A Novel Approach to Processing Fractal Signals Using the Yang-Fourier Transforms

Xiao-Jun Yang a*, Meng-Ke Liao b, Jiang-Wen Chen b

a Department of Mathematics and Mechanics, China University of Mining and Technology, Xuzhou, Jiangsu, 221008, P. R. C
b College of Water conservancy, Shihezhi University, Shihezhi, Xinjiang, 832003, P. R. C

Abstract

In the present paper, local fractional continuous non-differentiable functions in fractal space are studied, and the signals in fractal-time space are reflectively investigated using the Yang-Fourier transforms based on the local fractional calculus. Two illustrative examples are given to elaborate the signal process and reliable results.

© 2011 Published by Elsevier Ltd. Open access under the CC BY-NC-ND license.

Keywords: Non-differentiable function, fractal signals, Yang-Fourier transform, fractal-time space, local fractional calculus;

1. Introduction

Generally speaking, signal is that anything carries information. Fourier analysis is one of the most frequently used tools in signal processing [1]. A signal is defined as a function of time. However, some signal functions are fractal curves, which are everywhere continuous but nowhere differentiable [2-9]. As a result, we cannot employ the classical Fourier analysis, which requires that the defined functions should be differentiable, to describe the signals in fractal time–space.

Recently, local fractional calculus (fractal calculus), which is dealing with fractal functions, has been proposed and developed. For these merits, local fractional calculus was successfully applied in the fractal elasticity [4], the fractal wave equation [9], the Yang-Laplace transforms[6,7,9], the Yang-Fourier transforms[6-8], the local fractional short time transforms[6,7] and the local fractional wavelet transforms[6,7].

In this paper, we apply the Yang-Fourier transform to deal with the fractal signals.
2. Fractal-time signals and local fractional calculus

In this section, we mainly study the fractal-time signals and local fractional calculus. As is known, the signals are functions of time and the signals and functions are interchangeable. Here, both local fractional continuity of the functions and local fractional calculus are investigated.

2.1. Notations

Definition 1 If there exists the relation [6,7]
\[ |f(x) - f(x_0)| < \varepsilon^\alpha \]
with \(|x - x_0| < \delta\), for \(\varepsilon, \delta > 0\) and \(\varepsilon, \delta \in \mathbb{R}\). Now \(f(x)\) is called local fractional continuous at \(x = x_0\), denote by \(\lim_{x \to x_0} f(x) = f(x_0)\). Then \(f(x)\) is called local fractional continuous on the interval \((a, b)\), denoted by
\[ f(x) \in C_\alpha(a, b). \]

Definition 2 A function \(f(x)\) is called a non-differentiable function of exponent \(\alpha, 0 < \alpha \leq 1\), which satisfy Hölder function of exponent \(\alpha\), then for \(x, y \in X\) such that [6,7]
\[ |f(x) - f(y)| \leq C|x - y|^\alpha. \]

Definition 3 A function \(f(x)\) is called to be continuous of order \(\alpha, 0 < \alpha \leq 1\), or shortly \(\alpha\) continuous, when we have the following relation [6,7]
\[ f(x) - f(x_0) = o\left((x - x_0)^\alpha\right). \]

Remark 1. Compared with (2.4), (2.1) is standard definition of local fractional continuity. Here (2.3) is unified local fractional continuity.

Definition 4 Setting \(f(x) \in C_\alpha(a, b)\), local fractional derivative of \(f(x)\) of order \(\alpha\) at \(x = x_0\) is defined [6-9]
\[ f^{(\alpha)}(x_0) = \frac{d^\alpha f(x)}{dx^\alpha}\bigg|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha(f(x) - f(x_0))}{(x - x_0)^\alpha}, \]
where \(\Delta^\alpha(f(x) - f(x_0)) \equiv \Gamma(1+\alpha)\Delta(f(x) - f(x_0))\).

Definition 5 Setting \(f(x) \in C_\alpha(a, b)\), local fractional integral of \(f(x)\) of order \(\alpha\) in the interval \([a, b]\) is defined [6,9]
\[ \int_a^b f^{(\alpha)}(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta \to 0} \sum_{j=0}^{j=N-1} f(t_j)(\Delta t_j)^\alpha, \]
where \(\Delta t_j = t_{j+1} - t_j, \Delta t = \max \{\Delta t_1, \Delta t_2, \Delta t_j, \ldots\}\) and \([t_j, t_{j+1}]\). Here, \(f = 0, \ldots, N-1, t_0 = a, t_N = b\), is a partition of the interval \([a, b]\).

Here, it follows that
\[ I_a (\alpha) f(x) = 0 \text{ if } a = b \] (7)

and

\[ I_b (\alpha) f(x) = - I_a (\alpha) f(x) \text{ if } a < b. \] (8)

2.2. Recent results

Suppose that \( f(x), g(x) \in D_\alpha (a, b) \), the following differentiation rules are valid [6-9]:

\[
\frac{d^\alpha (f(x) \pm g(x))}{dx^\alpha} = \frac{d^\alpha f(x)}{dx^\alpha} \pm \frac{d^\alpha g(x)}{dx^\alpha};
\] (9)

\[
\frac{d^\alpha (f(x) g(x))}{dx^\alpha} = g(x) \frac{d^\alpha f(x)}{dx^\alpha} + f(x) \frac{d^\alpha g(x)}{dx^\alpha};
\] (10)

**Theorem 1** [6-9] Suppose that \( f(x), g(x) \in C_\alpha [a, b] \), then

\[ a I_b (\alpha) [f(x) \pm g(x)] = a I_b (\alpha) f(x) \pm a I_b (\alpha) g(x). \] (11)

**Theorem 2** [6-9] Suppose that \( f(x) = g^{(\alpha)}(x) \in C_\alpha [a, b] \), then we have

\[ a I_b (\alpha) f(x) = g(b) - g(a). \] (12)

3. Yang-Fourier transforms

In this section, we start with the Yang-Fourier transforms and some results.

**Definition 6** Suppose that \( f(x) \in C_\alpha [-\infty, \infty) \), the Yang-Laplace transform, denoted by \( \mathcal{F}_\alpha f(x) \equiv f^{F,\alpha}_\omega(\omega) \), is given in the form [6-8]

\[
F_\alpha f(x) = f^{F,\alpha}_\omega(\omega) := \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} E_\alpha (\tau^\alpha \omega^\alpha x^\alpha) f(x) (dx)^\alpha,
\] (12)

where the latter converges.

And of course, a sufficient condition for convergence is

\[
\left| \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(x) E_\alpha (\tau^\alpha \omega^\alpha x^\alpha) (dx)^\alpha \right| \leq \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} |f(x)|(dx)^\alpha < K < \infty.
\] (14)

**Definition 7** Suppose that \( F_\alpha f(x) \equiv f^{F,\alpha}_\omega(\omega) \), its inverse is given by the expression

\[
f(x) = F^{-1}_\alpha \left( f^{F,\alpha}_\omega(\omega) \right) := \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{\infty} F_\alpha \left( f^{F,\alpha}_\omega(\omega) \right) (d\omega)^\alpha, \quad x > 0.
\] (15)

Suppose that \( F_\alpha f(x) \equiv f^{F,\alpha}_\omega(\omega) \) and \( F_\alpha g(x) \equiv g^{F,\alpha}_\omega(\omega) \), the following formulas are valid [6-8]:

\[
F_\alpha \{af(x) + bg(x)\} = af^{F,\alpha}_\omega (\omega) + bg^{F,\alpha}_\omega (\omega);
\] (16)
If \( \lim_{x \to \infty} f(x) = 0 \), \( F_{\alpha} \{ f^{(\alpha)}(x) \} = i^{\alpha} \omega^{\alpha} f_{\omega^{\alpha}}^{(\alpha)}(\omega) \); \hspace{1cm} (17)

\[
F_{\alpha} \{ f(ax) \} = \frac{1}{a^{\alpha}} f_{\omega^{\alpha}}^{(\alpha)} \left( \frac{\omega}{a} \right), \; a > 0 ; \hspace{1cm} (18)
\]

\[
F_{\alpha} \{ f(x-c) \} = E_{\alpha} \left( -i^{\alpha} c^{\alpha} \omega^{\alpha} \right) F_{\alpha} \{ f(x) \} ; \hspace{1cm} (19)
\]

\[
F_{\alpha} \{ f(x) E_{\alpha} \left( -i^{\alpha} x^{\alpha} \omega^{\alpha} \right) \} = f_{\omega^{\alpha}}^{(\alpha)} (\omega-\omega_{0}) . \hspace{1cm} (20)
\]

4. Two illustrative examples

In this section, we give some applications of the local Yang-Fourier transforms to the fractal signals.

**Example 1.** Let a non-periodic signal \( X(t) \) be defined by the relation

\[
X(t) = \begin{cases} A_{0}, & -t_{0} \leq t < t_{0} ; \\ 0, & \text{else}. \end{cases} \hspace{1cm} (21)
\]

Taking the Yang-Fourier transforms, we have

\[
X_{\omega^{\alpha}}^{F\alpha}(\omega) = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} X(t) E_{\alpha} \left( -i^{\alpha} \omega^{\alpha} x^{\alpha} \right) (dx)^{\alpha}
\]

\[
= \frac{1}{\Gamma(1+\alpha)} \int_{-t_{0}}^{t_{0}} A E_{\alpha} \left( -i^{\alpha} \omega^{\alpha} x^{\alpha} \right) (dx)^{\alpha}
\]

\[
= \left. A E_{\alpha} \left( -i^{\alpha} \omega^{\alpha} x^{\alpha} \right) \right|_{-t_{0}}^{t_{0}}.
\]

Taking into account [6, 7] \( E_{\alpha} \left( -i^{\alpha} x^{\alpha} \right) = \cos_{\alpha} x^{\alpha} - i^{\alpha} \sin_{\alpha} x^{\alpha} \), we have

\[
X_{\omega^{\alpha}}^{F\alpha}(\omega) = \frac{2A \sin_{\alpha} \omega^{\alpha} t_{0}^{\alpha}}{\omega^{\alpha}} = 2A t_{0}^{\alpha} \sin_{\alpha C} \omega^{\alpha} t_{0}^{\alpha} . \hspace{1cm} (22)
\]

where \( \sin_{\alpha C} \omega^{\alpha} t_{0}^{\alpha} = \frac{\sin_{\alpha} \omega^{\alpha} t_{0}^{\alpha}}{\omega^{\alpha} t_{0}^{\alpha}} \).

Hence we obtain the pairs

\[
X(t) = \begin{cases} A_{0}, & -t_{0} \leq t < t_{0} ; \\ 0, & \text{else}. \end{cases} \hspace{1cm} \Leftrightarrow 2A t_{0}^{\alpha} \sin_{\alpha C} \omega^{\alpha} t_{0}^{\alpha} . \hspace{1cm} (23)
\]

**Example 2.** Let a non-periodic signal \( X(t) \) be defined by the relation

\[
X(t) = \frac{2 \sin_{\alpha} \omega_{0}^{\alpha} t^{\alpha}}{\omega_{0}^{\alpha} t^{\alpha}} . \hspace{1cm} (24)
\]

From (4.4), we directly obtain the relation
\[ X^{F,\alpha}_{\omega} (\omega) = \begin{cases} 1, & -\omega_0 \leq \omega < \omega_0; \\ 0, & \text{else.} \end{cases} \]  

Therefore we get the pairs

\[ X(t) = \frac{2 \sin \omega_0 t^\alpha}{\omega_0 t^\alpha} \leftrightarrow X^{F,\alpha}_{\omega} (\omega) = \begin{cases} 1, & -\omega_0 \leq \omega < \omega_0; \\ 0, & \text{else.} \end{cases} \]  

5. Conclusions

In this paper, we point out a novel method for processing the signals in fractal-time space are investigated using the Yang-Fourier transforms based on the local fractional calculus. Some typical examples are given to elaborate the signal process and reliable results.

Acknowledgements

This work is grateful for the finance supports of the National Natural Science Foundation of China (Grant No. 50904045).

References