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Methods for Assimilating Blood Velocity Measures in Hemodynamics Simulations: Preliminary Results

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Abstract

New measurement devices and techniques in biomedical images provide medical doctors with a huge amount of data on blood flow and vascular morphologies. These data are crucial for performing (and validating) individual-based simulations of hemodynamics (see e.g. [1]). Availability of velocity measures *inside* a region of interest poses problems that are new to the community of computational hemodynamics and however well known in other engineering fields. In particular, integration of data (measures) and numerical simulations has been an issue of utmost relevance in the prediction of fluid geophysics phenomena and, in particular, weather forecast. In computational hemodynamics a mathematically sound assimilation of data and numerical simulations is needed, on one hand for improving reliability of numerical results, on the other one for filtering noise and measurements errors. In this paper we consider and compare some possible methods for integrating numerical simulations and velocity measures in some internal points of the computational domain. Preliminary numerical results for a 2D Stokes problem are presented both for noise free and noisy data, investigating convergence rate and noise sensitivity.

Keywords: Computational fluid dynamics, Optimization techniques, Inverse problems, Data assimilation techniques, Hemodynamics

1. Introduction and Motivation

The development of numerical methods in incompressible fluid dynamics has received a strong impulse in the last 15 years from cardiovascular applications (see e.g. [1, 2, 3]). Specific numerical techniques have been proposed for the effective solution of fluid-structure interaction problems, the coupling of models featuring a different level of detail (lumped parameter, 1D, 3D), and the integration of medical images and numerical simulations. In particular, the last topic is crucial for the development of numerical solvers with a clinical impact. It is well known that vascular geometry plays a major role in the development of vascular pathologies. The combination of data coming from medical images and numerical simulations is a fundamental step for performing individual-based simulations. This integration process has been enhanced in the last years by the development of both new imaging devices and numerical methods for processing medical images (see e.g. [1, 4, 5]). Measurements devices have been improved also for flow

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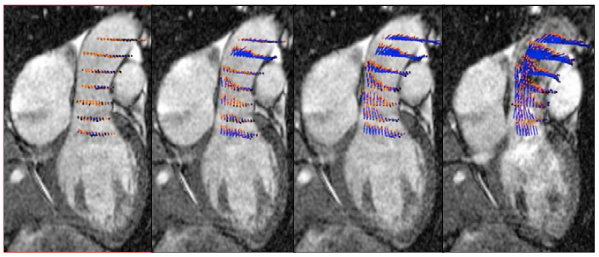

Figure 1: Blood velocity measured in the ascending aorta of a patient with MRI [6].

and pressure data; some of them make available measurements of the velocity field in the region of interest (see Fig. 1).

Availability of these data opens the problem of a mathematically sound integration of measures and simulations aiming at an overall improvement of reliability of numerical results. This problem is not new in other engineering fields. In particular, we mention geophysics and meteorology, where data come from different sources like Radar, Lidar or satellite images and are currently used for enhancing the quality of weather forecast. The procedure of including measured data into numerical simulations is called *Data Assimilation* (DA). Data are usually sparse, irregularly distributed in space and time and noisy for different reasons (e.g. instrumental noise, sampling and interpretation of instrumental measurements).

In this paper we present some possible methods for DA in computational hemodynamics and compare their performances on a simple test case. In particular, we move from velocity measurements in the computational domain (see Fig. 1) and assimilate these data in the simulation of the incompressible fluid dynamics in the region of interest. After a brief overview of DA techniques during the last 50 years and of recent developments of this procedure in biomedical applications, in Section 2 we report the mathematical formulation of the considered approaches. In Section 3 we discuss the numerical results and we identify the most effective methods; in Section 4 we draw conclusions and we present future research guidelines.

1.1. State of the art

DA was firstly introduced in fluid geophysics, where the aim was to use very sparse measurements of a physical variable in order to predict the weather evolution. The basis of DA is an automated procedure based on interpolation of the data, called *simple analysis*. This preliminary technique provided the ground for the development of new methods where the physics of the data analyzed was taken into account. The first proposed methods were based on the *Estimation Theory*: the analysis step is performed combining information from physical models and statistical correlations between observed data. The central method in this category is the *Kalman filter* introduced by Kalman in 1960; many variants of this technique have been lately introduced [7, 8]. In the meantime, the Dynamic Relaxation method was considered for time independent DA, based on the fact that the solution of a steady problem can be seen as the steady state of a time dependent problem. The basics of this method is the solution of the Partial Differential Equations (PDEs) governing the system where some artificial source terms are added in order to force the solution to match the observed data as time evolves. Later on, the numerical solution of PDEs was made simpler by the introduction of efficient discretization schemes; also, the theory of control and optimization became well known and used in many practical applications [9]. These facts allowed several methods for control problems to be used in the solution of DA problems. Nowadays, we can consider three groups of DA techniques:

- Kalman filter based approaches [8];
- control theory methods [9];
- stochastic methods [7].

The application of DA techniques to hemodynamics is a new field of investigation; so far, some DA techniques have been used in medicine in order to estimate model parameters (see [10]). More recently, DA has been taken into account also for blood flow: a control based approach has been chosen for recovering the velocity field [11] in an

unsteady simulation. The goal is to minimize a weighted difference of the predicted state variable and a measured field at a specific time, subject to the state equations using the initial condition as a control variable.

2. Proposed Approaches

In this Section we present the mathematical formulation of the problem and we describe several methods for its numerical solution. For the sake of simplicity, at this preliminary stage, we focus on the steady Stokes equations; this analysis will provide the ground for the solution of the Navier-Stokes (NS) equations for real applications, that will be addressed in future works.

2.1. Mathematical formulation

We introduce some notations: let $\Omega \subset \mathbb{R}^2$ represent the domain of interest, i.e. the vessel; Γ_w the vessel wall; Γ_{in} and Γ_{out} the inflow and outflow sections. Let $L^2(\Omega)$ be the usual Hilbert space of square summable functions and $\mathbf{H}^1(\Omega)$ be the space of vector functions whose components are in $L^2(\Omega)$ together with their first derivatives, $\mathbf{H}^1_\Gamma$ the corresponding space of $\mathbf{H}^1$ functions with null trace on the portion Γ of $\partial\Omega$. We denote by $\mathbf{u}(\mathbf{x}) \in \mathbf{H}^1_{\Gamma_w}(\Omega)$ and $p(\mathbf{x}) \in L^2(\Omega)$ the velocity and pressure fields respectively. The general statement of the problem is as follows. Let us denote by $\mathbf{d} \in \mathbb{R}^D$ ($D \in \mathbb{N}$) the vector of the measured data s.t. $\mathbf{d}_i$, $i = 1, ..., D$, is the measured velocity at point $\mathbf{x}_i$ (we assume measurement points to lie on some internal layers, which we denote Γ_d - see Fig. 2). We assume that $\Gamma_{in} \subset \Gamma_d$. Velocity and pressure fields are assumed to fulfill in Ω the equations

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{0}, \quad \nabla \cdot \mathbf{u} = 0. \tag{1}$$

On the boundary Γ_w we assume $\mathbf{u} = \mathbf{0}$, while on Γ_{out} we prescribe homogeneous Neumann boundary conditions. We aim at solving the following problem: *find* $\mathbf{u}$ *and* p *so that* $\mathbf{u}$ *fits the available measures* $\mathbf{d}$ *(in some sense) under the constraints (1).*

Different methods can be devised for solving this problem, with different ways for fitting the data. Here, we address in particular two techniques. The former is specifically devised for this problem and relies upon the assumption that available measures are taken over internal sections of the domain of interest, i.e. points $\mathbf{x}_i$ belong to a set of transversal sections of the vascular district at hand (see Fig. 2). Although this is a quite restrictive assumption, it reflects the way the measurements are actually taken (see Fig. 1). The latter method, on the contrary, is based on a more general approach, relying upon the theory of control for distributed systems.

2.2. Splitting techniques

The idea is to split the domain into subdomains (see Figs. 1 and 2), where the measurement sections act as domain interfaces. Measurements are used as boundary data for solving the global problem by subdomains. More precisely, let us assume to have s subdomains Ω_i ($i = 0, \ldots, s-1$), such that the measured data lie on the interfaces. Let us denote by Γ_i the interface, with $i = 0, 1, \ldots s-1$ so that $\Gamma_0 = \Gamma_{in}$ and $\Gamma_s = \Gamma_{out}$. Let $\mathbf{d}_i$ be the set of measures on interface Γ_i, or more precisely an interpolating function over Γ_i of the data (we will use piecewise linear interpolation). Denote by $\mathbf{u} = \sum_{i=0}^{s-1} \mathbf{u}_{0,i} + \mathcal{E}_i\mathbf{d}$ where $\mathbf{u} \in \mathbf{H}^1_{\Gamma_w}(\Omega)$, $\mathbf{u}_{0,i} \in \mathbf{H}^1_{\partial\Omega_i}(\Omega_i)$, and $\mathcal{E}_i\mathbf{d} \in \mathbf{H}^1(\Omega_i)$ is an arbitrary extension in Ω_i of the data $\mathbf{d}_i$ and $\mathbf{d}_{i+1}$. We want to solve the following problem. Find $\mathbf{u}$ such that for any $\mathbf{v}_i \in \mathbf{H}^1_{\partial\Omega_i}(\Omega_i)$, $q_i \in L^2(\Omega_i)$

$$\sum_{i=0}^{s-1} \left(\int_{\Omega_i} \nu\nabla\mathbf{u} \cdot \nabla\mathbf{v}_i \, d\mathbf{x} - \int_{\Omega_i} p\nabla \cdot \mathbf{v}_i \, d\mathbf{x} - \int_{\Omega_i} q_i \nabla \cdot \mathbf{u} \, d\mathbf{x} \right) = 0. \tag{2}$$

In principle, we can distinguish two approaches, according to the sequences "first split then discretize" or "first discretize then split". We focus on the latter approach, since the former has some strong drawbacks that prevent its use, as we point out later on.

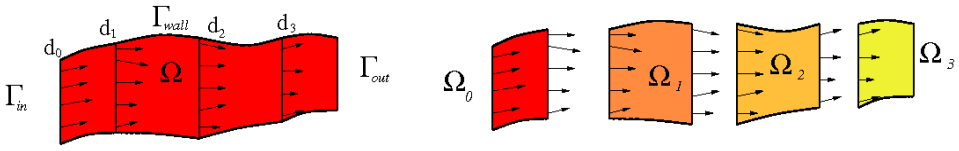

Figure 2: Schematic representation of the domain with velocity measures (left) and its subdomain splitting (right).

Matrix Updating. In this case, we first discretize problem (1) to the form $A\mathbf{U} = \mathbf{b}$ where $\mathbf{U}$ is the vector of the nodal unknowns. Then, the measurements are regarded as *conditions* in the form

$$c\mathbf{U}(\mathbf{x}_i) = c\mathbf{d}_i \tag{3}$$

where i ranges over the nodes where measures are available (which are assumed to be a subset of the discretization nodes) and c is a suitable coefficient. Conditions (3) are forced into the linear system similarly to Dirichlet conditions; this means that the i-th equations of the linear system corresponding to measurements points are replaced by (3). In practice, all the off-diagonal entries of the i-th rows are set to 0 and the diagonal one is set to c. The corresponding entries of the right hand side, $\mathbf{b}_i$ are set equal to $\mathbf{d}_i$. All the other entries $\mathbf{b}_j$ are updated accordingly, by setting $\mathbf{b}_j = \mathbf{b}_j - \sum_i a_{ji}\mathbf{d}_i$ (i still ranging on the measurements nodes). This perturbation of the matrix is expected to affect the condition number of the system. The coefficient c is tuned to minimize this effect. In the following table we report the condition number with the choice of $c = 1$ on two different meshes.

mesh	no update	$c = 1$
40×10	2.58e3	1.25e4
80×20	1.05e4	8.30e4

The condition number after the update is increased of about one order of magnitude (setting $c = \Delta x$ or $(\Delta x)^2$ yields same results). This has the clear drawback of requiring a higher computational effort in solving the linear systems. Even more important is the fact that the solution of the system becomes more sensitive to the perturbations on the data and in particular to the noise in the measures. Despite of its simplicity, this approach is therefore anticipated to be extremely sensitive to the noise, as we see in the Numerical Results section.

Remark *Domain Splitting* We can split the problem by subdomains and then perform the numerical discretization of subproblems separately. This corresponds to take in (2) alternatively all the test functions $\mathbf{v}_i = 0, q_i = 0$ apart from one. Then, we perform the discretization of each problem. Observe that in the first $s-1$ subdomains we have Dirichlet conditions over $\partial\Omega_i$. For this reason, we force the uniqueness of the pressure by taking a null average pressure or fixing the pressure in one node. Moreover, boundary data are supposed to be compatible with the incompressibility constraint $\int_{\partial\Omega_i} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}\gamma = 0$. In principle, this method is fast and prone to parallelism, since each subproblem is solved independently of the others. However, presence of the noise has two important drawbacks:

- the divergence compatibility condition is actually violated;
- measurements errors affect the solution inside each domain, without filtering.

Concerning the second point, it is known that perturbations on the boundary affect the solution of a Stokes problem with a decay featuring an exponential dependence on the distance from the boundary (see [12]). Presence of noise on the interfaces brings therefore errors inside the domain, yielding extremely inaccurate results. For this reason we do not dwell any longer with this approach, since it results more inaccurate than the others (errors are at least about ten times larger than the others).

2.3. Control based formulations

In this Section we present a different, more general, approach based on the theory of control for distributed systems. Let us introduce the problem: for all $\mathbf{v} \in \mathbf{H}^1_{\Gamma_w}(\Omega), q \in L^2(\Omega)$

$$\mathcal{S}: \int_\Omega \nu \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, \mathrm{d}\mathbf{x} - \int_\Omega p \nabla \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} - \int_\Omega q \nabla \cdot \mathbf{u} \, \mathrm{d}\mathbf{x} + \int_{\Gamma_{in}} \mathbf{h}\, \mathbf{v} \, d\gamma = 0. \tag{4}$$

Here, $\mathbf{h}(\mathbf{x}) \in \mathbf{H}^{-1/2}(\Gamma_{in})$ is the Neumann data at the inflow boundary, while homogeneous Neumann boundary conditions are still assumed at the outflow. Let us denote by $\mathcal{F}$ the set of solutions to $\mathcal{S}$ as a function of the inflow Neumann data $\mathbf{h}$. Correspondingly, the discretized Stokes problem is denoted by $\mathcal{S}_h$ and the solution set by $\mathcal{F}_h$. The basic formulation of the problem reads: find the inflow Neumann conditions such that the distance (to be defined) between velocity and data is minimal. As usual, we have two possible approaches, the *Optimize then Discretize* (OD) approach and the *Discretize then Optimize* (DO) one.

Optimize then Discretize. We define the distance between solution and data as

$$J(\mathbf{u}) = \frac{1}{2} \int_{\Gamma_d} (\mathbf{u} - \mathbf{d})^2 \mathrm{d}\gamma, \tag{5}$$

where $\mathbf{d}$ is here an extension of the vector data (for the sake of simplicity we do not change notation). We solve the problem: find $\mathbf{h}$ such that for the corresponding solution $\mathbf{u}$ to $\mathcal{S}$, $J(\mathbf{u}) \leq J(\mathbf{v})$, $\forall \mathbf{v} \in \mathcal{F}$. The Lagrangian of the problem regarded as a constrained optimization procedure reads

$$\mathcal{L}(\mathbf{u}, p, \boldsymbol{\lambda}_\mathbf{u}, \lambda_p, \mathbf{h}) = J(\mathbf{u}) + \int_\Omega \nu \nabla \mathbf{u} \cdot \nabla \boldsymbol{\lambda}_\mathbf{u} \mathrm{d}\mathbf{x} - \int_\Omega p \, \nabla \cdot \boldsymbol{\lambda}_\mathbf{u} \mathrm{d}\mathbf{x} - \int_{\Gamma_{in}} \mathbf{h} \boldsymbol{\lambda}_\mathbf{u} \mathrm{d}\gamma + \int_\Omega \lambda_p \nabla \cdot \mathbf{u} \, \mathrm{d}\mathbf{x}; \tag{6}$$

where $\boldsymbol{\lambda}_\mathbf{u}$ and λ_p are the Lagrange multipliers associated with velocity and pressure. The necessary conditions for optimality are (see [9])

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial(\boldsymbol{\lambda}_\mathbf{u}, \lambda_p)} = \int_\Omega \nu \nabla \mathbf{u} \cdot \nabla \mathbf{v} \mathrm{d}\mathbf{x} - \int_\Omega p \, \nabla \cdot \mathbf{v} \mathrm{d}\mathbf{x} - \int_{\Gamma_{in}} \mathbf{h} \mathbf{v} \mathrm{d}\gamma + \int_\Omega q \, \nabla \cdot \mathbf{u} \, \mathrm{d}\mathbf{x} = 0 \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_w}, \, q \in L^2 \\ \frac{\partial \mathcal{L}}{\partial(\mathbf{u}, p)} = \int_{\Gamma_d} (\mathbf{u} - \mathbf{d}) \, \mathbf{w} \mathrm{d}\gamma + \int_\Omega \nu \nabla \cdot \boldsymbol{\lambda}_\mathbf{u} \nabla \mathbf{w} \mathrm{d}\mathbf{x} + \int_\Omega \lambda_p \nabla \cdot \mathbf{w} \mathrm{d}\mathbf{x} - \int_\Omega r \, \nabla \cdot \boldsymbol{\lambda}_u \mathrm{d}\mathbf{x} = 0 \quad \forall \mathbf{w} \in \mathbf{H}^1_{\Gamma_w}, \, r \in L^2 \\ \frac{\partial \mathcal{L}}{\partial \mathbf{h}} = -\int_{\Gamma_{in}} \boldsymbol{\lambda}_\mathbf{u} \mathbf{s} \, \mathrm{d}\gamma = 0 \quad \forall \mathbf{s} \in \mathbf{H}^{1/2}(\Gamma_{in}). \end{cases} \tag{7}$$

Several methods are available for the discretization of this system; in this work we rely on the Finite Element (FE) method. Let $\mathbf{U} \in \mathbb{R}^{N_u}$, $\mathbf{P} \in \mathbb{R}^{N_p}$ and $\mathbf{H} \in \mathbb{R}^{N_h}$ be the vectors representing the computed solutions on the grid points, being N_u, N_p and N_h the dimensions of the FE spaces they belong to, they solve the following linear system:

$$\begin{bmatrix} \mathrm{S} & \mathrm{O} & \mathrm{N} \\ \mathrm{J} & \mathrm{S} & \mathrm{O} \\ \mathrm{O} & \mathrm{N}^T & \mathrm{O} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{V} \\ \boldsymbol{\Lambda} \\ \mathbf{H} \end{bmatrix} = \mathbf{F}, \quad \mathrm{S} = \begin{bmatrix} \mathrm{C} & \mathrm{B}^T \\ \mathrm{B} & \mathrm{O} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix}, \boldsymbol{\Lambda} = \begin{bmatrix} \mathbf{U}_\lambda \\ \mathbf{P}_\lambda \end{bmatrix}. \tag{8}$$

Here, $\mathbf{U}_\lambda$ and $\mathbf{P}_\lambda$ represent the discrete lagrangian multipliers; C, B, J and N are the FE matrices coming from the discretization of the Laplacian, divergence, derivative of the functional J and boundary conditions; it is worth specifying that the latter is associated with the integral on the inflow boundary and involves the control variable. Also, the right hand side comes from the evaluation of the derivative of the functional and involves the data.

Remark A precise characterization of the set $\mathcal{F}$ is strictly related to the efficiency of using $\mathbf{h}$ as control variable. In fact, the larger is $\mathcal{F}$ in the subset of solutions of the Stokes problem, the smaller can be the distance from the data. This problem at the best of our knowledge is still open and will be subject of future works (see [13, 14]).

Discretize then Optimize (DO). In this approach, we swap the discretization and optimization processes. We therefore resort to the problem $\mathcal{S}_h$ that in the algebraic form reads $\mathrm{S}\mathbf{V} = \mathrm{N}\mathbf{H}$. The distance between the (discrete) velocity and the data is defined as

$$J_h(\mathbf{U}) + \frac{a}{2}\,\|\mathrm{L}\mathbf{H}\|_2^2 = \frac{1}{2}\|\mathrm{Q}\mathbf{U} - \mathbf{d}\|_2^2 + \frac{a}{2}\,\|\mathrm{L}\mathbf{H}\|_2^2; \tag{9}$$

being Q the selection matrix extracting the velocity $\mathbf{U}$ at the points where measures are available. Since data are sparse and noisy and the problem can be ill-posed, a regularization term is added to the objective function, a is the regularization parameter (chosen according to some suitable techniques) and L is a discretized differential operator applied to the control variable $\mathbf{H}$ and will be specified later. In this discrete setting, we solve: find $\mathbf{H}$ such that for corresponding solution $\mathbf{U} \in \mathcal{F}_h$, $J_h(\mathbf{U}) \leq J_h(\mathbf{V})$, $\forall \mathbf{V} \in \mathcal{F}_h$. For the solution of this minimization problem, we write the Lagrangian, which reads

$$\mathcal{L}_h(\mathbf{V}, \mathbf{H}, \mathbf{\Lambda}) = \frac{1}{2}\|\mathrm{D}\mathbf{V} - \mathbf{d}\|_2^2 + \frac{a}{2}\,\|\mathrm{L}\mathbf{H}\|_2^2 + \mathbf{\Lambda}^T(\mathrm{S}\mathbf{V} - \mathrm{N}\mathbf{H}); \tag{10}$$

where D = [Q O] and all other terms are defined as in (8). The set of necessary conditions for the optimality is given by

$$\begin{cases} \frac{\partial \mathcal{L}_h}{\partial \mathbf{\Lambda}} = \mathrm{S}\mathbf{V} - \mathrm{N}\mathbf{H} = 0; \\ \frac{\partial \mathcal{L}_h}{\partial \mathbf{V}} = \mathrm{D}^T(\mathrm{D}\mathbf{V} - \mathbf{d}) + \mathrm{S}^T\mathbf{\Lambda} = 0; \\ \frac{\partial \mathcal{L}_h}{\partial \mathbf{H}} = a\mathrm{L}^T\mathrm{L}\mathbf{H} - \mathrm{N}^T\mathbf{\Lambda} = 0. \end{cases} \tag{11}$$

Upon block elimination, we get the reduced system

$$(\mathrm{N}^T\mathrm{S}^{-T}\mathrm{D}^T\mathrm{D}\mathrm{S}^{-1}\mathrm{N} + a\mathrm{L}^T\mathrm{L})\mathbf{H} = \mathrm{N}^T\mathrm{S}^{-T}\mathrm{D}^T\mathbf{d}. \tag{12}$$

By defining Z=$\mathrm{D}\mathrm{S}^{-1}\mathrm{N}$ system (12) reduces to $(\mathrm{Z}^T\mathrm{Z} + a\mathrm{L}^T\mathrm{L})\mathbf{H} = \mathrm{Z}^T\mathbf{d}$; Z is called *sensitivity* matrix, which is defined as Z=$\partial \mathrm{D}\mathbf{V}(\mathbf{H})/\partial\mathbf{H}$ and represents the derivative of the predicted data with respect to the control variable. The spectral properties of this matrix allow to determine the conditioning of the problem. Notice that, as long as the size of Z remains *small*, we can compute it explicitly; as its dimension grows we can solve the system efficiently employing methods that require only matrix-vector multiplications for Z and Z^T [15]. Finally, we point out that different techniques can be used for an optimal choice of the regularization parameter a. Among the others,we mention the Discrepancy Principle (DP) and the Generalized Cross Validation (GCV) [16].

3. Numerical results

In this Section we report numerical results obtained performing the presented approaches on a 2D test case. All the preliminary simulations presented in this work are based on the 2D Poiseuille flow in the rectangular domain $\Omega = [0,\, 5] \times [-0.5,\, 0.5]$, whose analytical velocity reads: $\mathbf{u}_{ex}(x, y) = \ [1 - 4y^2\ ,\ 0\ ]^T$.

Data generation We assume data to be given on three internal layers and on the inflow boundary in correspondence of discretization nodes (in this way the space discretization step Δx and N, the number of measurement nodes, are related). These data are generated adding to the exact solution a random noise uniformly distributed in space (globally) and normally distributed point-wise (locally) in such a way that the Signal-to-Noise Ratio, SNR, is fixed; this fact determines the parameters of the error probability distribution in the generation process. In real applications this value is strongly determined by the biomedical tools used to observe the data. In the case of noisy data from a 4D scan of the aorta SNR is of the order of 10 for flow measures.

Software details The preliminary comparison among the proposed approaches is performed combining two softwares: FreeFem++[2], for matrices generation, and Matlab, for the linear systems solution. Results presented in the last

[2]Free C++ like software available on www.FreeFem.org for the solution of 2D PDEs.

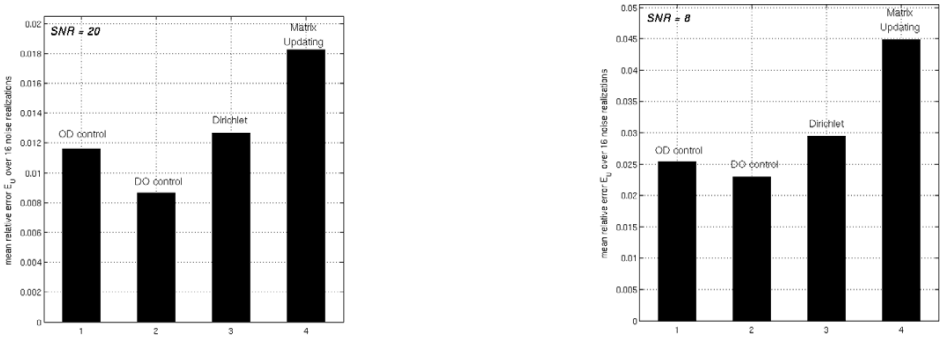

Figure 3: Mean relative error for OD, DO, MU and D with SNR = 20 (left) and 8 (right).

paragraph on the DO approach are obtained with the C++ FE library lifeV [3] and displayed with the software ParaView.

Computational issues We implement the FE method with choice of compatible FE spaces P1bubble-P1 for the velocity and pressure fields respectively. In the case of the DO method we choose the gradient as a smoothing regularizing operator and we generate the optimal parameter by means of DP and GCV methods (obtaining a parameter a of the order of 10^{-9}, since the conditioning of the problem is not critical).

Comparison results We compare the results with a direct approach, which we call "Dirichlet" (D). This consists in the solution of the forward Stokes problem with the prescription of the (noisy) velocity data only at the inflow and it is a consistency benchmark. We expect data assimilation techniques to give more accurate results than this solution (which is much cheaper). Fig. 3 reports the mean relative error

$$E_U = \sum_{k=1}^{k=16} \frac{||\mathbf{U}_k - \mathbf{u}_{ex}||_2}{||\mathbf{u}_e x||_2}, \tag{13}$$

being $\mathbf{U}_k$ the solution computed in correspondence of the k-th noise generation, out of 16 noise realizations with SNR = 20 (left) and SNR = 8 (right). Even if much cheaper in terms of computational saving, the MU method is not accurate enough since the sensitivity to the perturbation, incremented by the updating, makes it unreliable. For this reason, this is not considered for any further investigation. The OD and DO approaches yield better accuracy than the D one; the main issue related to them is the computational effort.

Error analysis for the OD and DO approaches According to numerical results presented, we focus on the control based techniques and we investigate in more details the dependence of the discretization error (with respect to the exact solution) on the discretization parameter Δx and on the amount of noise. In correspondence of noise free data and decreasing values of Δx we test the consistency of the method; in Fig. 4 we report for the OD and the DO approach the l_2 relative error versus Δx (in a logarithmic scale) and a quadratic reference curve; both methods preserve the $O(\Delta x^2)$ behavior predicted by the FE theory: embedding our problem into the optimization scheme does not affect the FE convergence rate. Also, numerical simulations have been performed in correspondence of different values of SNR. We obtain a linear behavior of the discretization error as the amount of noise increases.

DO simulations with noise free and noisy data Among the methods presented, the DO technique results to be the most accurate and cheaper than the OD one in terms of computational effort. We present more specific results

[3] Free C++ library, which the authors are developers of, available on www.lifev.org.

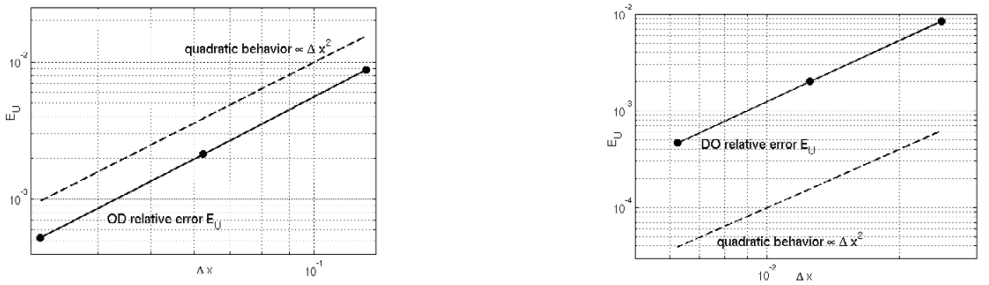

Figure 4: Relative error *vs* $\Delta x = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}$ in a loglog scale for OD (left) and DO (right).

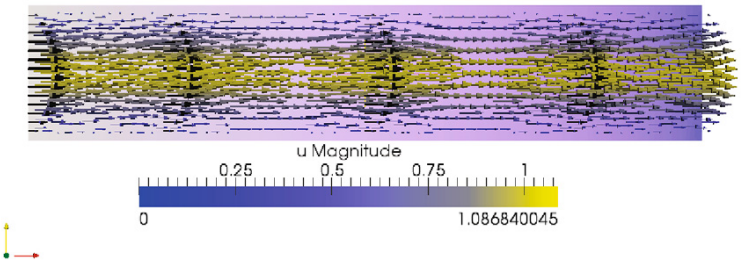

Figure 5: Pressure and velocity assimilated with noise free data (in black).

employing noise free and noisy data. In this analysis we are mainly interested in the velocity error behavior; nevertheless, in the noise free case, we also report the pressure error behavior. In Fig. 5, in correspondence of $\Delta x = \frac{1}{20}$, the recovered pressure and velocity vector field is displayed together with the noise free data: in correspondence of the internal layers the velocity matches the data (exact solution). Fig. 6 reports the computed velocity field and a noisy data generated with SNR = 10. We note the presence of the noise especially in the vertical component (zero in the exact solution). Notice in Fig. 6 that even if the data features a non-zero vertical component as a by-product of the noise, DA yields an almost horizontal velocity field. This points out the role of DA as a filtering process of noisy data.

4. Conclusion and future work

The present work represents an exploratory analysis of DA possible methods for incompressible fluid dynamics, with particular emphasis on cardiovascular applications. In particular, the comparison among different methods showed that control based approaches are robust and accurate, even if computationally expensive. Despite their straightaway implementation and low complexity, splitting methods are inaccurate because of their sensitivity to noise. The focus of the ongoing work is the implementation of an efficient solver for the nonlinear NS equations as a combination of the DO method introduced in the present paper and and the common Newton method for the treatment of the nonlinearity. In addition, we plan to analyze the dependence of the accuracy of the solution on the noise. In fact, preliminary results show that the problem at hand is highly sensitive to the noise nature. Another issue to be considered is the effectiveness of the control variable **h** over the entire minimization process and, more in general, its well posedness as a function of the data and of the control variable **h**. Finally, the ultimate goal consists in applying these techniques to real 3D geometries in order to used the recovered velocity field for the prediction of physical variables of medical relevance.

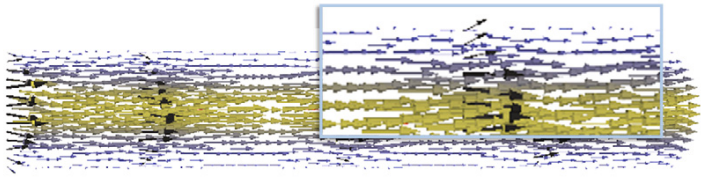

Figure 6: Velocity assimilated with noisy data (in black) with SNR = 10.

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