\hat{u}_h)}{\rho_h^2}\leqslant\lambda_{i+1}\lim_{h\to\infty}\|\hat{u}_h\|_2^2, \tag{4.9}$$

so that $\hat{u}_h\in E^{\varepsilon_{i+1}}$ for large h. Thus, from the definition of ω and from Proposition 2.4(a), we obtain

$$2f(u_h)-f'(u_h)(u_h)=\int\limits_{\Omega}\big(g(x,\rho_h\hat{u}_h)\rho_h\hat{u}_h-2G(x,\rho_h\hat{u}_h)\big)\geqslant\omega(\rho_h)\geqslant\rho_h^2\beta_\omega(\rho_h)^2$$

for large h. Taking Proposition 2.4(b) into account, this yields

$$\frac{2f(u_h)-f'(u_h)(u_h)}{\rho_h\beta_\omega(\rho_h)}\geqslant\rho_h\beta_\omega(\rho_h)\to+\infty,$$

while, on the other hand, (4.8) yields

$$\frac{2f(u_h)-f'(u_h)(u_h)}{\rho_h\beta_\omega(\rho_h)}\to 0,$$

which is the desired contradiction.

Thus, (u_h) is a bounded sequence with $f'(u_h)\to 0$, so that, since $p<2^*$, we conclude in a standard way (see, e.g., [15, Proposition B.35]) that (u_h) has a strongly convergent subsequence, the limit of which is a weak solution of (P). $\square$

In a (an essentially) symmetric way, we have the following result.

Theorem 4.2. *Assume* (g) *with* $p<2^*$. *Let further* $i\in\mathbb{N}$, *assume that*

$$\lambda_i\leqslant L\leqslant K\leqslant\lambda_{i+1},$$

and that conditions $(G.g)^-$, $(NQ_{i+1})^-$, *and* either $(G_{i+1})^+$ or (E_{i+1}) *and* $(G)^+$, *hold. Then, problem* (P) *has a weak solution.*

Proof. If condition $(G_{i+1})^+$ holds, the proof is quite similar to that of Theorem 4.1, observing that $L\geqslant\lambda_i$ and $(G.g)^-$ imply $(G_i)^-$, and using the function $\omega_{i+1,-}$ of Remark 3.1 in place of $\omega_{i+1,+}$. If conditions (E_{i+1}) and $(G)^+$ hold, we further have to observe, first, that f, which is coercive on X_2 according to Corollary 4.1(a), is bounded on bounded subsets of $H_0^1(\Omega)$, so that $\inf_{X_2}f>-\infty$; then, in order to obtain the estimate (4.9), we now use (4.4) together with $K\leqslant\lambda_{i+1}$. $\square$

The following is an immediate corollary of both Theorems 4.1 and 4.2.

Corollary 4.2. *Assume* (g) *with* $p<2^*$. *Assume further that for some* $i\in\mathbb{N}$ *we have*

$$L=K=\lambda_i,$$

and that one of the following sets of conditions is satisfied:

(i) $i\geqslant 2$, $(G.g)^+$, $(NQ_i)^+$, *and* $(G)^-$;
(ii) $(G.g)^-$, $(NQ_{i+1})^-$, *and* $(G)^+$.

Then, problem (P) *has a weak solution.*

(Indeed, in case (i) we may assume that $\lambda_{i-1} < \lambda_i$, so that (E_{i-1}) holds, while in case (ii) we may assume that $\lambda_i < \lambda_{i+1}$, so that (E_{i+1}) holds.)

Remark 4.3. (a) Theorems 4.1 and 4.2 extend the results of Furtado and Silva [10, Theorems 1.3, 4.1] where, using our notations, it is assumed that $(G_i)^-$ (resp., $(G_{i+1})^+$) holds whenever $L = \lambda_i$ (resp., $K = \lambda_{i+1}$), while the two-sided version of $(G)^\pm$ (that is, $(G)^+$ *and* $(G)^-$) is assumed, the latter being combined with the fact that g is assumed continuous on $\overline{\Omega} \times \mathbb{R}$ in order to get, in a rather involved way, the conclusions of Corollary 4.1. On the other hand, we note that Furtado and Silva were mainly concerned with the existence of a *nontrivial* solution, see [10, Theorem 1.2] (involving further assumptions on g), which is obtained using a Morse theoretic approach in the spirit of [11]. This type of result can probably also be revisited in the light of nonsmooth Morse theory — in that respect, see [4]. After this remark, we conclude the section with a refinement of [10, Theorem 1.4], concerning the existence of a nontrivial solution in the case $i = 1$ (resonance at the first eigenvalue).

(b) Theorems 4.1 and 4.2 also extend Costa and Magalhães' [7, Theorem 2], where $(WR)_{i,i+1}$ holds uniformly for almost every $x \in \Omega$ (which implies $(G)^+$ and $(G)^-$), and *either* $L \neq \lambda_i$ and $(H)^+$ holds, *or* $K \neq \lambda_{i+1}$ and $(H)^-$ holds (recall Remark 3.2(b)). It is indeed shown in [7] — using also the uniformity in $(WR)_{i,i+1}$ — that under condition $(H)^+$ (resp., $(H)^-$), the functional f is coercive on X_2 (resp., anticoercive on X_1), and satisfies the Palais–Smale condition (recall Remark 2.1) in $(H^1_0(\Omega), d_0)$, where d_0 is the Cerami metric (recall Remark 2.2). Thus, a solution of problem (P) can be obtained by applying the saddle point theorem of Rabinowitz [15, Theorem 4.6] in $(H^1_0(\Omega), d_0)$ (see [5, Theorem 3.7] for a general, metric version of the latter). We also note that an example of a function g which is not sublinear, but with a subquadratic potential G is given in [7].

Theorem 4.3. *Assume* (g) *with* $p < 2^*$, *and that* $K \leqslant \lambda_1$. *Assume further that one of the following sets of conditions is satisfied*:

(i) $(G.g)^+$ *and* $(NQ_1)^+$;
(ii) $(G.g)^-$, $(NQ_1)^-$, *and* either $(G_1)^+$ or (E_1) *and* $(G)^+$.

Then, problem (P) *has a weak solution.*

Proof. In both cases, arguing as in the proofs of Theorem 4.1 and 4.2, we see that the functional f is bounded below ($H^1_0(\Omega) = X_1^+$). Still arguing as in the proof of Theorem 4.1 (boundedness of the sequence (u_h)), we see that f indeed satisfies the $(\mathrm{PS})_c$ condition at any level c in the (complete) metric space $(H^1_0(\Omega), \tilde{d})$, where $\tilde{d}$ is the metric of Proposition 2.2 corresponding to the function β_ω in Proposition 2.4, with $\omega := \omega_{1,+}$ given by Lemma 3.2 in case (i), and with $\omega := \omega_{1,-}$ given by Remark 3.1 in case (ii).

Thus, the functional f attains its global minimum at some $u \in H^1_0(\Omega)$, which is a weak solution of problem (P). □

Slightly refining the arguments in [10, Theorem 1.4], it is possible to obtain a nontrivial solution in the previous result, through additional assumptions on the behavior of $G(x,s)$ for $s > 0$ and small. Namely, set

$$L_0(x) := \liminf_{s \to 0^+} \frac{2G(x,s)}{s^2},$$

and consider the local conditions:

For almost every $x \in \Omega$ and for all $s > 0$ and small,

$$(G_1)^0 \qquad 2G(x,s) \geqslant \lambda_1 s^2;$$

There exists $b_0 \in L^1(\Omega)$ such that for almost every $x \in \Omega$ and for all $s > 0$ and small,

$$(G)^0 \qquad 2G(x, s) \geqslant -b_0(x)s^2.$$

Lemma 4.3. *Let $0 < s_h \to 0$. If $(G)^0$ holds, then*

$$\liminf_{h\to\infty} \int_\Omega \frac{2G(x, s_h e_1)}{s_h^2} \geqslant \int_\Omega L_0 e_1^2.$$

If, moreover, $L_0 \geqslant \lambda_1$ and $\limsup_{h\to\infty} \frac{f(s_h e_1)}{s_h^2} \geqslant 0$, then $L_0 = \lambda_1$.

Proof. The first conclusion follows from Fatou's lemma. Under the further assumptions made, we thus obtain

$$\|e_1\|^2 \geqslant \liminf_{h\to\infty} \int_\Omega \frac{2G(x, s_h e_1)}{s_h^2} \geqslant \int_\Omega L_0 e_1^2 \geqslant \lambda_1 \|e_1\|_2^2 = \|e_1\|^2,$$

so that $L_0 = \lambda_1$. □

In the following result, $L_0 \ngeqslant \lambda_1$ means that $L_0 \geqslant \lambda_1$ and $L_0 \neq \lambda_1$.

Corollary 4.3. *Under the assumptions of Theorem* 4.3, *assume further that* either $(G_1)^0$ *holds,* or $L_0 \ngeqslant \lambda_1$ *and* $(G)^0$ *holds. Then, problem* (P) *has a nontrivial weak solution.*

Proof. If $(G_1)^0$ holds, since e_1 is a bounded, positive function on Ω, we have

$$2f(se_1) = \int_\Omega \left(\lambda_1 s^2 e_1^2 - 2G(x, se_1)\right) \leqslant 0$$

for $s > 0$ and small enough. In the other case, it follows from Lemma 4.3 that $f(se_1) < 0$ for all $s > 0$ and small enough. Thus, the functional f attains its minimun at a nonzero $u \in H_0^1(\Omega)$ (since $f(0) = 0$). □

Remark 4.4. In [10, Theorem 1.4], it is assumed that $(G)^0$ holds with $b_0 := 0$, while only case (i) of Theorem 4.3 is considered, under the additional (explicit) assumption that $(G)^+$ holds. As a matter of fact, $(G.g)^+$ and $K \leqslant \lambda_1$ imply $(G_1)^+$ (with $\delta_1 := a_0$), which is stronger than $(G)^+$.

5. A strongly doubly resonant problem

In this section, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for almost every $x \in \Omega$ and all $s \in \mathbb{R}$,

$$(\hat{g}) \qquad \left|g(x, s)\right| \leqslant a(x) + b(x)|s|,$$

where $a \in L^{r_1}(\Omega)$ and $b \in L^{r_2}(\Omega)$, with $r_1 \geqslant 2^{*\prime}$ and $r_2 \geqslant (2^*/2)'$. Set

$$r := \min\left\{r_1, \frac{2^* r_2}{2^* + r_2}\right\}.$$

Since $\frac{2^* r_2}{2^*+r_2} = (\frac{1}{2^*} + \frac{1}{r_2})^{-1} \geqslant (\frac{1}{2^*} + \frac{1}{(2^*/2)'})^{-1} = 2^{*\prime}$, we have $r \geqslant 2^{*\prime}$, and

$$r > 2^{*\prime} \quad \Longleftrightarrow \quad r_1 > 2^{*\prime} \text{ and } r_2 > \big(2^*/2\big)'.$$

We shall use the standard notations $l := \min\{l_+, l_-\}$, $k := \max\{k_+, k_-\}$, where

$$l_\pm(x) := \liminf_{s\to\pm\infty} \frac{g(x,s)}{s}, \qquad k_\pm(x) := \limsup_{s\to\pm\infty} \frac{g(x,s)}{s}.$$

It is easy to see that $l_\pm \leqslant L_\pm \leqslant K_\pm \leqslant k_\pm$ ($L_\pm$ and $K_\pm$ as defined in the previous section). Arguing as in the proof of Lemma 4.2, but using condition $(\hat{g})$ in place of condition $(G)^\pm$, readily yields the following.

Lemma 5.1. *Let $(\hat{u}_h)$ be a sequence in $H_0^1(\Omega)$, weakly convergent to a function u, and let $0 < \rho_h \to +\infty$. Then*:

(a) *For every $v \in L^{2^*}(\Omega)$ with $v \geqslant 0$, we have*

$$\limsup_{h\to\infty} \int_\Omega \frac{g(x, \rho_h \hat{u}_h)v}{\rho_h} \leqslant \int_\Omega \big(k_+ u^+ - l_- u^-\big)v,$$

$$\liminf_{h\to\infty} \int_\Omega \frac{g(x, \rho_h \hat{u}_h)v}{\rho_h} \geqslant \int_\Omega \big(l_+ u^+ - k_- u^-\big)v;$$

(b) *We have*:

$$\limsup_{h\to\infty} \int_\Omega \frac{2G(x, \rho_h \hat{u}_h)}{\rho_h^2} \leqslant \int_\Omega \big(K_+ \big(u^+\big)^2 + K_- \big(u^-\big)^2\big),$$

$$\liminf_{h\to\infty} \int_\Omega \frac{2G(x, \rho_h \hat{u}_h)}{\rho_h^2} \geqslant \int_\Omega \big(L_+ \big(u^+\big)^2 + L_- \big(u^-\big)^2\big).$$

The following lemma builds on some well-known facts (see, e.g., [1,13]).

Lemma 5.2. *Let $i \in \mathbb{N}$, let $(\hat{u}_h)$ be a sequence in $H_0^1(\Omega)$ with $\|\hat{u}_h\| = 1$ for every h, and weakly convergent to a function u, and let $0 < \rho_h \to +\infty$. Assume that*

$$\frac{f'(\rho_h \hat{u}_h)}{\rho_h} \to 0.$$

Then:

(a) *$u \neq 0$, and if $\lambda_i \leqslant l \leqslant k \leqslant \lambda_{i+1}$ then*, either *$u \in E_i$ and $(l_+ - \lambda_i)u^+ + (l_- - \lambda_i)u^- = 0$ (whence $l = \lambda_i$)*, or *$u \in E_{i+1}$ and $(\lambda_{i+1} - k_+)u^+ + (\lambda_{i+1} - k_-)u^- = 0$ (whence $k = \lambda_{i+1}$).*
(b) *If $r > 2^{*\prime}$, $\hat{u}_h \to u$ strongly in $H_0^1(\Omega)$. Moreover, if $\lambda_i \leqslant l \leqslant k \leqslant \lambda_{i+1}$ and if*

$$\frac{f(\rho_h \hat{u}_h)}{\rho_h^2} \to 0,$$

then, either $u \in E_i$ *and* $(L_+ - \lambda_i)u^+ + (L_- - \lambda_i)u^- = 0$ *(whence* $L = \lambda_i$*),* or $u \in E_{i+1}$ *and* $(\lambda_{i+1} - K_+)u^+ + (\lambda_{i+1} - K_-)u^- = 0$ *(whence* $K = \lambda_{i+1}$*).*

Proof. (a) Since

$$\frac{|g(\cdot, \rho_h \hat{u}_h)\hat{u}_h|}{\rho_h} \leqslant \frac{a|\hat{u}_h|}{\rho_h} + b\hat{u}_h^2 \to bu^2 \quad \text{strongly in } L^1(\Omega),$$

assuming that $u = 0$ yields the contradiction

$$1 = \|\hat{u}_h\|^2 = \frac{f'(\rho_h \hat{u}_h)(\hat{u}_h)}{\rho_h} + \int_\Omega \frac{g(x, \rho_h \hat{u}_h)\hat{u}_h}{\rho_h} \to 0.$$

Moreover, we have

$$\left\| \frac{g(\cdot, \rho_h \hat{u}_h)}{\rho_h} \right\|_r \leqslant \frac{\|a\|_r}{\rho_h} + C\|b\|_{r_2}\|\hat{u}_h\|_{2^*} \leqslant C$$

(for some constants $C > 0$), so that, up to a subsequence,

$$\frac{g(\cdot, \rho_h \hat{u}_h)}{\rho_h} \to z \quad \text{weakly in } L^r(\Omega) \tag{5.1}$$

(for some function z). It follows that for every $v \in H_0^1(\Omega)$:

$$\begin{aligned} \int_\Omega \nabla u \cdot \nabla v &= \lim_{h\to\infty} \int_\Omega \nabla \hat{u}_h \cdot \nabla v \\ &= \lim_{h\to\infty} \left[\frac{f'(\rho_h \hat{u}_h)(v)}{\rho_h} + \int_\Omega \frac{g(x, \rho_h \hat{u}_h)v}{\rho_h} \right] \\ &= \int_\Omega zv \end{aligned}$$

while, according to Lemma 5.1(a), we have

$$l_+u^+ - k_-u^- \leqslant z \leqslant k_+u^+ - l_-u^-. \tag{5.2}$$

If $\lambda_i \leqslant l \leqslant k \leqslant \lambda_{i+1}$, we thus have $z = mu$ for some (measurable) function m such that $\lambda_i \leqslant m \leqslant \lambda_{i+1}$, so that

$$-\Delta u = mu \quad \text{in } \mathcal{D}'(\Omega). \tag{5.3}$$

It follows that either $m = \lambda_i$, or $m = \lambda_{i+1}$ – for, otherwise, denoting by $\hat{\lambda}_j(m)$ the eigenvalues of the weighted eigenvalue problem (5.3), the monotonicity property of these eigenvalues with respect to m yields

$$\hat{\lambda}_i(m) < \hat{\lambda}_i(\lambda_i) = 1 = \hat{\lambda}_{i+1}(\lambda_{i+1}) < \hat{\lambda}_{i+1}(m),$$

contradicting the fact that u is an eigenfunction of (5.3). Thus, (5.3) reads $u \in E_i \cup E_{i+1}$, and the fact that $(l_+ - \lambda_i)u^+ + (l_- - \lambda_i)u^- = 0$, or $(\lambda_{i+1} - k_+)u^+ + (\lambda_{i+1} - k_-)u^- = 0$ (depending on which eigenspace u belongs to) follows from (5.2) again.

(b) If $r > 2^{*\prime}$, up to a subsequence we have $\hat{u}_h \to u$ strongly in $L^{r'}(\Omega)$ and we obtain from (5.1) and the fact that $\frac{f'(\rho_h \hat{u}_h)}{\rho_h} \to 0$, that

$$\begin{aligned}\|u\|^2 &= \lim_{h\to\infty} \int_\Omega \nabla \hat{u}_h \cdot \nabla u \\ &= \lim_{h\to\infty} \int_\Omega \frac{g(x, \rho_h \hat{u}_h) u}{\rho_h} \\ &= \lim_{h\to\infty} \int_\Omega \frac{g(x, \rho_h \hat{u}_h) \hat{u}_h}{\rho_h} \\ &= \lim_{h\to\infty} \|\hat{u}_h\|^2 \quad (=1),\end{aligned}$$

so that $\hat{u}_h \to u$ strongly in $H_0^1(\Omega)$.

If $\lambda_i \leqslant l \leqslant k \leqslant \lambda_{i+1}$, we know from part (a) that $u \in E_i \cup E_{i+1}$, and the alternative follows as in Lemma 4.2, thanks to Lemma 5.1(b). $\square$

Remark 5.1. The conclusion of Lemma 5.2(b) also holds in the case $r = 2^{*\prime}$ provided $l = k$. Indeed, if $m := l = k$, we have

$$\frac{g(\cdot, \rho_h \hat{u}_h)}{\rho_h} \to mu \quad \text{strongly in } L^{2^{*\prime}}(\Omega),$$

which yields $\|\hat{u}_h\| \to \|u\|$ as in the above proof, so that again $\hat{u}_h \to u$ in $H_0^1(\Omega)$. Observe that if $l = k$, the alternative in part (b) of Lemma 5.2 is the same as in part (a).

Theorem 5.1. *Assume $(\hat{g})$ with $r > 2^{*\prime}$ and $(G.g)^+$. Let further $i \in \mathbb{N}$, assume that*

$$(SR)_{i,i+1} \qquad \lambda_i \leqslant l \leqslant k \leqslant \lambda_{i+1},$$

and that one of the following sets of conditions is satisfied:

(i) $(NQ_1)^+$, *and* $(G_i)^-$;
(ii) $(NQ_{i+1})_0^+$ *and* (E_i).

Then, problem (P) has a weak solution.

Proof. Consider the decomposition (4.6). Arguing as in the proof of Theorem 4.1 (note that in case (ii), the second inequality in Lemma 5.1(b) yields the conclusion of Corollary 4.1(b)), we have (4.7) again.

In case (i), let, for $j := i, i+1$, $E_j^{\varepsilon_j}$ and $\hat{\omega}_{j,+}$ be the set and function defined in Lemma 3.4, set $\omega := \min\{\hat{\omega}_{i,+}, \hat{\omega}_{i+1,+}\}$, and let β_ω be given by Proposition 2.4. Applying Theorem 2.3, we find a sequence $(u_h) \subset H_0^1(\Omega)$ such that

$$\big(f(u_h)\big) \text{ is bounded} \quad \text{and} \quad \frac{f'(u_h)}{\beta_\omega(\|u_h\|)} \to 0, \tag{5.4}$$

and we show that (u_h) is bounded. Arguing by contradiction, assume that, up to a subsequence, $0 < \rho_h := \|u_h\| \to \infty$ and $\hat{u}_h := \frac{u_h}{\rho_h}$ converges weakly in $H_0^1(\Omega)$ to some function u. Recalling that β_ω is bounded, (5.4) yields

$$\frac{f(\rho_h \hat{u}_h)}{\rho_h^2} \to 0 \quad \text{and} \quad \frac{f'(\rho_h \hat{u}_h)}{\rho_h} \to 0,$$

so that, according to Lemma 5.2(b), $(\hat{u}_h)$ strongly converges to u and $u \in E_i \cup E_{i+1}$. Thus, $\|u\| = 1$, and either $\lambda_i \|u\|_2^2 = 1$, or $\lambda_{i+1}\|u\|_2^2 = 1$, so that $\hat{u}_h \in E_i^{\varepsilon_i} \cup E_{i+1}^{\varepsilon_{i+1}}$ for large h. We conclude similarly as in the proof of Theorem 4.1.

In case (ii), taking Remark 3.2 into account, we proceed as above, but with $\omega := \hat{\omega}_{i+1,+}$ only, since condition (E_i) excludes the possibility that $u \in E_i$ (recall also Remark 4.1). □

In a dual way, we have the following result.

Theorem 5.2. *Assume $(\hat{g})$ with $r > 2^{*\prime}$ and $(G.g)^-$. Let further $i \in \mathbb{N}$, assume that*

$$\lambda_i \leqslant l \leqslant k \leqslant \lambda_{i+1},$$

and that one of the following sets of conditions is satisfied:

(i) $(NQ_1)^-$, *and* $(G_{i+1})^+$;
(ii) $(NQ_i)_0^-$ *and* (E_{i+1}).

Then, problem (P) has a weak solution.

Remark 5.2. In view of Remark 5.1, if $l = k$, Theorems 5.1 and 5.2 hold also with $r = 2^{*\prime}$. In particular, we have the following, which complements Corollary 4.2 and Theorem 4.3.

Corollary 5.1. *Assume $(\hat{g})$. Assume further that for some $i \in \mathbb{N}$ we have*

$$l = k = \lambda_i,$$

and that one of the following sets of conditions is satisfied:

(i) $i \geqslant 2$, $(G.g)^+$, *and* $(NQ_i)_0^+$;
(ii) $(G.g)^-$ *and* $(NQ_i)_0^-$.

Then, problem (P) has a weak solution.

Remark 5.3. In the classical case $\lambda_i \leqslant l \leqslant k \leqslant \lambda_{i+1}$ with $L \neq \lambda_i$ and $K \neq \lambda_{i+1}$, the problem (P_w) of the existence of a weak solution $u \in H_0^1(\Omega)$ of the semilinear equation

$$-\Delta u = g(\cdot, u) + w \quad \text{in } \mathcal{D}'(\Omega), \tag{5.5}$$

can be studied for any $w \in H^{-1}(\Omega)$. Precisely, assuming $(\hat{g})$ and $(SR)_{i,i+1}$, and one of the following sets of conditions:

(i) $r > 2^{*\prime}$, $(L_+ - \lambda_i)u^+ + (L_- - \lambda_i)u^- \neq 0$ for every $u \in E_i \setminus \{0\}$, and $(\lambda_{i+1} - K_+)u^+ + (\lambda_{i+1} - K_-)u^- \neq 0$ for every $u \in E_{i+1} \setminus \{0\}$;
(ii) $(l_+ - \lambda_i)u^+ + (l_- - \lambda_i)u^- \neq 0$ for every $u \in E_i \setminus \{0\}$, and $(\lambda_{i+1} - k_+)u^+ + (\lambda_{i+1} - k_-)u^- \neq 0$ for every $u \in E_{i+1} \setminus \{0\}$,

then, for any $w \in H^{-1}(\Omega)$ problem (P_w) has a weak solution. Indeed, according to Corollary 4.1 (but using $(\hat{g})$ instead of (g) and $(G)^{\pm}$), the functional f is coercive on X_2 and anticoercive on X_1, while according to Lemma 5.2, f satisfies the Palais–Smale condition in $(H_0^1(\Omega), \|\cdot\|)$ (the "usual" Palais–Smale condition); whence it is clear that for every $w \in H^{-1}(\Omega)$, the functional $f - w$ also satisfies these properties. Thus, the saddle point theorem of Rabinowitz provides a solution to problem (P_w): see, e.g., Costa and Oliveira [8, Theorem 1] (where it is assumed that $(SR)_{i,i+1}$ holds uniformly with respect to x).

6. A result of mixed type

In this section as in the previous one, the Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies $(\hat{g})$. Let $0 < \mu \leqslant 2$. We consider the conditions:

There exist $a_\mu \in L^1(\Omega)$ and $b_\mu \in L^{(2^*/\mu)'}(\Omega)$ such that for almost every $x \in \Omega$ and all $s \in \mathbb{R}$,

$$(G.g)^{\pm}_{\mu} \qquad \pm\big(g(x,s)s - 2G(x,s)\big) \geqslant -a_\mu(x) - b_\mu(x)|s|^\mu.$$

For $x \in \Omega$, we define

$$\underline{H}_{\mu,\pm}(x) := \liminf_{s \to -\infty} \frac{\pm(g(x,s)s - 2G(x,s))}{|s|^\mu},$$
$$\overline{H}_{\mu,\pm}(x) := \liminf_{s \to +\infty} \frac{\pm(g(x,s)s - 2G(x,s))}{|s|^\mu},$$

and we consider the conditions

$$(NQ)^{\pm}_{\mu} \qquad H_{\mu,\pm}(x) := \min\big\{\overline{H}_{\mu,\pm}(x), \underline{H}_{\mu,\pm}(x)\big\} > 0 \quad \text{for a.e. } x \in \Omega.$$

Lemma 6.1. *Let $0 < \mu \leqslant 2$, let $(\hat{u}_h)$ be a sequence in $H_0^1(\Omega)$, weakly convergent to a function u, and let $0 < \rho_h \to +\infty$. If $(G.g)^{+}_{\mu}$ holds, then*

$$\liminf_{h \to \infty} \frac{1}{\rho_h^\mu} \int_\Omega \big(g(x,\rho_h\hat{u}_h)\rho_h\hat{u}_h - 2G(x,\rho_h\hat{u}_h)\big) \geqslant \int_\Omega \big(\overline{H}_{\mu,+}\big(u^+\big)^\mu + \underline{H}_{\mu,+}\big(u^-\big)^\mu\big).$$

Proof. Argue as in the proof of (4.4) (just note that $b_\mu|\hat{u}_h|^\mu \to b_\mu|u|^\mu$ strongly in $L^1(\Omega)$). □

For $j \in \mathbb{N}$, we further consider the conditions:

$$(G_j)^{\pm}_{\mu} \quad \begin{cases} \bullet \text{ There exist } \tilde{a}_\mu \in L^1(\Omega) \text{ and } \tilde{b}_\mu \in L^{(2^*/\mu)'}(\Omega) \text{ such that for almost every } x \in \Omega \text{ and} \\ \quad \text{for all } s \in \mathbb{R}, \\ \qquad \pm 2G(x,s) \leqslant \pm\lambda_j s^2 + \tilde{b}_\mu(x)|s|^\mu + \tilde{a}_\mu(x); \\ \bullet \quad \limsup_{|s| \to \infty} \dfrac{\pm(2G(x,s) - \lambda_j s^2)}{|s|^\mu} \leqslant 0 \quad \text{for a.e. } x \in \Omega. \end{cases}$$

$$(NQ_j)^{\pm}_{\mu} \qquad \int_\Omega \big(\overline{H}_{\mu,\pm}\big(u^+\big)^\mu + \underline{H}_{\mu,\pm}\big(u^-\big)^\mu\big) > 0 \quad \text{for every } u \in E_j \setminus \{0\}.$$

Remark 6.1. (a) If $\mu < 2$, arguing as in Lemma 4.1 and in Remark 4.1 shows again that conditions $(G.g)^{\pm}_{\mu}$ imply $K_+ = L_+$ and $K_- = L_-$. Moreover, the first properties in $(G_j)^{\pm}_{\mu}$ respectively imply

$K \leqslant \lambda_j$, $L \geqslant \lambda_j$. On the other hand, for every $0 < \mu \leqslant 2$, the second properties in $(G_j)^{\pm}_{\mu}$ do (respectively) imply $K \leqslant \lambda_j$, $L \geqslant \lambda_j$ — while the converse is true only when $\mu = 2$. A condition which is stronger than $(G_j)^{\pm}_{\mu}$ is the following:

For every $\varepsilon > 0$, there exists $a_{\mu,\varepsilon} \in L^1(\Omega)$ such that for almost every $x \in \Omega$ and for all $s \in \mathbb{R}$,

$$\pm 2G(x,s) \leqslant \pm\lambda_j s^2 + \varepsilon|s|^{\mu} + a_{\mu,\varepsilon}(x).$$

(b) Conditions $(NQ)^{\pm}_{\mu}$ are (respectively) stronger than conditions $(NQ_j)^{\pm}_{\mu}$; indeed, intermediate properties are:

$$H_{\mu,\pm} \geqslant 0 \quad \text{and} \quad \left|\left\{x \in \Omega\colon\ H_{\mu,\pm}(x) > 0\right\}\right| > 0.$$

Lemma 6.2. *Let $0 < \mu \leqslant 2$, let $j \in \mathbb{N}$, let $(\hat{u}_h) \subset H^1_0(\Omega)$ be a weakly convergent sequence, and let $0 < \rho_h \to +\infty$. If $(G_j)^{+}_{\mu}$ holds, then*

$$\liminf_{h\to\infty} \frac{1}{\rho_h^{\mu}} \int_{\Omega} \left(\lambda_j \rho_h^2 \hat{u}_h^2 - 2G(x, \rho_h \hat{u}_h)\right) \geqslant 0, \tag{6.1}$$

so that

$$\liminf_{h\to\infty} \left(\frac{1}{h^{\mu}} \inf_{D_{2,h}} f\right) \geqslant 0, \tag{6.2}$$

where $D_{2,h}$ denotes the closed ball in X_j^{+}, centered at the origin and of radius h.

Proof. Once again, arguing as in the proof of (4.4) yields (6.1); using it with $\hat{u}_h := \frac{u_h}{h}$ and $\rho_h := h$, where $u_h \in D_{2,h}$ is such that $f(u_h) \leqslant \inf_{D_{2,h}} f + \frac{1}{h}$, and taking (4.2) into account, we obtain

$$2\liminf_{h\to\infty}\left(\frac{1}{h^{\mu}} \inf_{D_{2,h}} f\right) = \liminf_{h\to\infty} \frac{2f(u_h)}{h^{\mu}} \geqslant \liminf_{h\to\infty} \int_{\Omega} \frac{\lambda_j u_h^2 - 2G(x,u_h)}{h^{\mu}} \geqslant 0,$$

establishing (6.2) (where indeed, the lower limit is a limit). □

Before stating the main result of this section, we introduce a last condition, for $j \in \mathbb{N}$:

$(E_j)^-$ $\qquad L \geqslant \lambda_j$, and $(L_+ - \lambda_j)u^+ + (L_- - \lambda_j)u^- \neq 0$ for every $u \in E_j \setminus \{0\}$.

Theorem 6.1. *Let $0 < \mu \leqslant 2$ and $i \in \mathbb{N}$. Assume that conditions $(\hat{g})$, $(G.g)^{+}_{\mu}$, $(G_{i+1})^{+}_{\mu}$, and* either $(G_i)^-$ or $(E_i)^-$, *hold, as well as one of the following*:

(i) $(NQ)^{+}_{\mu}$;
(ii) $(SR)_{i,i+1}$, $(NQ_i)^{+}_{\mu}$, *and* $(NQ_{i+1})^{+}_{\mu}$.

Then, problem (P) has a weak solution.

Proof. Consider again the decomposition (4.6). As in the proof of Theorem 4.1, using either $(G_i)^-$ or $(E_i)^-$ we have that f is bounded above on X_1. On the other hand, from $(G_{i+1})^+_\mu$ we have (6.2) (with $j := i+1$). Thus, applying Theorem 2.4, we obtain a sequence $(u_h) \subset H^1_0(\Omega)$ such that

$$\limsup_{h\to\infty} f(u_h) \leqslant \sup_{X_1} f \quad \text{and} \quad \frac{f'(u_h)}{(1+\|u_h\|)^{\mu-1}} \to 0. \tag{6.3}$$

We show that (u_h) is bounded. Arguing by contradiction, assume that, up to a subsequence, $0 < \rho_h := \|u_h\| \to \infty$ and $\hat{u}_h := \frac{u_h}{\rho_h}$ converges weakly in $H^1_0(\Omega)$ to some function u. Since $\mu \leqslant 2$, we see from (6.3) that

$$\frac{f'(\rho_h \hat{u}_h)}{\rho_h} \to 0,$$

so that, according to Lemma 5.2(a), $u \neq 0$, and $u \in E_i \cup E_{i+1}$ if $\lambda_i \leqslant l \leqslant k \leqslant \lambda_{i+1}$. On the other hand, from (6.3) and Lemma 6.1 we have

$$\begin{aligned} 0 \geqslant \liminf_{h\to\infty} \frac{2f(u_h) - f'(u_h)(u_h)}{\|u_h\|^\mu} &= \liminf_{h\to\infty} \frac{1}{\rho_h^\mu} \int_\Omega \big(g(x,u_h)u_h - 2G(x,u_h)\big) \\ &\geqslant \int_\Omega \big(\overline{H}_{\mu,+}\big(u^+\big)^\mu + \underline{H}_{\mu,+}\big(u^-\big)^\mu\big), \end{aligned} \tag{6.4}$$

which contradicts $(NQ)^+_\mu$ in case (i), $(NQ_i)^+_\mu$ and $(NQ_{i+1})^+_\mu$ in case (ii). □

Remark 6.2. As in the previous sections, we obtain results dual to Lemmas 6.1 and 6.2, and to Theorem 6.1, by appropriately inverting the "minus" and the "plus" conditions. Theorem 6.2 below, dealing with weak resonance at the first eigenvalue, features such dual conditions. Several variants of our results can also be stated: for example, we can replace condition $(NQ_i)^+_\mu$ in (ii) of Theorem 6.1 by

$$(l_+ - \lambda_i)u^+ + (l_- - \lambda_i)u^- \neq 0 \quad \text{for every } u \in E_i \setminus \{0\}$$

(recall Lemma 5.2(a)), in which case $(E_i)^-$ is also automatically satisfied.

For $0 < \mu \leqslant 2$, we consider the condition

$$(G_1)_\mu \quad \begin{cases} \bullet \text{ There exist } \tilde{a}_\mu \in L^1(\Omega) \text{ and } \tilde{b}_\mu \in L^{(2^*/\mu)'}(\Omega) \text{ such that for almost every } x \in \Omega \text{ and} \\ \quad \text{for all } s \in \mathbb{R}, \\ \qquad \big|2G(x,s) - \lambda_1 s^2\big| \leqslant \tilde{b}_\mu(x)|s|^\mu + \tilde{a}_\mu(x); \\ \bullet \quad \lim_{|s|\to\infty} \dfrac{2G(x,s) - \lambda_1 s^2}{|s|^\mu} = 0 \quad \text{for a.e. } x \in \Omega. \end{cases}$$

Of course, $(G_1)_\mu$ is equivalent to $(G_1)^+_\mu$ *and* $(G_1)^-_\mu$.

Lemma 6.3. *Let $0 < \mu \leqslant 2$, and assume that $(G_1)^+_\mu$ holds. Then,*

$$\liminf_{\|u\|\to\infty} \frac{f(u)}{\|u\|^\mu} \geqslant 0.$$

Proof. Taking (4.2) into account, for $u \in H_0^1(\Omega)$, $u \neq 0$, we have

$$\frac{2f(u)}{\|u\|^\mu} \geqslant -\frac{1}{\|u\|^\mu} \int_\Omega \left(2G(x,u) - \lambda_1 u^2\right),$$

while, arguing as in Lemma 6.2, $(G_1)_\mu^+$ implies

$$\limsup_{\|u\|\to\infty} \frac{1}{\|u\|^\mu} \int_\Omega \left(2G(x,u) - \lambda_1 u^2\right) \leqslant 0. \qquad \square$$

Theorem 6.2. *Let $0 < \mu \leqslant 2$, and assume that conditions $(\hat{g})$, $(G.g)_\mu^-$, $(G_1)_\mu$, and $(NQ)_\mu^-$ hold. We have*:

(a) *Problem (P) has a weak solution*;
(b) *If $1 < \mu \leqslant 2$, then for every $w \in H^{-1}(\Omega)$, problem (P_w) (recall (5.5)) has a weak solution.*

Proof. Considering the decomposition $H_0^1(\Omega) = X_1^- \oplus X_2^+$, we obtain from $(G_1)_\mu$ that

$$\liminf_{h\to\infty}\left(\frac{1}{h^\mu} \inf_{D_{2,h}} f\right) \geqslant 0 \quad \text{and} \quad \limsup_{h\to\infty}\left(\frac{1}{h^\mu} \sup_{D_{1,h}} f\right) \leqslant 0.$$

Indeed, the first inequality follows from $(G_2)_\mu^+$ (see Lemma 6.2), which is of course weaker than $(G_1)_\mu^+$, and the second one similarly follows from $(G_1)_\mu^-$. Note that if $1 < \mu \leqslant 2$, these inequalities also hold replacing f by $f - w$, for arbitrary $w \in H^{-1}(\Omega)$. In that case, applying Theorem 2.4 to the functional $f - w$, we obtain a sequence $(u_h) \subset H_0^1(\Omega)$ such that

$$\frac{f'(u_h) - w}{(1 + \|u_h\|)^{\mu-1}} \to 0. \tag{6.5}$$

We need to show that (u_h) is bounded. Assume, for a contradiction, and up to a subsequence, that $0 < \rho_h := \|u_h\| \to \infty$, and that $\hat{u}_h := \frac{u_h}{\rho_h}$ converges weakly in $H_0^1(\Omega)$ to some function u. As in the proof of Theorem 6.1, we have $u \neq 0$ and we deduce from (6.5), Lemma 6.3, and $(G.g)_\mu^-$ that

$$\begin{aligned} 0 &\geqslant \liminf_{h\to\infty} \frac{f'(u_h)(u_h) - 2f(u_h)}{\|u_h\|^\mu} \\ &= \liminf_{h\to\infty} \frac{1}{\rho_h^\mu} \int_\Omega \left(2G(x,u_h) - g(x,u_h)u_h\right) \\ &\geqslant \int_\Omega \left(\overline{H}_{\mu,-}\left(u^+\right)^\mu + \underline{H}_{\mu,-}\left(u^-\right)^\mu\right), \end{aligned} \tag{6.6}$$

which contradicts $(NQ)_\mu^-$. In the case $0 < \mu \leqslant 1$, we have (6.5) for $w = 0$, and the rest of the proof is the same. $\square$

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