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Central invariants and Frobenius–Schur indicators for semisimple quasi-Hopf algebras

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Abstract

In this paper, we obtain a canonical central element ν_H for each semi-simple quasi-Hopf algebra H over any field k and prove that ν_H is invariant under gauge transformations. We show that if k is algebraically closed of characteristic zero then for any irreducible representation of H which affords the character χ, $\chi(\nu_H)$ takes only the values 0, 1 or -1, moreover if H is a Hopf algebra or a twisted quantum double of a finite group then $\chi(\nu_H)$ is the corresponding Frobenius–Schur indicator. We also prove an analog of a theorem of Larson–Radford for split semi-simple quasi-Hopf algebras over any field k. Using this result, we establish the relationship between the antipode S, the values of $\chi(\nu_H)$, and certain associated bilinear forms when the underlying field k is algebraically closed of characteristic zero.

Keywords: Frobenius–Schur indicators; Gauge equivalence; Tensor categories; Quasi-Hopf algebras

1. Introduction

In the paper [20], Linchenko and Montgomery introduced and studied Frobenius–Schur indicators for irreducible representations of a semi-simple Hopf algebra H

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over an algebraically closed field of characteristic $p \neq 2$. If Λ is the *unique* normalized left integral of H, i.e. $\epsilon(\Lambda) = 1$, set

$$v = v_H = \sum_{(\Lambda)} \Lambda_1 \Lambda_2. \tag{1.1}$$

Here we have used Sweedler notation $\Delta(\Lambda) = \sum_{(\Lambda)} \Lambda_1 \otimes \Lambda_2$, so that if m is multiplication in H then $v = m \circ \Delta(\Lambda)$. Then v is a central element of H and the Frobenius–Schur indicator v_χ of an irreducible H-module M with character χ is defined via

$$v_\chi = \chi(v). \tag{1.2}$$

In case H is a group algebra $k[G]$, $v = |G|^{-1} \sum_{g \in G} g^2$ and $v_\chi = |G|^{-1} \sum_{g \in G} \chi(g^2)$ reduces to the original definition of Frobenius and Schur (cf. [6] or [27], for example). Generalizing the famous result of Frobenius and Schur for group algebras, Linchenko and Montgomery show that for general semi-simple H, v_χ can take only the values 0, 1, or -1. Moreover $v_\chi \neq 0$ if, and only if, $M \cong M^*$, and in this case M admits a non-degenerate H-invariant bilinear form $\langle \cdot, \cdot \rangle$ satisfying

$$\langle u, v \rangle = v_\chi \langle v, u \rangle \tag{1.3}$$

for $u, v \in M$. Recall that $\langle \cdot, \cdot \rangle$ is H-invariant if

$$\sum_{(h)} \langle h_1 u, h_2 v \rangle = \varepsilon(h) \langle u, v \rangle \tag{1.4}$$

for $h \in H$ and $u, v \in M$.

In a recent paper [16] the authors showed how one may effectively compute Frobenius–Schur indicators for a certain class of Hopf algebras. Their work applies, in particular, to the case of the *quantum double* $D(G)$ of a finite group G, and it was shown [16] how the indicators for irreducible modules over $D(G)$ may be given in terms of purely group-theoretic invariants associated to G and its subgroups. The algebra $D(G)$ is of interest in orbifold conformal field theory [23], indeed in this context there is a more general object, the *twisted quantum double* $D^\omega(G)$, that arises naturally [7]. (Here, $\omega \in Z^3(G, \mathbb{C}^\times)$ is a normalized 3-cocycle about which we shall have more to say below.) The present work originated with a natural problem: understand Frobenius–Schur indicators for twisted quantum doubles.

$D^\omega(G)$ is a semi-simple *quasi-Hopf* algebra (over $\mathbb{C}$, say), but is generally not a Hopf algebra. One of the difficulties this imposes is that the antipode S is not necessarily involutive (something that is always true for semi-simple Hopf algebras by a theorem of Larson and Radford [18]), whereas having $S^2 = id$ is fundamental for the Linchenko–Montgomery approach and therefore for the calculations in [16]. If it happens that $S^2 = id$ then Theorem 4.4 of [16] can be used to obtain indicators given by

$$v_\chi = |G|^{-1} \sum_{x^{-1}gx=g^{-1}} \gamma_x(g,g^{-1})\theta_g(x,x)\chi(e(g)\otimes x^2). \tag{1.5}$$

(Undefined notation is explained below; γ_x and θ_g are certain 2-cochains determined by ω.)

If G is *abelian* then $D^\omega(G)$ is a Hopf algebra [21], though perhaps with a non-trivial β element, and for any G it turns out that one can always *gauge* ω, i.e. replace it by a cohomologous 3-cocycle ω', in such a way that the antipode for $D^{\omega'}(G)$ is an involution. So (1.5) provides a preliminary solution to our problem, but it is unsatisfactory for the following reason: if we gauge ω, the new 3-cocycle ω' will give new values for the Frobenius–Schur indicators which in general are not the same as the original values. While this may not be an issue if one is interested in a fixed $D^\omega(G)$, there are both mathematical and physical reasons for insisting that the FS indicators for $D^\omega(G)$ be *robust*, that is they depend only on the cohomology class of ω. From this standpoint, (1.5) is generally not what we are looking for. We need a more *functorial* approach.

One knows that if ω and ω' are cohomologous then $D^\omega(G)$ and $D^{\omega'}(G)$ are gauge-equivalent and that therefore the corresponding module categories are tensor equivalent (cf. [7,8,17]). Indeed, it follows from a result of Etingof and Gelaki [9] that the converse is also true, so that gauge-equivalence of the twisted doubles is the *same* as tensor equivalence of the module categories. So we are looking for invariants of such module categories with respect to tensor equivalence. Because Hopf algebras and twisted doubles are not closed with respect to gauge equivalence, this means that we have to work with the module categories of *arbitrary* semi-simple quasi-Hopf algebras.

Peter Bantay has introduced a notion of indicator into rational conformal field theory from a rather different point-of-view [3,4]. His point of departure is the Verlinde formula and the S and T matrices associated to a RCFT. To this modular data together with an irreducible character he associates a certain numerical expression and shows that it is equal once again to either $0, 1$ or -1. It is possible to evaluate Bantay's indicator in case the matrices S and T are associated to a twisted double $D^\omega(G)$ [5] and one obtains the expression

$$|G|^{-1} \sum_{x^{-1}gx=g^{-1}} \omega(g^{-1},g,g^{-1})\gamma_x(g,g^{-1})\theta_g(x,x)\chi(e(g)\otimes x^2). \tag{1.6}$$

Compared to (1.5), (1.6) contains an extra term $\omega(g^{-1},g,g^{-1})$. Furthermore, it is easy to see that (1.6) is robust in the previous sense.

Suppose that H is any semi-simple quasi-Hopf algebra, and let M be an irreducible H-module with character χ. In the present paper we will construct a *canonical* central element v_H of H with the following properties:

(a) v_H is invariant under *any* gauge transformation of H.

(b) If H is a Hopf algebra then ν_H coincides with (1.1).
(c) If $H = D^{\omega}(G)$ then $\chi(\nu_H)$ coincides with Bantay's indicator (1.6).
(d) Assume that k is algebraically closed and char $k = 0$.
  (i) $\chi(\nu_H) = 0, 1$, or -1.
  (ii) $\chi(\nu_H) \neq 0$ if, and only if, ${}^*M \cong M$. In this case, M admits a certain non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ such that

$$\langle x, y \rangle = \langle y, g^{-1}x \rangle \tag{1.7}$$

  for all $x,\ y \in M$. Here, g is a distinguished element of H, which we call the *trace element*, which is independent of M.
  (iii) $\mathrm{Tr}(S) = \sum_{\chi \in \mathrm{Irr}(H)} \chi(\nu_H)\chi(g^{-1})$.

Part (d) is the analog for general semi-simple quasi-Hopf algebras of the corresponding result in [20] for Hopf algebras. The bilinear form $\langle \cdot, \cdot \rangle$ has a certain adjointness property with respect to the antipode S of H, and there are relations to an analog of a theorem of Larson–Radford (S is involutive for semi-simple Hopf algebras). Namely, we show that for a semi-simple quasi-Hopf algebra the antipode is involutive up to conjugation. The trace element g plays an important role in our discussion of (d), in particular its properties lead to the fact that the category H-$\mathbf{mod}_{\mathrm{fin}}$ of finite-dimensional H-modules is a pivotal category in the sense of Joyal and Street. For twisted doubles, g coincides with β, while for Hopf algebras the Larson–Radford theorem implies that $g = 1$.

The proof that $\chi(\nu_H)$ takes only the values 0, 1 or -1 is somewhat elaborate. Indeed, in an earlier version of the present paper [22] this had been left open. Subsequently, Pavel Etingof alerted us to the existence of his recent preprint with Nikshych and Ostrik [10] on fusion categories, and suggested that some of the results obtained there could be used to help settle the issue of the values of our indicator. More precisely, Etingof pointed out that our trace element g defines an isomorphism of tensor functors $Id \to {}^{**}?$. This together with $S(g) = g^{-1}$ are the main ingredients in the proof.

The paper is organized as follows: we cover some basic facts about quasi-Hopf algebras in Section 2, including several strategically important elements in $H \otimes H$ introduced by Hausser and Nill [15]. In Sections 3 and 4 we define the central element ν_H and establish that the family of Frobenius–Schur indicators $\chi(\nu_H)$ is a gauge invariant for semi-simple quasi-Hopf algebras. In Section 5 we show that our indicators coincide with those of Bantay in the case of a twisted double. In Section 6 we introduce the trace element g and establish the analog of the Larson–Radford theorem, while Section 7 is devoted to further properties of g as discussed above. Section 8 covers the relation of indicators to bilinear forms and completes the proof of (d)(i), and in Section 9 we return to the case of twisted doubles to complete the analysis in that case. For simplicity, we will only work over algebraically closed fields of characteristic zero in Sections 7–9.

2. Quasi-Hopf algebras

In this section we recall the definition of quasi-Hopf algebras and their properties described in [8,17]. Moreover, we recall some interesting results recently obtained in [13–15,25]. In the sequel, we will use the notation introduced in this section. Throughout this paper, we will always assume that k is a field and any algebras and vector spaces are over k. In Sections 7–9, we will further assume k to be an algebraically closed field of characteristic zero.

A *quasi-bialgebra* over k is a 4-tuple $(H, \Delta, \epsilon, \Phi)$, in which H is an algebra over k, $\Delta : H \to H \otimes H$ and $\epsilon : H \to k$ are algebra maps, and Φ is an invertible element in $H \otimes H \otimes H$ satisfying the following conditions:

$$(\epsilon \otimes id)\Delta(h) = h = (id \otimes \epsilon)\Delta(h), \tag{2.1}$$

$$\Phi(\Delta \otimes id)\Delta(h)\Phi^{-1} = (id \otimes \Delta)\Delta(h) \quad \text{for all } h \in H, \tag{2.2}$$

$$(id \otimes id \otimes \Delta)(\Phi)(\Delta \otimes id \otimes id)(\Phi) = (1 \otimes \Phi)(id \otimes \Delta \otimes id)(\Phi)(\Phi \otimes 1), \tag{2.3}$$

$$(id \otimes \epsilon \otimes id)(\Phi) = 1 \otimes 1. \tag{2.4}$$

The maps Δ, ϵ and Φ are, respectively, called the diagonal map, counit, and associator of the quasi-bialgebra. If there is no ambiguity, we will simply write H for the quasi-bialgebra $(H, \Delta, \epsilon, \Phi)$. Using (2.3), one can also easily see that

$$(\epsilon \otimes id \otimes id)(\Phi) = 1 \otimes 1 = (id \otimes id \otimes \epsilon)(\Phi). \tag{2.5}$$

Moreover, the module category H-**mod** of the quasi-bialgebra H is a tensor category (cf. [8,17] for the details).

Following [17], a *gauge transformation* on a quasi-bialgebra $H = (H, \Delta, \epsilon, \Phi)$ is an invertible element F of $H \otimes H$ such that

$$(\epsilon \otimes id)(F) = 1 = (id \otimes \epsilon)(F).$$

Using a gauge transformation on H, one can define an algebra map $\Delta_F : H \to H \otimes H$ by

$$\Delta_F(h) = F\Delta(h)F^{-1} \tag{2.6}$$

for any $h \in H$, and an invertible element Φ_F of $H \otimes H \otimes H$ by

$$\Phi_F = (1 \otimes F)(id \otimes \Delta)(F)\Phi(\Delta \otimes id)(F^{-1})(F^{-1} \otimes 1). \tag{2.7}$$

Then $H_F = (H, \Delta_F, \epsilon, \Phi_F)$ is also a quasi-bialgebra.

Two quasi-bialgebras A and B are said to be *gauge equivalent* if there exists a gauge transformation F on B such that A and B_F are isomorphic as quasi-bialgebras.

If A and B are gauge equivalent quasi-bialgebras, A-**mod** and B-**mod** are equivalent tensor categories (cf. [17]). Conversely, if A, B are finite-dimensional semi-simple quasi-bialgebras such that A-**mod** and B-**mod** are equivalent tensor categories, then A and B are gauge equivalent quasi-bialgebras (cf. [9]).

A quasi-bialgebra $(H, \Delta, \epsilon, \Phi)$ is called a *quasi-Hopf algebra* if there exist an anti-algebra automorphism S of H and elements $\alpha, \beta \in H$ such that for all element $h \in H$, we have

$$\sum_{(h)} S(h_1)\alpha h_2 = \epsilon(h)\alpha, \quad \sum_{(h)} h_1 \beta S(h_2) = \epsilon(h)\beta \tag{2.8}$$

and

$$\sum_i X_i \beta S(Y_i)\alpha Z_i = 1, \quad \sum_i S(\bar{X}_i)\alpha \bar{Y}_i \beta S(\bar{Z}_i) = 1, \tag{2.9}$$

where $\Phi = \sum_i X_i \otimes Y_i \otimes Z_i$, $\Phi^{-1} = \sum_i \bar{X}_i \otimes \bar{Y}_i \otimes \bar{Z}_i$ and $\sum_{(h)} h_1 \otimes h_2 = \Delta(h)$. We shall write $(H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$ for the complete data of the quasi-Hopf algebra; S is called the *antipode* of H. When the context is clear, we will simply write H for the quasi-Hopf algebra $(H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$. One can easily see that a Hopf algebra is a quasi-Hopf algebra with $\Phi = 1 \otimes 1 \otimes 1$ and $\alpha = \beta = 1$.

Unlike a Hopf algebra, the antipode for a quasi-Hopf algebra is generally not unique.

Proposition 2.1 (Drinfel'd [8, Proposition 1.1]). *Let $H = (H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$ be a quasi-Hopf algebra. If u is a unit of H then $H_u = (H, \Delta, \epsilon, \Phi, u\alpha, \beta u^{-1}, S_u)$ is also a quasi-Hopf algebra, where $S_u(h) = uS(h)u^{-1}$ for all $h \in H$. Conversely, for any $\alpha', \beta' \in H$ and for any algebra anti-automorphism S' of H such that $H' = (H, \Delta, \epsilon, \Phi, \alpha', \beta', S')$ is a quasi-Hopf algebra, then there exists a unique invertible element u of H such that*

$$H_u = H'.$$

If F is a gauge transformation on the quasi-Hopf algebra $H = (H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$, we can define α_F and β_F by

$$\alpha_F = \sum_i S(d_i)\alpha e_i \quad \text{and} \quad \beta_F = \sum_i f_i \beta S(g_i),$$

where $F = \sum_i f_i \otimes g_i$ and $F^{-1} = \sum_i d_i \otimes e_i$. Then, $H_F = (H, \Delta_F, \epsilon, \Phi_F, \alpha_F, \beta_F, S)$ is also a quasi-Hopf algebra.

The antipode of a Hopf algebra is known to be an anti-coalgebra map. For a quasi-Hopf algebra H, this is true up to conjugation. Following [8], we define $\gamma, \delta \in H \otimes H$ by the formulae

$$\gamma = \sum_i S(U_i)\alpha V_i \otimes S(T_i)\alpha W_i, \tag{2.10}$$

$$\delta = \sum_j K_j\beta S(N_j)\otimes L_j\beta S(M_j), \tag{2.11}$$

where

$$\sum_i T_i\otimes U_i\otimes V_i\otimes W_i = (1\otimes\Phi^{-1})(id\otimes id\otimes\Delta)(\Phi),$$

$$\sum_j K_j\otimes L_j\otimes M_j\otimes N_j = (\Delta\otimes id\otimes id)(\Phi)(\Phi^{-1}\otimes 1).$$

Then,

$$F_H = \sum_i (S\otimes S)(\Delta^{op}(\bar{X}_i))\cdot\gamma\cdot\Delta(\bar{Y}_i\beta S(\bar{Z}_i)) \tag{2.12}$$

is an invertible element of $H\otimes H$, where $\Phi^{-1} = \sum_i \bar{X}_i\otimes\bar{Y}_i\otimes\bar{Z}_i$. Moreover,

$$F_H\Delta(S(h))F_H^{-1} = (S\otimes S)\Delta^{op}(h)$$

for all $h\in H$.

The category of finite-dimensional left H-modules of a quasi-Hopf algebra H with antipode S, denoted by H-**mod**$_{\text{fin}}$ is a rigid tensor category. Let M be a finite-dimensional left H-module and M' its k-linear dual. Then the H-action on M', given by

$$(h\cdot f)(m) = f(S(h)m)$$

for any $f\in M'$ and $m\in M$, defines a left H-module structure on M'. We shall denote by ${}^*M$ the left dual of M in H-**mod**$_{\text{fin}}$. Similarly, the right dual of M, denoted by M^*, is the H-module with the underlying k-linear space M' with the H-action given by

$$(h\cdot f)(m) = f(S^{-1}(h)m)$$

for any $f\in M'$ and $m\in M$ (cf. [8]).

In [13–15], Frank Hausser and Florian Nill introduced some interesting elements in $H\otimes H$ for any arbitrary quasi-Hopf algebra $H = (H,\Delta,\epsilon,\Phi,\alpha,\beta,S)$ in the course of studying the corresponding theories of quantum double, integral and the fundamental theorem for quasi-Hopf algebras. These elements of $H\otimes H$ are given by

$$q_R = \sum X_i\otimes S^{-1}(\alpha Z_i)Y_i,\quad p_R = \sum \bar{X}_i\otimes\bar{Y}_i\beta S(\bar{Z}_i), \tag{2.13}$$

$$q_L = \sum S(\bar{X}_i)\alpha\bar{Y}_i\otimes\bar{Z}_i,\quad p_L = \sum Y_i S^{-1}(X_i\beta)\otimes Z_i, \tag{2.14}$$

where $\Phi = \sum_i X_i \otimes Y_i \otimes Z_i$ and $\Phi^{-1} = \sum_i \bar{X}_i \otimes \bar{Y}_i \otimes \bar{Z}_i$. One can show easily (cf. [15]) that they obey the relations (for all $a \in H$)

$$(a \otimes 1)q_R = \sum (1 \otimes S^{-1}(a_2))q_R \Delta(a_1), \tag{2.15}$$

$$(1 \otimes a)q_L = \sum (S(a_1) \otimes 1)q_L \Delta(a_2), \tag{2.16}$$

$$p_R(a \otimes 1) = \sum \Delta(a_1)p_R(1 \otimes S(a_2)), \tag{2.17}$$

$$p_L(1 \otimes a) = \sum \Delta(a_2)p_L(S^{-1}(a_1) \otimes 1), \tag{2.18}$$

where $\Delta(a) = \sum a_1 \otimes a_2$. Suppressing the summation symbol and indices, we write $q_R = q_R^1 \otimes q_R^2$, etc. These elements also satisfy the identities (cf. [15]):

$$\Delta(q_R^1)p_R(1 \otimes S(q_R^2)) = 1 \otimes 1, \tag{2.19}$$

$$(1 \otimes S^{-1}(p_R^2))q_R \Delta(p_R^1) = 1 \otimes 1, \tag{2.20}$$

$$\Delta(q_L^2)p_L(S^{-1}(q_L^1) \otimes 1) = 1 \otimes 1, \tag{2.21}$$

$$(S(p_L^1) \otimes 1)q_L \Delta(p_L^2) = 1 \otimes 1. \tag{2.22}$$

We will use these equations in the sequel.

3. Central gauge invariants for semi-simple quasi-Hopf algebras

Suppose that $H = (H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$ is a finite-dimensional quasi-Hopf algebra. A left integral of H is an element l of H such that $hl = \epsilon(h)l$ for all $h \in H$. A right integral of H can be defined similarly. It follows from [15] that the subspace of left (right) integrals of H is of dimension 1. Moreover, if H is semi-simple, the subspace of left integrals is identical to the space of right integrals of H and $\epsilon(\Lambda) \neq 0$ for any non-zero left integral Λ of H (see also [25]). We will call the two-sided integral Λ of H *normalized* if $\epsilon(\Lambda) = 1$.

Let Λ be a left integral of H. Then for any $a \in H$,

$$\epsilon(a)\Delta(\Lambda) = \Delta(a)\Delta(\Lambda). \tag{3.1}$$

Similarly, if Λ' is a right integral of H, then we have

$$\epsilon(a)\Delta(\Lambda') = \Delta(\Lambda')\Delta(a). \tag{3.2}$$

We then have the following lemma.

Lemma 3.1. *Let $H = (H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$ be a finite-dimensional quasi-Hopf algebra.*

(i) *If Λ is a left integral of H, then for any $a \in H$,*

$$(1 \otimes a)q_R\Delta(\Lambda) = (S(a) \otimes 1)q_R\Delta(\Lambda), \tag{3.3}$$

$$(1 \otimes a)q_L\Delta(\Lambda) = (S(a) \otimes 1)q_L\Delta(\Lambda), \tag{3.4}$$

$$\text{and} \quad (\beta \otimes 1)q_L\Delta(\Lambda) = (\beta \otimes 1)q_R\Delta(\Lambda) = \Delta(\Lambda). \tag{3.5}$$

(ii) *If Λ' is a right integral of H, then for any $a \in H$,*

$$\Delta(\Lambda')p_R(a \otimes 1) = \Delta(\Lambda')p_R(1 \otimes S(a)), \tag{3.6}$$

$$\Delta(\Lambda')p_L(a \otimes 1) = \Delta(\Lambda')p_L(1 \otimes S(a)), \tag{3.7}$$

$$\text{and} \quad \Delta(\Lambda')p_L(1 \otimes \alpha) = \Delta(\Lambda')p_R(1 \otimes \alpha) = \Delta(\Lambda'). \tag{3.8}$$

Proof. (i) By Eqs. (3.1) and (2.15), for any $a \in H$,

$$\begin{aligned}(a \otimes 1)q_R\Delta(\Lambda) &= (1 \otimes S^{-1}(a_2))q_R\Delta(a_1)\Delta(\Lambda)\\ &= (1 \otimes S^{-1}(a_2\epsilon(a_1)))q_R\Delta(\Lambda)\\ &= (1 \otimes S^{-1}(a))q_R\Delta(\Lambda).\end{aligned}$$

Hence, by substituting a with $S(a)$, we prove Eq. (3.3). Here the summation symbols are suppressed. Now we have

$$\begin{aligned}\Delta(\Lambda) &= (1 \otimes S^{-1}(p_R^2))q_R\Delta(p_R^1)\Delta(\Lambda) \quad \text{(by (2.20))}\\ &= (1 \otimes S^{-1}(p_R^2\epsilon(p_R^1))q_R\Delta(\Lambda) \quad \text{(by (3.1))}\\ &= (1 \otimes S^{-1}(\beta))q_R\Delta(\Lambda) \quad \text{(by (2.5))}\\ &= (\beta \otimes 1)q_R\Delta(\Lambda) \quad \text{(by (3.3))}.\end{aligned}$$

The remaining formulae in (i) and (ii) can be proved similarly using Eqs. (2.5), (2.15)–(2.22), (3.1) and (3.2). □

Lemma 3.2. *Let $H = (H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$ be a finite-dimensional quasi-Hopf algebra and F a gauge transformation on H. Suppose that $q_R^F, q_L^F, p_R^F, p_L^F$ are the corresponding p's and q's for H_F defined in* (2.13) *and* (2.14).

(i) *If Λ is a left integral of H, then*

$$q_R^F \Delta_F(\Lambda) = q_R \Delta(\Lambda) F^{-1} \quad \textit{and} \quad q_L^F \Delta_F(\Lambda) = q_L \Delta(\Lambda) F^{-1}.$$

(ii) *If Λ' is a right integral of H, then*

$$\Delta_F(\Lambda') p_R^F = F\Delta(\Lambda') p_R \quad \textit{and} \quad \Delta_F(\Lambda') p_L^F = F\Delta(\Lambda') p_L.$$

Proof. (i) Let $\Phi^{-1} = \sum_j \bar{X}_j \otimes \bar{Y}_j \otimes \bar{Z}_j$, $F = \sum_i f_i \otimes g_i$ and $F^{-1} = \sum_l d_l \otimes e_l$. Then, we obtain

$$\begin{aligned}\Phi_F^{-1} &= (F \otimes 1)(\Delta \otimes id)(F)\Phi^{-1}(id \otimes \Delta)(F^{-1})(1 \otimes F^{-1}) \\ &= (F \otimes 1)\left(\sum_{i,j,l} f_{i,1}\bar{X}_j d_l \otimes f_{i,2}\bar{Y}_j e_{l,1} \otimes g_i \bar{Z}_j e_{l,2}\right)(1 \otimes F^{-1}),\end{aligned}$$

where $\Delta(f_i) = \sum f_{i,1} \otimes f_{i,2}$ and $\Delta(e_l) = \sum e_{l,1} \otimes e_{l,2}$. Thus, we have

$$\begin{aligned} & q_L^F \Delta_F(\Lambda) \\ &= \left(\sum S(f_{i'} f_{i,1} \bar{X}_j d_l)\alpha_F g_{i'} f_{i,2} \bar{Y}_j e_{l,1} \otimes g_i \bar{Z}_j e_{l,2}\right) F^{-1} F \Delta(\Lambda) F^{-1} \\ &= \left(\sum S(f_{i,1}\bar{X}_j d_l)\alpha f_{i,2}\bar{Y}_j e_{l,1} \otimes g_i \bar{Z}_j e_{l,2}\right)\Delta(\Lambda)F^{-1} \quad \left(\text{since } \sum S(f_{i'})\alpha_F g_{i'} = \alpha\right) \\ &= \left(\sum S(f_{i,1}\bar{X}_j d_l \epsilon(e_l))\alpha f_{i,2}\bar{Y}_j \otimes g_i \bar{Z}_j\right)\Delta(\Lambda)F^{-1} \quad (\text{by } (3.1)) \\ &= \left(\sum S(f_{i,1}\bar{X}_j)\alpha f_{i,2}\bar{Y}_j \otimes g_i \bar{Z}_j\right)\Delta(\Lambda)F^{-1} \quad \left(\text{since } \sum d_l \epsilon(e_l) = 1_H\right) \\ &= \left(\sum S(\bar{X}_j)\alpha \epsilon(f_i)\bar{Y}_j \otimes g_i \bar{Z}_j\right)\Delta(\Lambda)F^{-1} \quad \left(\text{since } \sum S(f_{i,1})\alpha f_{i,2} = \epsilon(f_i)\alpha\right) \\ &= \left(\sum S(\bar{X}_j)\alpha \bar{Y}_j \otimes \bar{Z}_j\right)\Delta(\Lambda)F^{-1} \quad \left(\text{since } \sum \epsilon(f_i) g_i = 1_H\right) \\ &= q_L \Delta(\Lambda) F^{-1}.\end{aligned}$$

The other three equations can be proved similarly. □

Theorem 3.3. *Let $H = (H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$ be a finite-dimensional quasi-Hopf algebra. Suppose that Λ is a two-sided integral of H. Then, the elements*

$$q_R\Delta(\Lambda)p_R, \quad q_R\Delta(\Lambda)p_L, \quad q_L\Delta(\Lambda)p_R, \quad \textit{and} \quad q_L\Delta(\Lambda)p_L$$

in $H \otimes H$ are invariant under gauge transformations. Moreover,

$$m(q_R\Delta(\Lambda)p_R) = m(q_R\Delta(\Lambda)p_L) = m(q_L\Delta(\Lambda)p_R) = m(q_L\Delta(\Lambda)p_L),$$

where m denotes the multiplication of H. In addition, $m(q_R\Delta(\Lambda)p_R)$ is a central element of H.

Proof. It follows from Lemma 3.2 that for any gauge transformation F on H,

$$q_*^F\Delta_F(\Lambda) = q_*\Delta(\Lambda)F^{-1}, \quad \text{and} \quad \Delta_F(\Lambda)p_*^F = F\Delta(\Lambda)p_*,$$

where $q_*^F = q_L^F$ or q_R^F and $p_*^F = p_L^F$ or p_R^F. Thus we have

$$\begin{aligned} q_*^F\Delta_F(\Lambda)p_*^F &= q_*\Delta(\Lambda)F^{-1}p_*^F \\ &= q_*F^{-1}\Delta_F(\Lambda)p_*^F \\ &= q_*F^{-1}F\Delta(\Lambda)p_* \\ &= q_*\Delta(\Lambda)p_*. \end{aligned}$$

Let m denote the multiplication of H and let $m_{RR}, \ m_{RL}, \ m_{LR}$ and m_{LL} denote the elements

$$m(q_R\Delta(\Lambda)p_R), m(q_R\Delta(\Lambda)p_L), m(q_L\Delta(\Lambda)p_R), \quad \text{and} \quad m(q_L\Delta(\Lambda)p_L),$$

respectively. Then for any $a \in H$,

$$\begin{aligned} S(a)m_{RR} &= m((S(a) \otimes 1)q_R\Delta(\Lambda)p_R) \\ &= m((1 \otimes a)q_R\Delta(\Lambda)p_R) \quad \text{by Lemma 3.1(i)} \\ &= m(q_R\Delta(\Lambda)p_R(a \otimes 1)) \\ &= m(q_R\Delta(\Lambda)p_R(1 \otimes S(a))) \quad \text{by Lemma 3.1(ii)} \\ &= m_{RR}S(a). \end{aligned}$$

As S is an automorphism, the above equation implies that m_{RR} is in the center of H. Using the same kind of arguments, one can show that m_{RL}, m_{LR} and m_{LL} are each in the center of H.

Let Q_R, Q_L, P_R and P_L denote the elements

$$m(q_R\Delta(\Lambda)), m(q_L\Delta(\Lambda)), m(\Delta(\Lambda)p_R), \text{ and } m(\Delta(\Lambda)p_L),$$

respectively. Then, we have

$$\begin{aligned} m_{RR} &= \sum q_R^1 \Lambda_1 p_R^1 q_R^2 \Lambda_2 p_R^2 \\ &= \sum S(p_R^1) q_R^1 \Lambda_1 q_R^2 \Lambda_2 q_R^2 \quad \text{by Lemma 3.1(i)} \\ &= S(p_R^1) Q_R p_R^2 \\ &= \sum_j S(\bar{X}_j) Q_R \beta \bar{Y}_j S(\bar{Z}_j) \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} m_{RR} &= \sum q_R^1 \Lambda_1 p_R^1 q_R^2 \Lambda_2 p_R^2 \\ &= \sum q_R^1 \Lambda_1 p_R^1 \Lambda_2 p_R^2 S(q_R^2) \quad \text{by Lemma 3.1(ii)} \\ &= q_R^1 P_R S(q_R^2) \\ &= \sum X_i P_R S(Y_i) \alpha Z_i, \end{aligned} \tag{3.10}$$

where $\Phi^{-1} = \sum_j \bar{X}_j \otimes \bar{Y}_j \otimes \bar{Z}_j$ and $\Phi = \sum_i X_i \otimes Y_i \otimes Z_i$. Similarly,

$$m_{LL} = S(\bar{X}_j)\alpha \bar{Y}_j P_L S(\bar{Z}_j) = \sum X_i \beta S(Y_i) Q_L Z_i. \tag{3.11}$$

By (3.5) and (3.8), we have

$$Q_R = m_{RL}\alpha, \quad Q_L = m_{LR}\alpha,$$

$$P_L = \beta m_{RL}, \quad P_R = \beta m_{LR}. \tag{3.12}$$

Therefore, using Eq. (2.9), we have

$$m_{RR} = S(\bar{X}_j) m_{RL} \alpha \bar{Y}_j \beta S(\bar{Z}_j) = m_{RL} S(\bar{X}_j) \alpha \bar{Y}_j \beta S(\bar{Z}_j) = m_{RL}.$$

Similarly, using Eqs. (3.10)–(3.12) we can prove

$$m_{RR} = m_{LR} = m_{LL}. \qquad \square$$

In [15,25], it is shown that a finite-dimensional quasi-Hopf algebra H is semi-simple if, and only, if there exists a unique normalized two-sided integral. In this case, we have the following:

Definition 3.4. Let $H = (H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$ be a finite-dimensional semi-simple quasi-Hopf algebra and let Λ be the unique normalized two-sided integral of H. We denote by v_H the central element

$$m(q_L \Delta(\Lambda) p_L)$$

discussed in Theorem 3.3.

Corollary 3.5. *Let $H = (H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$ be a finite-dimensional semi-simple quasi-Hopf algebra and Λ the normalized two-sided integral of H. Then v_H is invariant under gauge transformations, that is*

$$v_H = v_{H_F}$$

for any gauge transformation F on H. Moreover,

$$\beta\alpha v_H = v_H \beta\alpha = \sum(\Lambda_1 \Lambda_2),$$

where $\sum \Lambda_1 \otimes \Lambda_2 = \Delta(\Lambda)$. In particular, if both α and β are units of H, then

$$v_H = \sum(\Lambda_1 \Lambda_2)(\beta\alpha)^{-1} = (\beta\alpha)^{-1} \sum(\Lambda_1 \Lambda_2).$$

Proof. The first statement follows immediately from Theorem 3.3. By Eqs. (3.5) and (3.8), we have $\beta v_H \alpha = \sum(\Lambda_1 \Lambda_2)$. Since v_H is central, then the result follows. $\square$

Corollary 3.6. *Let $H = (H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$, $H' = (H', \Delta', \epsilon', \Phi', \alpha', \beta', S')$ be semi-simple quasi-Hopf algebras. If H and H' are gauge equivalent quasi-bialgebras via the gauge transformation F on H and the quasi-bialgebra isomorphism $\sigma: H_F \to H'$, then*

$$\sigma(v_H) = v_{H'}.$$

In particular, if u is a unit of H, then $v_{H_u} = v_H$.

Proof. Since H_F and H' are isomorphic quasi-bialgebras, $(H', \Delta', \epsilon', \Phi', \sigma(\alpha_F), \sigma(\beta_F), \sigma S \sigma^{-1})$ is a quasi-Hopf algebra. By Proposition 2.1, there exists a unit u of H' such that

$$\sigma S \sigma^{-1}(a) = u S'(a) u^{-1}, \quad \sigma(\alpha_F) = u\alpha' \quad \text{and} \quad \sigma(\beta_F) = \beta' u^{-1} \tag{3.13}$$

for all $a \in H'$. Then, we have

$$\sigma S^{-1} \sigma^{-1}(a) = S'^{-1}(u) S'^{-1}(a) S'^{-1}(u^{-1}). \tag{3.14}$$

Let Λ be the normalized two-sided integral of H. Since σ is a quasi-bialgebra isomorphism, $\sigma(\Lambda)$ is then a two-sided integral of H' and

$$\epsilon'(\sigma(\Lambda)) = \epsilon(\Lambda) = 1.$$

Therefore, $\Lambda' = \sigma(\Lambda)$ is the unique normalized integral of H'. In particular, we have

$$(\sigma \otimes \sigma)\Delta_F(\Lambda) = \sum \Lambda'_1 \otimes \Lambda'_2 \quad \text{and} \quad (\sigma \otimes \sigma \otimes \sigma)(\Phi_F) = \Phi',$$

where $\sum \Lambda'_1 \otimes \Lambda'_2 = \Delta'(\Lambda')$. Let

$$\begin{aligned}
&\Phi_F = \sum X_i^F \otimes Y_i^F \otimes Z_i^F, \quad \Phi_F^{-1} = \sum \bar{X}_j^F \otimes \bar{Y}_j^F \otimes \bar{Z}_j^F,\\
&\Phi' = \sum X'_i \otimes Y'_i \otimes Z'_i, \quad \Phi'^{-1} = \sum \bar{X}'_j \otimes \bar{Y}'_j \otimes \bar{Z}'_j,\\
&\text{and} \quad \Delta_F(\Lambda) = \sum \Lambda_1^F \otimes \Lambda_2^F.
\end{aligned}$$

Then,

$$\begin{aligned}
\sigma(v_H) &= \sigma(v_{H_F}) \quad \text{(by Corollary 3.5)}\\
&= \sigma\Big(\sum X_i^F \Lambda_1^F \bar{X}_j^F S^{-1}(\alpha_F Z_i^F) Y_i^F \Lambda_2^F \bar{Y}_j^F \beta_F S(\bar{Z}_j^F)\Big)\\
&= \sum X'_i \Lambda'_1 \bar{X}'_j (\sigma S^{-1})(\alpha_F Z_i^F) Y'_i \Lambda'_2 \bar{Y}'_j \sigma(\beta_F)(\sigma S)(\bar{Z}_j^F)\\
&= \sum X'_i \Lambda'_1 \bar{X}'_j (\sigma S^{-1}\sigma^{-1})(Z'_i)(\sigma S^{-1}\sigma^{-1})(\sigma(\alpha_F)) Y'_i \Lambda'_2 \bar{Y}'_j \sigma(\beta_F)(\sigma S \sigma^{-1})(\bar{Z}'_j)\\
&= \sum X'_i \Lambda'_1 \bar{X}'_j S'^{-1}(u) S'^{-1}(Z'_i) S'^{-1}(\alpha') Y'_i \Lambda'_2 \bar{Y}'_j \beta' S'(\bar{Z}'_j) u^{-1} \quad \text{(by Eqs. (3.13) and (3.14))}\\
&= \sum X'_i \Lambda'_1 \bar{X}'_j S'^{-1}(\alpha' Z'_i) Y'_i \Lambda'_2 \bar{Y}'_j \beta' S'(\bar{Z}'_j) u u^{-1} \quad \text{(by Lemma 3.1(ii))}\\
&= v_{H'}.
\end{aligned}$$

For any unit u of H, H and H_u are obviously gauge equivalent as quasi-bialgebras under the gauge transformation $1 \otimes 1$ and the quasi-bialgebra isomorphism id_H. Hence, the second statement follows. □

4. Frobenius–Schur indicators

Let $(H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$ be a semi-simple quasi-Hopf algebra over the field k. Let M be an irreducible H-module with character χ. We call $\chi(v_H)$ the Frobenius–Schur indicator of χ (or M). The family of Frobenius–Schur indicators $\{\chi(v_H)\}$ is in fact an invariant of the tensor category H-**mod** for any semi-simple quasi-Hopf algebra H.

Theorem 4.1. *Let $H = (H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$ and $H' = (H', \Delta', \epsilon', \Phi', \alpha', \beta', S')$ be finite-dimensional semi-simple quasi-Hopf algebras over an algebraically closed field k of characteristic zero. If H-**mod** and H'-**mod** are equivalent as k-linear tensor categories, then the families of Frobenius–Schur indicators for H and H' are identical.*

Proof. If H-**mod** and H'-**mod** are equivalent as k-linear tensor categories, then, by Etingof and Gelaki [9, Theorem 6.1], H and H' are gauge equivalent quasi-bialgebras. Suppose that F is a gauge transformation on H and $\sigma: H_F \to H'$ is a quasi-bialgebra isomorphism. It follows from Corollary 3.5 that

$$\sigma(v_H) = v_{H'}.$$

Let $Irr(H)$, $Irr(H')$ be the sets of irreducible characters of H and H' respectively. Then, the map $\chi' \mapsto \chi' \circ \sigma$ is a bijection from $Irr(H')$ onto $Irr(H)$. Moreover, for any irreducible character χ' of H',

$$\chi' \circ \sigma(v_H) = \chi'(v_{H'}).$$

Thus, $\{\chi'(v_{H'})\}_{\chi' \in Irr(H')}$ is identical to the family $\{\chi(v_H)\}_{\chi \in Irr(H)}$. □

Remark 4.2. If H is a semi-simple Hopf algebra, then $\Phi = 1 \otimes 1 \otimes 1$ and $\alpha = \beta = 1$. It follows from Corollary 3.5 that

$$v_H = \sum \Lambda_1 \Lambda_2,$$

where $\sum \Lambda_1 \otimes \Lambda_2 = \Delta(\Lambda)$ and Λ is the normalized two-sided integral of H. Thus, $\chi(v_H)$ coincides with the Frobenius–Schur indicator defined in [20].

As an application of Theorem 4.1, we give a simple alternative proof of the fact that $\mathbb{C}[Q_8]$-**mod** and $\mathbb{C}[D_8]$-**mod** are not equivalent as $\mathbb{C}$-linear tensor categories where Q_8 and D_8 are the quaternion group and the dihedral group of order 8, respectively (cf. [28]).

Proposition 4.3 (Tambara and Yamagami [28]). *The $\mathbb{C}$-linear categories $\mathbb{C}[Q_8]$-**mod** and $\mathbb{C}[D_8]$-**mod** are not equivalent as tensor categories.*

Proof. Let $G = D_8$ or Q_8. Then, G has four degree 1 characters and one degree 2 irreducible character χ_2. Let z be the non-trivial central element of G. Then $\chi_2(z) = -2$ and $\chi(z) = 1$ for any character χ of G of degree 1. Since $v_G = \frac{1}{8}\sum_{g \in G} g^2$, one can easily obtain that

$$v_{Q_8} = \tfrac{1}{8}(6z + 2e) \quad \text{and} \quad v_{D_8} = \tfrac{1}{8}(2z + 6e),$$

where e is the identity of the group. Thus, the family of Frobenius–Schur indicators for Q_8 is $\{1, 1, 1, 1, -1\}$ but the family of Frobenius–Schur indicators for D_8 is

$\{1, 1, 1, 1, 1\}$. By virtue of Theorem 4.1, $\mathbb{C}[Q_8]$-**mod** and $\mathbb{C}[D_8]$-**mod** are not equivalent as $\mathbb{C}$-linear tensor categories. □

5. Bantay's formula for indicators of twisted quantum doubles

In this section, we will show that if H is a twisted quantum double of a finite group G over the field k such that $|G|^{-1}$ exists in k, then for any irreducible character χ of H, $\chi(v_H)$ is identical to Bantay's formula (1.6). We begin with the definition of twisted quantum doubles of finite groups.

Let G be a finite group and $\omega: G \times G \times G \to k^{\times}$ a normalized 3-cocycle; that is a function such that $\omega(x, y, z) = 1$ whenever one of x, y or z is equal to the identity element 1 of G and which satisfies the functional equation

$$\omega(g, x, y)\omega(g, xy, z)\omega(x, y, z) = \omega(gx, y, z)\omega(g, x, yz) \quad \text{for any } g, x, y, z \in G. \tag{5.1}$$

For any $g \in G$, define the functions $\theta_g, \gamma_g: G \times G \to k^{\times}$ as follows:

$$\theta_g(x, y) = \frac{\omega(g, x, y)\omega(x, y, (xy)^{-1}gxy)}{\omega(x, x^{-1}gx, y)}, \tag{5.2}$$

$$\gamma_g(x, y) = \frac{\omega(x, y, g)\omega(g, g^{-1}xg, g^{-1}yg)}{\omega(x, g, g^{-1}yg)}. \tag{5.3}$$

Let $\{e(g) \mid g \in G\}$ be the dual basis of the canonical basis of $k[G]$. The *twisted quantum double* $D^{\omega}(G)$ of G with respect to ω is the quasi-Hopf algebra with underlying vector space $k[G]' \otimes k[G]$. The multiplication, comultiplication and associator are given, respectively, by

$$(e(g) \otimes x)(e(h) \otimes y) = \theta_g(x, y)\delta_{g, xhx^{-1}} e(g) \otimes xy, \tag{5.4}$$

$$\Delta(e(g) \otimes x) = \sum_{hk=g} \gamma_x(h, k) e(h) \otimes x \otimes e(k) \otimes x, \tag{5.5}$$

$$\Phi = \sum_{g,h,k \in G} \omega(g, h, k)^{-1} e(g) \otimes 1 \otimes e(h) \otimes 1 \otimes e(k) \otimes 1. \tag{5.6}$$

The counit and antipode are given by

$$\epsilon(e(g) \otimes x) = \delta_{g,1} \tag{5.7}$$

and

$$S(e(g)\otimes x)=\theta_{g^{-1}}(x,x^{-1})^{-1}\gamma_x(g,g^{-1})^{-1}e(x^{-1}g^{-1}x)\otimes x^{-1}, \tag{5.8}$$

where $\delta_{g,1}$ is the Kronecker delta. The corresponding elements α and β are $1_{D^\omega(G)}$ and $\sum_{g\in G}\omega(g,g^{-1},g)e(g)\otimes 1$, respectively (cf. [7]). Verification of the details involves the following identities, which result from the 3-cocycle identity for ω:

$$\theta_z(a,b)\theta_z(ab,c)=\theta_{a^{-1}za}(b,c)\theta_z(a,bc), \tag{5.9}$$

$$\theta_y(a,b)\theta_z(a,b)\gamma_a(y,z)\gamma_b(a^{-1}ya,a^{-1}za)=\theta_{yz}(a,b)\gamma_{ab}(y,z), \tag{5.10}$$

$$\gamma_z(a,b)\gamma_z(ab,c)\omega(z^{-1}az,z^{-1}bz,z^{-1}cz)=\gamma_z(b,c)\gamma_z(a,bc)\omega(a,b,c), \tag{5.11}$$

for all $a,b,c,y,z\in G$.

Remark 5.1. The algebra $D^\omega(G)$ is semi-simple (cf. [7]). If $\omega=1$, then the twisted quantum double $D^\omega(G)$ identical to the Drinfeld double of the group algebra $k[G]$. However, $D^\omega(G)$ is not a Hopf algebra in general. Moreover, even if $\omega,\ \omega'$ differ by a coboundary, $D^\omega(G)$ and $D^{\omega'}(G)$ are not isomorphic as quasi-bialgebras. Nevertheless, they are *gauge equivalent*. In addition, if G is abelian, $D^\omega(G)$ also admits a Hopf algebra structure with the same underlying Δ, ϵ and S (cf. [21]).

Let

$$\Lambda=\frac{1}{|G|}\sum_{x\in G}e(1)\otimes x\in D^\omega(G). \tag{5.12}$$

It is straightforward to show that Λ is a left integral of $D^\omega(G)$. Moreover,

$$\epsilon(\Lambda)=1.$$

After [15,26], this gives another proof of the semi-simplicity of $D^\omega(G)$. Note that

$$\Delta(\Lambda)=\sum\Lambda_1\otimes\Lambda_2=\frac{1}{|G|}\sum_{g,x\in G}\gamma_x(g,g^{-1})e(g)\otimes x\otimes e(g^{-1})\otimes x.$$

Since $\beta\alpha=\beta$ is invertible, it follows from Corollary 3.5 that

$$v_{D^\omega(G)}$$
$$=\frac{1}{|G|}\left(\sum_{g\in G}\omega(g,g^{-1},g)^{-1}e(g)\otimes 1\right)\left(\sum_{g,x\in G}\gamma_x(g,g^{-1})(e(g)\otimes x)(e(g^{-1})\otimes x)\right)$$

$$= \frac{1}{|G|}\left(\sum_{g\in G}\omega(g^{-1},g,g^{-1})e(g)\otimes 1\right)\left(\sum_{x^{-1}gx=g^{-1}}\gamma_x(g,g^{-1})\theta_g(x,x)e(g)\otimes x^2\right)$$
$$= |G|^{-1}\sum_{x^{-1}gx=g^{-1}}\omega(g^{-1},g,g^{-1})\gamma_x(g,g^{-1})\theta_g(x,x)(e(g)\otimes x^2).$$

Here we have used the equality

$$\omega(g,g^{-1},g)^{-1} = \omega(g^{-1},g,g^{-1})$$

which is readily derived from Eq. (5.1). Thus for any irreducible character χ of $D^\omega(G)$, the Frobenius–Schur indicator of χ is

$$\chi(v_{D^\omega(G)}) = |G|^{-1}\sum_{x^{-1}gx=g^{-1}}\omega(g^{-1},g,g^{-1})\gamma_x(g,g^{-1})\theta_g(x,x)\chi(e(g)\otimes x^2),$$

as given by Bantay.

6. Trace elements and antipodes of semi-simple quasi-Hopf algebras

It is proved by Larson and Radford [18,19] that if char $k=0$, the antipode of a semi-simple Hopf algebra over k is an involution. However, the antipode of a semi-simple quasi-Hopf algebra H could be of any order. Nevertheless, we prove an analog of the Larson–Radford theorem for split semi-simple quasi-Hopf algebras H over any field k: there exists a unit $u\in H$ such that the antipode of H_u is an involution. To this end we introduce the *trace element* g of a semi-simple quasi-Hopf algebra. This element will play a rôle throughout the remaining sections of the paper.

Let $H=(H,\Delta,\epsilon,\Phi,\alpha,\beta,S)$ be a finite-dimensional semi-simple quasi-Hopf algebra over k and Λ the normalized two-sided integral of H. By Hausser and Nill [15], there exists a functional $\lambda\in H'$, called the normalized left cointegral of H, given by the formula

$$\lambda(x) = \sum_i b^i(xS^2(b_i)S(\beta)\alpha) \tag{6.1}$$

for all $x\in H$, where $\{b_i\}$ is a basis of H and $\{b^i\}$ is its dual basis (see [15] for the details of cointegral). The normalized left cointegral λ admits the following properties:

(i) $\lambda(\Lambda)=1$.
(ii) $\lambda(ab)$ $(a,b\in H)$ defines a non-degenerate bilinear form on H.
(iii) For all $a,b\in H$,

$$\lambda(ab) = \lambda(bS^2(a)). \tag{6.2}$$

Let χ_{reg} denote the character of the left regular representation of H. The bilinear form on H defined by $\langle a, b \rangle_{\text{reg}} := \chi_{\text{reg}}(ab)$ is then symmetric and non-degenerate. By the non-degeneracy of λ, there exists a unique element g of H such that

$$\chi_{\text{reg}}(x) = \lambda(xg) \tag{6.3}$$

for all $x \in H$. We call g the *trace element*.

Example 6.1. If char $k = 0$, and H is a finite-dimensional semi-simple Hopf algebra over k, then $S^2 = id_H$. By (6.1),

$$\lambda(x) = \sum_i b^i(xb_i) = \chi_{\text{reg}}(x).$$

Thus, the trace element of H is 1.

Lemma 6.2. *Let $H = (H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$ be a finite-dimensional semi-simple quasi-Hopf algebra. Then the trace element g of H is invertible and*

$$S^2(a) = g^{-1}ag$$

for all $a \in H$. Moreover, $gS(g)$ is in the center of H and $gS(g) = S(g)g$.

Proof. By (6.3), the left annihilator of g in H is a subset of ker χ_{reg}. Since H is semi-simple, ker χ_{reg} does not contain any non-trivial left ideals of H. Therefore, the left annihilator of g is trivial. Since the left regular representation of H is faithful and finite-dimensional, g is invertible. Thus, we have

$$\lambda(ab) = \lambda(abg^{-1}g) = \chi_{\text{reg}}(abg^{-1}) = \chi_{\text{reg}}(bg^{-1}a) = \lambda(bg^{-1}ag)$$

for all $a, b \in H$. By the non-degeneracy of λ and (6.2), we obtain

$$S^2(a) = g^{-1}ag$$

for all $a \in H$. In particular,

$$S(g^{-1}ag) = S^3(a) = g^{-1}S(a)g.$$

Therefore,

$$gS(g)S(a) = S(a)gS(g) \tag{6.4}$$

for all $a \in H$ and hence $gS(g)$ is in the center of H. Taking $a = g^{-1}$ in (6.4), the result in the last statement follows. □

Lemma 6.3. *Let A be a finite-dimensional split semi-simple algebra over k and S an algebra anti-automorphism on A such that S^2 is inner. Then there exists a unit $u \in A$*

such that $S_u^2 = id_A$ *where*

$$S_u(x) = uS(x)u^{-1}$$

for all $x \in A$.

Proof. Without loss of generality, we can assume that A is a direct sum of full matrix rings over k, say $A = \oplus_{i=1}^d M_{n_i}(k)$. Let $\imath_i$ denote the natural embedding from $M_{n_i}(k)$ into A, p_i the natural surjection from A onto $M_{n_i}(k)$, and A_i the image of $\imath_i$. Then, $A_1, \dots, A_d$ is the complete set of minimal ideals of A. Since S is an algebra anti-automorphism, there exists a permutation σ on $\{1, \dots, d\}$ such that $S(A_i) = A_{\sigma(i)}$ for all $i = 1, \dots, d$. As S^2 is inner, $S^2(A_i) = A_i$ for all i and so $\sigma^2 = id$.

Since $S(A_i) = A_{\sigma(i)}$, $M_{n_i}(k) = M_{n_{\sigma(i)}}(k)$. Moreover, $p_j \circ S \circ \imath_i = 0$ for $j \neq \sigma(i)$ and $p_{\sigma(i)} \circ S \circ \imath_i$ is an algebra anti-automorphism on $M_{n_i}(k)$. By the Skolem–Noether theorem, there exists an invertible matrix $u_i \in M_{n_{\sigma(i)}}(k)$ such that $p_{\sigma(i)} \circ S \circ \imath(x) = u_i^{-1} x^t u_i$ for any $x \in M_{n_i}(k)$ where x^t is the transpose of x.

Let $u = \sum_{i=1}^d \imath_{\sigma(i)}(u_i)$. Since u_i is invertible in $M_{n_{\sigma(i)}}(k)$ for all i, u is invertible in A. Since $S(A_i) = A_{\sigma(i)}$ is an ideal of A, $S_u(A_i) = A_{\sigma(i)}$. Then for any $x \in M_{n_i}(k)$,

$$p_{\sigma(i)}(S_u(\imath_i(x))) = u_i(p_{\sigma(i)} \circ S \circ \imath_i(x))u_i^{-1} = x^t.$$

Thus,

$$\imath_{\sigma(i)}(x^t) = \imath_{\sigma(i)} \circ p_{\sigma(i)}(S_u(\imath_i(x))) = S_u(\imath_i(x)),$$

and hence

$$S_u^2(\imath_i(x)) = S_u(\imath_{\sigma(i)}(x^t)) = \imath_{\sigma^2(i)}((x^t)^t) = \imath_i(x)$$

as $\sigma^2 = id$. Therefore, $S_u^2(a) = a$ for all $a \in A_i$, $i = 1, \dots, d$. Since $A = A_1 \oplus \cdots \oplus A_d$, $S_u^2 = id_A$. □

Theorem 6.4. *Let* $H = (H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$ *be a finite-dimensional split semi-simple quasi-Hopf algebra over* k. *Then there exists an invertible element* u *of* H *such that the antipode of* H_u *is an involution.*

Proof. It follows from Lemma 6.2 or [15, Proposition 5.6] that S^2 is inner. By Lemma 6.3, the result follows. □

Remark 6.5. Suppose u is an invertible element of H and M a finite-dimensional left H-module. Let ${}^+M$, ${}^*M$ denote the left dual of M in H_u-**mod**$_{\text{fin}}$ and H-**mod**$_{\text{fin}}$ respectively. Then, ${}^+M$ and ${}^*M$ are isomorphic left H-modules under the map $\phi_u: {}^+M \to {}^*M$ defined by

$$\phi_u(f)(x) = f(ux)$$

for all $x \in M$ and $f \in M'$. In particular, $M \cong {}^*M$ if, and only if, $M \cong {}^+M$ as left H-modules (cf. [8, p. 1425]). □

7. Pivotal category structure of H-$\mathbf{mod}_{\mathrm{fin}}$

We begin with Etingof's observation (Theorem 7.1) that the trace element g of a finite-dimensional semi-simple quasi-Hopf algebra $H = (H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$ over an algebraically closed field of characteristic zero defines an isomorphism of tensor functors

$$j: Id \to {}^{**}?.$$

Moreover, we prove that $S(g) = g^{-1}$, a fact that we will need in Section 8. A direct result of this is that H-$\mathbf{mod}_{\mathrm{fin}}$ is a pivotal category in the sense of Joyal and Street (cf. [11]). For the remainder of this paper we will assume that k is an algebraically closed field of characteristic zero.

For simplicity, we write $\mathscr{C}$ for the semi-simple rigid tensor category H-$\mathbf{mod}_{\mathrm{fin}}$ in this section. Obviously, $\mathscr{C}$ is a fusion category over k (cf. [10]). Recall from [2] that if $V \in \mathscr{C}$ and $f: V \to {}^{**}V$ then the categorical trace of f is the scalar $\mathrm{tr}_V(f)$ defined by

$$ev_{^*V} \circ (f \otimes id) \circ coev_V, \tag{7.1}$$

where $ev_V: {}^*V \otimes V \to k$ and $coev_V: k \to V \otimes {}^*V$ are evaluation and coevaluation maps.

Following [24], for any simple object V in $\mathscr{C}$ and an isomorphism $f: V \to {}^{**}V$, we define

$$|V|^2 = tr_V(f)\, tr_{^*V}({}^*(f^{-1})). \tag{7.2}$$

Clearly, $|V|^2$ is independent of the choice of f.

By Etingof et al. [10], there exists an isomorphism of tensor functors

$$j: Id \to {}^{**}?$$

such that for any simple object V of $\mathscr{C}$,

$$tr_V(j) = FPdim(V) = dim(V) \tag{7.3}$$

where $FPdim(V)$ is the *Frobenius–Perron dimension* of V. Moreover,

$$|V|^2 = dim(V)^2. \tag{7.4}$$

Let a be the unique invertible element of H such that

$$j_H(1)(f) = f(a) \tag{7.5}$$

for all $f \in {}^*H$. By the naturality of j, one can show that

$$S^2(x) = axa^{-1} \quad \text{for all} \quad x \in H, \tag{7.6}$$

and for any $V \in \mathscr{C}$, $j: V \to {}^{**}V$ is given by

$$j_V(x)(f) = f(ax) \tag{7.7}$$

for all $x \in V$ and $f \in {}^*V$. Thus, by (7.1)–(7.3), for any simple objective V in $\mathscr{C}$ with character χ,

$$dim(V) = \chi(a\beta S(\alpha)) \tag{7.8}$$

and

$$|V|^2 = \chi(a\beta S(\alpha))\chi(a^{-1}S(\beta)\alpha). \tag{7.9}$$

Hence, by (7.8) and (7.4), we also have

$$dim(V) = \chi(a^{-1}S(\beta)\alpha) \tag{7.10}$$

In fact, a^{-1} is the trace element of H.

Theorem 7.1 (Etingof). *Let $H = (H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$ be a finite-dimensional semi-simple quasi-Hopf algebra over k and g the trace element of H. Then the natural isomorphism $j_V: V \to {}^{**}V$ for any V in H-$\mathbf{mod}_{\mathrm{fin}}$, given by*

$$j_V(x)(f) = f(g^{-1}x)$$

for all $x \in V$ and $f \in {}^*V$, defines an isomorphism of the tensor functors Id and **? such that

$$dim(V) = \chi(g^{-1}\beta S(\alpha))$$

for any simple H-module V with character χ.

Proof. By the preceding discussion, it suffices to show that the element a defined in (7.5) is identical to g^{-1}. By Lemma 6.2 and (7.6), ag is in the center of H. Therefore, it is enough to show that for any simple H-module V with character χ,

$$\chi(a\beta S(\alpha)) = \chi(g^{-1}\beta S(\alpha)).$$

Let e_V be the central idempotent of H such that

$$\chi(x)dim(V) = \chi_{\mathrm{reg}}(e_V x)$$

for all $x \in H$. Thus, we obtain

$$\begin{aligned}\chi(g^{-1}\beta S(\alpha))dim(V) &= \chi_{\text{reg}}(e_V g^{-1}\beta S(\alpha))\\ &= \chi_{\text{reg}}(e_V \beta S(\alpha) g^{-1})\\ &= \lambda(e_V \beta S(\alpha)),\end{aligned}$$

where λ is the normalized left cointegral of H. Let $\{b^i\}$ be the dual basis of the basis $\{b_i\}$ of H. Then, we have

$$\begin{aligned}\chi(g^{-1}\beta S(\alpha))dim(V) &= \sum_i b^i(e_V \beta S(\alpha) S^2(b_i) S(\beta)\alpha)\\ &= \chi_{\text{reg}}(e_V \beta S(\alpha) a b_i a^{-1} S(\beta)\alpha)\\ &= \chi(\beta S(\alpha) a)\chi(a^{-1}S(\beta)\alpha)\\ &= |V|^2 = dim(V)^2 \quad \text{by (7.9) and (7.4).}\end{aligned}$$

Therefore, by (7.8), we obtain

$$\chi(g^{-1}\beta S(\alpha)) = dim(V) = \chi(a\beta S(\alpha)). \quad \square$$

Theorem 7.2. *Let $H = (H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$ be a finite-dimensional semi-simple quasi-Hopf algebra over k and g the trace element of H. Then $S(g) = g^{-1}$, and hence*

$$^*(j_V) \circ j_{^*V} = id_{^*V} \tag{7.11}$$

for any $V \in H$-$\mathbf{mod}_{\text{fin}}$.

Proof. Since $gS(g)$ is central, $gS(g)$ acts on any simple H-module V as multiplication by a scalar $c_V \in k$. In order to show that $S(g) = g^{-1}$, it suffices to prove that

$$c_V = 1$$

for any simple H-module V.

Let V be a simple H-module with character χ. Then the character of $^*V$ is $^*\chi$ given by

$$^*\chi = \chi \circ S.$$

By Theorem 7.1, (7.8) and (7.10), we have

$$dim(^*V) = {}^*\chi(gS(\beta)\alpha) = \chi(S(\alpha)S^2(\beta)S(g))$$

$$= \chi(S(\alpha)g^{-1}\beta g S(g)) = c_V \chi(S(\alpha)g^{-1}\beta)$$

$$= c_V dim(V)$$

Therefore, $c_V = 1$. Eq. (7.11) follows easily from $S(g) = g^{-1}$. $\square$

Theorem 7.1 and (7.11) implies that H-$\mathbf{mod}_{\mathrm{fin}}$ is indeed a pivotal category as defined by Joyal–Street (cf. [11]). Nikshych also pointed out that (7.11) can be proved using *weak Hopf algebras*.

8. Frobenius–Schur indicators via bilinear forms with adjoint S

Let $H = (H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$ be a finite-dimensional semi-simple quasi-Hopf algebra over k, and g the trace element of H. In this section, we will prove that for any simple left H-module M with character χ, the Frobenius–Schur indicator $\chi(v_H)$ of χ can only be 0, 1 or -1. It is non-zero if, and only if $M \cong {}^*M$. Moreover in this case, M admits a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ such that $\langle hu, v \rangle = \langle u, S(h)v \rangle$ for all $h \in H$, $u, v \in M$, and

$$\langle u, v \rangle = \chi(v_H) \langle v, g^{-1}u \rangle.$$

Definition 8.1. Let $H = (H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$ be a quasi-Hopf algebra over k, M be a left H-module and $\langle \cdot, \cdot \rangle$ a bilinear form on M.

(i) The form is said to be H-invariant if

$$\sum \langle h_1 u, h_2 v \rangle = \epsilon(h) \langle u, v \rangle$$

for all $h \in H$ and $u, v \in V$ where $\sum h_1 \otimes h_2 = \Delta(h)$.

(ii) The antipode S is said to be the adjoint of the form if

$$\langle hu, v \rangle = \langle u, S(h)v \rangle$$

for all $h \in H$ and $u, v \in V$.

Lemma 8.2. *Let $H = (H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$ be a quasi-Hopf algebra over k and M a simple left H-module. If $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ are non-degenerate bilinear forms on M with the same adjoint S, then there exists a non-zero element $c \in k$ such that*

$$\langle u, v \rangle_1 = c \langle u, v \rangle_2$$

for all $u, v \in M$.

Proof. Define $J_i: M \to {}^*M$ $(i = 1, 2)$ by

$$J_i(u)(v) = \langle u, v \rangle_i$$

for $u, v \in M$. Since $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ are non-degenerate bilinear forms on M with the adjoint S, J_1, J_2 are isomorphisms of H-modules. In particular, M and ${}^*M$ are isomorphic simple H-modules. By Schur's lemma, $J_1 = cJ_2$ for some non-zero element $c \in k$ and so the result follows. □

Lemma 8.3. *Let $H = (H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$ be a finite-dimensional semi-simple quasi-Hopf algebra, Λ the normalized two-sided integral of H and g the trace element of H. Suppose that*

$$q_R \Delta(\Lambda) p_R = \sum_{i=1}^{n} x_i \otimes y_i,$$

where $\{x_i\}$ is a basis of H. Then $\{S(x_i)g^{-1}, y_i\}$ is a pair of dual bases with respect to $\langle \cdot, \cdot \rangle_{\mathrm{reg}}$.

Proof. Following [15], we define the elements $U, V \in H \otimes H$ by

$$U = F_H^{-1}(S \otimes S)(q_R^{21}), \tag{8.1}$$

$$V = (S^{-1} \otimes S^{-1})(F_H^{21} p_R^{21}), \tag{8.2}$$

where F_H, $q_R, p_R \in H \otimes H$ are defined in (2.12) and (2.13). By Hausser and Nill [15, (7.3) and (7.4)],

$$\begin{aligned} q_R \Delta(\Lambda) p_R &= (q_L^2 \otimes 1) V \Delta(S^{-1}(q_L^1)) \Delta(\Lambda) \Delta(S(p_L^1))\, U(p_L^2 \otimes 1) \\ &= (q_L^2 \epsilon(S^{-1}(q_L^1)) \otimes 1) V \Delta(\Lambda) U(\epsilon(S(p_L^1)) p_L^2 \otimes 1). \end{aligned}$$

By Drinfel'd [8, Remark 7], $\epsilon \circ S = \epsilon = \epsilon \circ S^{-1}$. Therefore,

$$q_L^2 \epsilon(S^{-1}(q_L^1)) = \epsilon(\alpha) 1_H \quad \text{and} \quad \epsilon(S(p_L^1)) p_L^2 = \epsilon(\beta) 1_H.$$

It follows from (2.9) that $\epsilon(\alpha\beta) = 1$ and so

$$q_R \Delta(\Lambda) p_R = \epsilon(\alpha)\epsilon(\beta) V \Delta(\Lambda) U = V \Delta(\Lambda) U.$$

Let λ be the normalized left cointegral of H. By Hausser and Nill [15, Proposition 5.5],

$$\sum_i S(x_i) \lambda(y_i a) = a$$

for all $a \in H$. In particular,

$$a = (ag)g^{-1} = \sum_i S(x_i)g^{-1}\lambda(y_i ag) = \sum_i S(x_i)g^{-1}\chi_{\text{reg}}(y_i a).$$

Since $\{S(x_i)g^{-1}\}$ is also a basis of H, $\chi_{\text{reg}}(y_i S(x_j)g^{-1}) = \delta_{ij}$ and so $\{S(x_i)g^{-1}, y_i\}$ is a pair of dual bases of H with respect to $\langle \cdot, \cdot \rangle_{\text{reg}}$. $\square$

Lemma 8.4. *Let A be a finite-dimensional semi-simple algebra over k and $\{a_i, b_i\}$ a pair of dual bases with respect to the form $\langle \cdot, \cdot \rangle_{\text{reg}}$. Then*

$$\sum_i a_i b_i = 1_A.$$

Proof. Without loss of generality, we may assume that $A = \oplus_{i=1}^d M_{n_i}(k)$. Then $\chi_{\text{reg}}(x) = \sum_{i=1}^d n_i\, tr_i(x)$, where $tr_i(x)$ is the trace of the ith component matrix of x. Let $\{e^i_{lm}\}$ be the set of matrix units for the ith summand $M_{n_i}(k)$ of A. Following [20], $\{n_i^{-1}e^i_{lm}, e^i_{ml}\}$ is a pair of dual bases with respect to $\langle \cdot, \cdot \rangle_{\text{reg}}$. Thus,

$$\sum_{i,l,m} n_i^{-1} e^i_{lm} e^i_{ml} = \sum_{i,l} e^i_{ll} = 1_A.$$

It follows from [20, Lemma 2.6] that

$$\sum_i a_i b_i = \sum_{i,l,m} n_i^{-1} e^i_{lm} e^i_{ml} = 1_A. \quad \square$$

Corollary 8.5. *Let $H = (H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$ be a finite-dimensional semi-simple quasi-Hopf algebra over k. Then the trace element g of H is given by*

$$g = m\tau(S \otimes id)(q_R \Delta(\Lambda) p_R),$$

where Λ is the normalized integral of H, m is multiplication and τ the usual flip map.

Proof. Let

$$q_R \Delta(\Lambda) p_R = \sum_i x_i \otimes y_i.$$

By Lemmas 8.3 and 8.4, we have

$$\sum_i y_i S(x_i) g^{-1} = 1$$

and so the result follows. $\square$

Let $\{a_i, b_i\}$ be dual bases of the semi-simple quasi-Hopf algebra H with respect to the form $\langle \cdot, \cdot \rangle_{\text{reg}}$ discussed in Lemma 8.3. For any k-involution $\mathscr{I}$ on H and for any character χ of H, we define

$$\mu_2(\chi, \mathscr{I}) = \chi\left(\sum_i \mathscr{I}(a_i)b_i\right).$$

Remark 8.6. Since $\sum_i a_i b_i = 1_H$ by Lemma 8.4, the μ_2 defined in [20, Theorem 2.7] with respect to the k-involution $\mathscr{I}$ is given by

$$\frac{\chi(1_H)}{\chi(\sum_i a_i b_i)}\chi\left(\sum_i \mathscr{I}(a_i)b_i\right) = \chi\left(\sum_i \mathscr{I}(a_i)b_i\right)$$

which coincides with $\mu_2(\chi, \mathscr{I})$.

Lemma 8.7. *Let $H = (H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$ be a finite-dimensional semi-simple quasi-Hopf algebra over k, g the trace element of H, and M an irreducible H-module with character χ. Then for any unit $u \in H$ such that S_u is an involution,*

$$\mu_2(\chi, S_u) = c\chi(v_H),$$

where c is the non-zero scalar given by

$$c = \frac{\chi(uS(u^{-1})g^{-1})}{dim\, M}.$$

Proof. If u is a unit of H such that S_u is an involution, then for any $x \in H$,

$$x = S_u^2(x) = uS(u^{-1})S^2(x)S(u)u^{-1},$$

or equivalently

$$S^2(x) = S(u)u^{-1}xuS(u^{-1}).$$

By Lemma 6.2, $uS(u^{-1})g^{-1}$ is in the center of H. Thus, $uS(u^{-1})g^{-1}$ acts on M as multiplication by the non-zero scalar

$$c = \frac{\chi(uS(u^{-1})g^{-1})}{dimM}.$$

Suppose that

$$q_R\Delta(\Lambda)p_R = \sum_i x_i \otimes y_i$$

as in Lemma 8.3, where Λ is the normalized two-sided integral of H. Then we have

$$\begin{aligned}\mu_2(\chi, S_u) &= \chi\left(\sum_i S_u(S(x_i)g^{-1})y_i\right)\\ &= \chi\left(\sum_i uS(g^{-1})S^2(x_i)u^{-1}y_i\right)\\ &= \chi\left(\sum_i uS(g^{-1})g^{-1}x_i g u^{-1}y_i\right)\\ &= \chi\left(\sum_i uS(g^{-1})g^{-1}S(u^{-1})S(g)x_i y_i\right) \quad \text{(by Lemma 3.1)}\\ &= \chi\left(\sum_i uS(u^{-1})g^{-1}x_i y_i\right) \quad \text{(by Lemma 6.2)}\\ &= c\chi(v_H). \qquad \square\end{aligned}$$

Theorem 8.8. *Let $H = (H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$ be a finite-dimensional semi-simple quasi-Hopf algebra over k, g the trace element of H, and M a simple H-module with character χ. Then the Frobenius–Schur indicator $\chi(v_H)$ of χ satisfies the following properties*:

(i) $\chi(v_H) \neq 0$ *if, and only if,* $M \cong {}^*M$ *as left H-modules.*
(ii) *For any non-zero $\kappa \in k$, $\chi(v_H) = \kappa$ if, and only if, M admits a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ with the adjoint S such that*

$$\langle x, y \rangle = \kappa \langle y, g^{-1}x \rangle$$

for all $x, y \in M$.
(iii) *The values of $\chi(v_H)$ can only be* 0, 1 *or* −1.

Moreover,

$$Tr(S) = \sum_{\chi \in Irr(H)} \chi(v_H)\chi(g^{-1}).$$

Proof. By Theorem 6.4, there exists a unit $u \in H$ such that S_u is an involution. As in the proof of Lemma 8.7, $uS(u^{-1})g^{-1}$ is a central unit of H. Thus, $uS(u^{-1})g^{-1}$ acts on M as multiplication by the non-zero scalar

$$c = \frac{\chi(uS(u^{-1})g^{-1})}{dim M}.$$

Also, by Linchenko and Montgomery [20, Theorem 2.7] and Remark 8.6, the element $\mu_2(\chi, S_u) \neq 0$ if, and only if $M \cong {}^+M$ as left H-modules where ${}^+M$ is the left

H-module with underlying space M' and the H-action given by

$$(hf)(x) = f(S_u(h)x)$$

for all $f \in M'$ and $h \in H$. Actually, ${}^+M$ is the left dual of M in H_u-$\mathbf{mod}_{\mathrm{fin}}$. It follows from Remark 6.5 that $\mu_2(\chi, S_u) \neq 0$ if, and only if $M \cong {}^*M$ as left H-modules. Hence, by Lemma 8.7, statement (i) follows.

If $\chi(v_H) \neq 0$, then $\mu_2(\chi, S_u) \neq 0$ by Lemma 8.7. By Remark 8.6 and [20, Theorem 2.7(ii)], M admits a non-degenerate bilinear form $(\cdot, \cdot)$ with adjoint S_u such that

$$(x, y) = \mu_2(\chi, S_u)(y, x)$$

for any $x, y \in M$. Define

$$\langle x, y \rangle = (x, uy)$$

for any $x, y \in M$. One can easily see that $\langle \cdot, \cdot \rangle$ is a non-degenerate bilinear form on M with adjoint S. Moreover, for any $x, y \in M$,

$$\langle x, y \rangle = (x, uy) = \mu_2(\chi, S_u)(uy, x) = \mu_2(\chi, S_u)(y, S_u(u)x).$$

Thus, by Lemma 8.7, we obtain

$$\langle x, y \rangle = c\chi(v_H)(y, S_u(u)x) = \chi(v_H)\langle y, S(u)u^{-1}cx \rangle = \chi(v_H)\langle y, g^{-1}x \rangle.$$

Conversely, suppose M admits a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ with adjoint S and that there exists a non-zero element κ of k such that

$$\langle x, y \rangle = \kappa \langle y, g^{-1}x \rangle$$

for all $x, y \in M$. Then the map $J: M \to {}^*M$, defined by

$$J(x)(y) = \langle x, y \rangle, \quad x, y \in M$$

is an isomorphism of left H-modules. Thus, by (i), $\chi(v_H) \neq 0$. Hence, by above arguments, M admits a non-degenerate bilinear form $\langle \cdot, \cdot \rangle_0$ with adjoint S such that

$$\langle x, y \rangle_0 = \chi(v_H)\langle y, g^{-1}x \rangle_0$$

for all $x, y \in M$. By Lemma 8.2, $\langle \cdot, \cdot \rangle$ is a non-zero scalar multiple of $\langle \cdot, \cdot \rangle_0$. Therefore,

$$\kappa = \chi(v_H)$$

and this finishes the proof statement (ii).

(iii) If M is a simple H-module with character χ such that $\chi(v_H) \neq 0$, by (ii), M admits a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ with adjoint S such that

$$\langle x, y \rangle = \chi(v_H) \langle y, g^{-1} x \rangle$$

for all $x, y \in M$. Thus, we have

$$\begin{aligned} \langle x, y \rangle &= \chi(v_H)^2 \langle g^{-1} x, g^{-1} y \rangle = \chi(v_H)^2 \langle x, S(g^{-1}) g^{-1} y \rangle \\ &= \chi(v_H)^2 \langle x, y \rangle \quad \text{(by Theorem 7.2)}. \end{aligned}$$

Therefore, $\chi(v_H)^2 = 1$ or equivalently $\chi(v_H) = \pm 1$.

Let $\sum_i x_i \otimes y_i = q_R \Delta(\Lambda) p_R$, where Λ is the normalized two-sided integral of H. By Lemma 8.3, $\{S(x_i) g^{-1}, y_i\}$ is a pair of dual bases of H with respect to the form $\langle \cdot, \cdot \rangle_{\text{reg}}$ on H. Therefore, we obtain

$$\begin{aligned} \mathrm{Tr}(S) &= \sum_i \langle S(S(x_i) g^{-1}), y_i \rangle_{\text{reg}} \\ &= \sum_i \chi_{\text{reg}}(S(g^{-1}) S^2(x_i) y_i) \\ &= \sum_i \chi_{\text{reg}}(S(g^{-1}) g^{-1} x_i g y_i) \\ &= \sum_i \chi_{\text{reg}}(S(g^{-1}) g^{-1} S(g) x_i y_i) \quad \text{by Lemma 3.1} \\ &= \sum_i \chi_{\text{reg}}(g^{-1} x_i y_i) \quad \text{by Lemma 6.2} \\ &= \chi_{\text{reg}}(g^{-1} v_H). \end{aligned}$$

Since v_H is in the center of H, for any irreducible H-module M with character χ, v_H acts on M as a multiplication by the scalar

$$c_\chi = \chi(v_H)/\chi(1_H).$$

Since $\chi_{\text{reg}} = \sum_{\chi \in Irr(H)} \chi(1_H) \chi$, we have

$$\begin{aligned} Tr(S) &= \sum_{\chi \in Irr(H)} \chi(1_H) \chi(g^{-1} v_H) \\ &= \sum_{\chi \in Irr(H)} \chi(1_H) c_\chi \chi(g^{-1}) \\ &= \sum_{\chi \in Irr(H)} \chi(v_H) \chi(g^{-1}). \quad \square \end{aligned}$$

Remark 8.9. In [12], Fuchs et al. also define a notion of Frobenius–Schur indicator for simple objects in a sovereign C^*-category $\mathscr{C}$ such that

$$id: {}^*M \to M^*$$

defines an isomorphism of the tensor functors ${}^*?$ and $?^*$. Let $k_M: M \to ({}^*M)^* = {}^{**}M$ be the natural isomorphism of the underlying autonomous structure of $\mathscr{C}$. Then for any simple object M in $\mathscr{C}$, the Frobenius–Schur indicator c_M of M is defined to be 0 if $M \ncong {}^*M$ and c if there exists an H-module isomorphism $J: M \to {}^*M$, where c is given by the equation

$$J^* \circ k_M = cJ. \tag{8.3}$$

Then the values of c_M can only be 0, 1 or -1.

The category H-$\mathbf{mod}_{\mathrm{fin}}$ is not of this kind in general. Nevertheless, if one replaces k_M in (8.3) by $j_M: M \to {}^{**}M$, given by

$$j_M(f)(x) = f(g^{-1}x) \quad \text{for all} \quad x \in M \quad \text{and} \quad f \in {}^*M,$$

one can still define *Frobenius–Schur indicator* c_M for any simple H-module M to be 0 if $M \ncong {}^*M$ and c if there exists a H-module isomorphism $J: M \to {}^*M$ where c given by the equation

$$J^* \circ j_M = cJ.$$

Theorem 8.8(i) and (ii) implies $c_M = \chi(v_H)$. $\square$

Before closing this section, we will show that if α is a central unit, a bilinear form on an H-module M is H-invariant if, and only if, S is the adjoint of the form. Both semi-simple Hopf algebras over k and twisted quantum doubles of finite groups are of this type.

Proposition 8.10. *Let $H = (H, \Delta, \epsilon, \Phi, \alpha, \beta, S)$ be a quasi-Hopf algebra over k and M an H-module. Then, the set $Inv(M)$ of H-invariant forms on M and the set $Adj_S(M)$ of forms on M with adjoint S are isomorphic as k-spaces. In addition, if α is a central unit of H, then*

$$Inv(M) = Adj_S(M).$$

Proof. Note that both $Inv(M)$ and $Adj_S(M)$ are k-subspaces of $(M \otimes M)^*$. We define $\phi: Adj_S(M) \to (M \otimes M)^*$ and $\psi: Inv(M) \to (M \otimes M)^*$ by

$$\phi(\mathbf{b})(x \otimes y) = \mathbf{b}(x \otimes \alpha y), \tag{8.4}$$

$$\psi(\mathbf{b}')(x \otimes y) = \mathbf{b}'(p_L(x \otimes y)) \tag{8.5}$$

for any $x, y \in M$, $\mathbf{b} \in Adj_S(M)$ and $\mathbf{b}' \in Inv(M)$. Using (2.8), one can easily see that

$$Im(\phi) \subseteq Inv(M).$$

By (2.18), $\psi(\mathbf{b}')$ has adjoint S for any H-invariant form $\mathbf{b}'$ on M and so

$$Im(\psi) \subseteq Adj_S(M).$$

It follows easily from (2.21) that for any $\mathbf{b}' \in Inv(M)$ and $x, y \in M$,

$$\begin{aligned}\mathbf{b}'(x \otimes y) &= \mathbf{b}'(\Delta(q_L^2)p_L(S^{-1}(q_L^1)x \otimes y)) \\ &= \mathbf{b}'(p_L(S^{-1}(q_L^1\epsilon(q_L^2))x \otimes y) \\ &= \psi(\mathbf{b}')((S^{-1}(\alpha)x \otimes y).\end{aligned}$$

Since $\psi(\mathbf{b}') \in Adj_S(M)$,

$$\phi \circ \psi = id_{Inv(M)}.$$

On the other hand, by (2.9), for any $\mathbf{b} \in Adj_S(M)$ and $x, y \in M$,

$$\psi \circ \phi(\mathbf{b})(x \otimes y) = \mathbf{b}(p_L^1 x \otimes \alpha p_L^2 y) = \mathbf{b}(x \otimes S(p_L^1)\alpha p_L^2 y) = \mathbf{b}(x \otimes y).$$

Therefore, $\phi: Adj_S(M) \to Inv(M)$ is a k-linear isomorphism.

If α is a central unit, we consider the quasi-Hopf algebra $H_{\alpha^{-1}}$. Then, the corresponding ϕ is the identity map and so

$$Adj_{S_{\alpha^{-1}}}(M) = Inv(M).$$

Since $S_{\alpha^{-1}} = S$, the second statement follows. □

9. Frobenius–Schur indicators of twisted quantum doubles of finite groups

We showed in Section 5 that for any simple module M of $D^\omega(G)$ with character χ, Bantay's formula for the indicator of χ is $\chi(v_{D^\omega(G)})$. In this section, we will prove that the trace element of $D^\omega(G)$ is β and the Frobenius–Schur indicator $\chi(v_{D^\omega(G)})$ of χ is non-zero if, and only if, ${}^*M \cong M$. Moreover, the indicator of χ is 1 (respectively -1) if and only if M admits a β^{-1}-symmetric (resp. β^{-1}-skew symmetric) non-degenerate $D^\omega(G)$-invariant bilinear form $\langle \cdot, \cdot \rangle$, that is

$$\langle x, y \rangle = \langle y, \beta^{-1}x \rangle \quad (resp. \quad \langle x, y \rangle = -\langle y, \beta^{-1}x \rangle)$$

for all $x, y \in M$.

We first need the following formula (cf. [1]) to compute the trace element of $D^\omega(G)$.

Lemma 9.1. *Let $\omega{:}G\times G\times G\to k^{\times}$ be a normalized 3-cocycle of a finite group G and let S be the antipode of the quasi-Hopf algebra $D^{\omega}(G)$ defined in Section 5. Then for any $g, x\in G$,*

$$S^2(e(g)\otimes x) = \frac{\omega(g^x,(g^{-1})^x,g^x)}{\omega(g,g^{-1},g)}\,e(g)\otimes x, \\ = \beta^{-1}(e(g)\otimes x)\beta.$$

Proof. It follows from (5.8) that

$$S^2(e(g)\otimes x) = (\theta_{g^{-1}}(x,x^{-1})\gamma_x(g,g^{-1})\theta_{g^x}(x^{-1},x)\gamma_{x^{-1}}((g^{-1})^x,g^x))^{-1}e(g)\otimes x, \quad (9.1)$$

where g^x denotes the product $x^{-1}gx$. By the normality of ω and (5.9),

$$\theta_g(x,x^{-1}) = \theta_{g^x}(x^{-1},x).$$

Thus, we have

$$\theta_{g^{-1}}(x,x^{-1})\gamma_x(g,g^{-1})\theta_{g^x}(x^{-1},x)\gamma_{x^{-1}}((g^{-1})^x,g^x) \\ = \theta_{g^{-1}}(x,x^{-1})\theta_g(x,x^{-1})\gamma_x(g,g^{-1})\gamma_{x^{-1}}((g^{-1})^x,g^x).$$

By the normality of ω and Eq. (5.10), we have

$$\theta_{g^{-1}}(x,x^{-1})\gamma_x(g,g^{-1})\theta_{g^x}(x^{-1},x)\gamma_{x^{-1}}((g^{-1})^x,g^x) \\ = \frac{\gamma_x(g,g^{-1})\gamma_{x^{-1}}((g^{-1})^x,g^x)}{\gamma_x(g,g^{-1})\gamma_{x^{-1}}(g^x,(g^{-1})^x)} \\ = \frac{\gamma_{x^{-1}}((g^{-1})^x,g^x)}{\gamma_{x^{-1}}(g^x,(g^{-1})^x)}.$$

By Eq. (5.11), for any $z, a\in G$ we have

$$\gamma_z(a,a^{-1})\omega(a^z,(a^{-1})^z,a^z) = \gamma_z(a^{-1},a)\omega(a,a^{-1},a).$$

Hence we have

$$\frac{\gamma_{x^{-1}}((g^{-1})^x,g^x)}{\gamma_{x^{-1}}(g^x,(g^{-1})^x)} = \frac{\omega((g^x)^{x^{-1}},((g^{-1})^x)^{x^{-1}},(g^x)^{x^{-1}})}{\omega(g^x,(g^{-1})^x,g^x)} \\ = \frac{\omega(g,g^{-1},g)}{\omega(g^x,(g^{-1})^x,g^x)}. \quad (9.2)$$

The second equation in the statement of the lemma follows immediately from (5.4). □

Proposition 9.2. *Let $\omega: G \times G \times G \to k^{\times}$ be a normalized 3-cocycle of a finite group G. Then the trace element of the quasi-Hopf algebra $D^{\omega}(G)$ is β.*

Proof. Using (5.8), $S(\beta) = \beta^{-1}$. Suppose that $\{f_{g,x}\}_{g,x\in G}$ is the dual basis of $\{e(g)\otimes x\}_{g,x\in G}$. Then, by (6.1), the normalized left cointegral of $D^{\omega}(G)$ is given by

$$\lambda(e(g)\otimes x) = \sum_{h,y\in G} f_{h,y}((e(g)\otimes x)S^2(e(h)\otimes y)\beta^{-1}).$$

Using Lemma 9.1, we have

$$\begin{aligned}\lambda(e(g)\otimes x) &= \sum_{h,y\in G} f_{h,y}((e(g)\otimes x)\beta^{-1}(e(h)\otimes y))\\ &= \chi_{\mathrm{reg}}((e(g)\otimes x)\beta^{-1})\\ &= \lambda((e(g)\otimes x)\beta^{-1}g).\end{aligned}$$

By the non-degeneracy of λ, $\beta^{-1}g = 1$ and so $g = \beta$. $\square$

Corollary 9.3. *Let $\omega: G \times G \times G \to k^{\times}$ be a normalized 3-cocycle of a finite group G. Suppose that M is a simple $D^{\omega}(G)$-module with character χ. Then the Frobenius–Schur indicator $\chi(v_{D^{\omega}(G)})$ of χ satisfies the following properties:*

(i) $\chi(v_{D^{\omega}(G)}) = 0, 1,$ *or* -1.
(ii) $\chi(v_{D^{\omega}(G)}) \neq 0$ *if, and only if,* ${}^{*}M \cong M$.
(iii) $\chi(v_{D^{\omega}(G)}) = 1$ (*respectively* -1) *if and only if M admits a β^{-1}-symmetric* (*resp. β^{-1}-skew symmetric*) *non-degenerate $D^{\omega}(G)$-invariant bilinear form.*

Moreover,

$$\mathrm{Tr}(S) = \sum_{\chi\in Irr(D^{\omega}(G))} \chi(v_{D^{\omega}(G)})\chi(\beta^{-1}).$$

Proof. Statements (i), (ii) and the last statement are immediate consequences of Theorem 8.8. Since $\alpha = 1$, by Proposition 8.10, a bilinear form $\langle \cdot, \cdot \rangle$ on M is $D^{\omega}(G)$-invariant if, and only if, S is the adjoint of $\langle \cdot, \cdot \rangle$. Thus, by Theorem 8.8 (ii), the result in statement (iii) follows. $\square$

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