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A convergence analysis of block accelerated over-relaxation iterative methods for weak block H-matrices to partition π

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Abstract

The aim of this paper is to establish the convergence of the block iteration methods such as the block successively accelerated over-relaxation method (BAOR) and the symmetric block successively accelerated over-relaxation method (BSAOR): Let $A \in \mathbf{C}^{\mathbf{m},\mathbf{m}}_{\pi,\mathbf{n}}$ be a weak block H-matrix to partition π, then for $0 \leqslant r \leqslant \omega \leqslant \frac{2}{1+\rho(|B_J(A)|)}$,

$$\rho(B_{\mathscr{L}_{r,\omega}}) \leqslant |1-\omega| + \omega\rho(|B_J(A)|), \quad \rho(B_{\mathscr{S}_{r,\omega}}) \leqslant [|1-\omega| + \omega\rho(|B_J(A)|)]^2,$$

and exact convergence and divergence domains for the block SOR and block SSOR iterative methods are obtained as it has been obtained to H-matrices. Based on these results, the main results in Bai [Parallel Computing 25 (1999)] and Cvetković [Appl. Numer. Math. 41 (2002)] can be improved.

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Keywords: Weak block diagonally dominant matrix to partition π; Weak block H-matrix to partition π; Spectral radius; Generalized ultrametric matrix

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1. Introduction

Consider a linear system $Ax = b$ where A is an $m \times m$ matrix, partitioned into block matrix form

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}, \tag{1.1}$$

where A_{ij} is of order $m_i \times m_j$, $1 \leqslant m_i \leqslant m$, $1 \leqslant m_j \leqslant m$. Let $\mathbf{C}^{\mathbf{m},\mathbf{m}}_{\pi,\mathbf{n}}$ denote the set of all matrices in $\mathbf{C}^{\mathbf{m},\mathbf{m}}$ which are of form (1.1) relative to some given block partitioning π (we will only consider π for which the diagonal blocks are square matrices). Here we shall consider simultaneously block versions of stationary iterative methods. Let $A \in \mathbf{C}^{\mathbf{m},\mathbf{m}}_{\pi,\mathbf{n}}$ be written as $A = D - L - U$, where $D = \operatorname{diag}(A_{11}, A_{22}, \ldots, A_{nn})$ and

$$L = (L_{ij}) = \begin{cases} -A_{ij} & \text{for } j < i, \\ 0 & \text{for } j \geqslant i \end{cases}, \quad U = (U_{ij}) = \begin{cases} -A_{ij} & \text{for } j > i \\ 0 & \text{for } j \leqslant i \end{cases} \tag{1.2}$$

are block matrices consisting of the block diagonal, strict block lower triangular, and strict block upper triangular parts of A respectively. Here the diagonal entries A_{ii} are assumed to be nonsingular.

The block Jacobi iteration matrix is $B_J = D^{-1}(L + U)$, the block Gauss-Seidel iteration matrix is $B_{GS} = (D - L)^{-1}U$, the block successive over-relaxation method (BSOR) iteration matrix is $B_{\mathscr{L}_\omega} = (D - \omega L)^{-1}[(1 - \omega)D + \omega U]$, the block SSOR iteration matrix is $B_{\mathscr{S}_\omega} = B_{\mathscr{U}_\omega} \cdot B_{\mathscr{L}_\omega}$, where $B_{\mathscr{U}_\omega} = (D - \omega U)^{-1}[(1 - \omega)D + \omega L]$, the block accelerated over-relaxation method (BAOR) iteration matrix is $B_{\mathscr{L}_{r,\omega}} = (D - rL)^{-1}[(1 - \omega)D + (\omega - r)L + \omega U]$, and the block SAOR iteration matrix is $B_{\mathscr{S}_{r,\omega}} = B_{\mathscr{U}_{r,\omega}} \cdot B_{\mathscr{L}_{r,\omega}}$, where $B_{\mathscr{U}_{r,\omega}} = (D - rU)^{-1}[(1 - \omega)D + (\omega - r)U + \omega L]$.

For a recent survey, the convergence has been discussed in the case where the coefficient matrix A is a block diagonally dominant matrix or a block H-matrix (cf. [1–4,6,10–14]). For block H-matrices, there are few estimations about the upper bound of the spectral radius of block iteration methods and few descriptions about the domains of convergence. And for many strictly diagonally dominant matrices, their partitions are not block H-matrices (cf. [13]). So we do not know the convergence of the block iterative methods for H-matrices. Here we use the concepts of *weak block diagonally dominant matrix to partition* π and *weak block H-matrix to partition* π including the block diagonally dominant matrix defined as Feingold and Varga [6] and Robert [14]. All the concepts have been already introduced elsewhere (see e.g. [17]). We will establish estimations about the spectral radius of block iteration matrices similar to Varga (cf. [15]) and Varga, Niethammer and Cai (cf. [16]), and the exact domains for the convergence and divergence of the BSOR and BSSOR iterative methods similar to Neumann and Varga (cf. [12]) and Neumaier and Varga (cf. [10]). From these results, it is easy to deduce that the classical block iterations are convergent for any partitioned block form of a pointwise H-matrix. Finally we give some applications to generalized ultrametric matrices.

In this paper, we confine ourselves to the vector norm $\|x\|_\infty = \max_i |x_i|$ and the matrix norm $\|A\|_\infty = \max_i \sum_{j=1}^{m} |a_{ij}|$, denote by $|A| = (|a_{ij}|)$ and by $J(A)$ the Jacobi iterative matrix of A and by $B_J(A)$ the block Jacobi iterative matrix of A.

2. Weak block diagonally dominant matrices and weak block H-matrices to partition π

Let $A \in \mathbf{C}^{\mathbf{m},\mathbf{m}}$. Then its comparison matrix $\mathscr{U}(A) = [b_{ij}]$ is defined by

$$b_{ij} = \begin{cases} |a_{ij}|, & i = j, \\ -|a_{ij}|, & i \neq j. \end{cases}$$

A is said to be an H-matrix if its comparison matrix is an M-matrix. Due to Fan [5], A is an H-matrix if and only if there exists a positive vector v such that $\mathscr{U}(A)v > 0$. Then there exists a diagonal matrix D such that AD is a strictly diagonally dominant matrix if and only if A is an H-matrix.

Let $A = [A_{ij}] \in \mathbf{C}^{\mathbf{m},\mathbf{m}}_{\pi,\mathbf{n}}$ and A_{ii} $(i = 1, 2, \ldots, n)$ be nonsingular. Then its block comparison matrix $\mathscr{U}_b(A) = [b_{ij}]$ is defined by

$$b_{ij} = \begin{cases} \|A_{ij}^{-1}\|_\infty^{-1}, & i = j, \\ -\|A_{ij}\|_\infty, & i \neq j. \end{cases}$$

We can reformulate the definition of block diagonally dominant matrix due to Feingold and Varga [6] as follows:

Definition 2.1 [6]. A is called a strictly block diagonally dominant matrix if its block comparison matrix $\mathscr{U}_b(A)$ exists and is strictly diagonally dominant.

$E = [E_{ij}]$ in $\mathbf{C}^{\mathbf{m},\mathbf{m}}_{\pi,\mathbf{n}}$ is said to be block diagonal if $E_{ij} = 0$ for all $i \neq j$.

Definition 2.2 [13]. $A \in \mathbf{C}^{\mathbf{m},\mathbf{m}}_{\pi,\mathbf{n}}$ is said to be a block H-matrix if there exist nonsingular block diagonal matrices $E_1, E_2 \in \mathbf{C}^{\mathbf{m},\mathbf{m}}_{\pi,\mathbf{n}}$ such that the block comparison matrix $\mathscr{U}_b(E_1AE_2)$ is an M-matrix.

Feigold and Varga used the term of strictly block diagonally dominant matrices, while Robert [14] introduced block diagonally dominant matrices. $A \in \mathbf{C}^{\mathbf{m},\mathbf{m}}_{\pi,\mathbf{n}}$ is called a block diagonally dominant matrix if $N(A)$ is a strict diagonally dominant matrix, where $N(A) = [b_{ij}]$ and

$$b_{ij} = \begin{cases} 1, & i = j, \\ -\|A_{ii}^{-1}A_{ij}\|_\infty, & i \neq j. \end{cases}$$

Polman [13] pointed out that some strictly diagonally dominant matrices partitioned into block forms are not block H-matrices.

Definition 2.3 [17]. $A \in \mathbf{C}^{\mathbf{m},\mathbf{m}}_{\pi,\mathbf{n}}$ is said to be weak block diagonally dominant to partition π if $D = \mathrm{diag}(A_{11}, A_{22}, \ldots, A_{nn})$ is nonsingular and $D^{-1}A$ is a strictly diagonally dominant matrix.

Note that if $A \in \mathbf{C}^{\mathbf{m},\mathbf{m}}_{\pi,\mathbf{n}}$ is strictly block diagonally dominant according to the definition by Feingold and Varga [6] or by Robert [14], then A must be a weak block diagonally dominant matrix to partition π (cf. [17]).

Definition 2.4 [17]. $A \in \mathbf{C}^{\mathbf{m},\mathbf{m}}_{\pi,\mathbf{n}}$ is called weak block H-matrix to partition π if there are nonsingular block diagonal matrices E_1 and E_2 such that E_1AE_2 is a weak block diagonally dominant matrix to partition π.

Partition π' is said to be finer than partition π in the sense that $n' > n$ and π has been obtained from π' by gathering block entries.

Lemma 2.5 [17]. *$A \in \mathbf{C}_{\pi,\mathbf{n}}^{\mathbf{m},\mathbf{m}}$ is weak block diagonally dominant to partition π if and only if every vector $x \in \mathbf{C}^{\mathbf{m}}$ with $x \neq 0$, $x^T = \left(x_1^T, x_2^T, \ldots, x_n^T\right)$, $x_i \in \mathbf{C}^{\mathbf{m_i}}$ satisfies that for every i, $\|x_i\|_\infty < \|x\|_\infty$ whenever $\sum_{j=1}^n A_{ij}x_j = 0$.*

Theorem 2.6. *Suppose that π' is a finer partition than π and $A \in \mathbf{C}_{\pi',\mathbf{n}'}^{\mathbf{m},\mathbf{m}}$ is a weak block diagonally dominant matrix to partition π', then for partitioned block matrix $A \in \mathbf{C}_{\pi,\mathbf{n}}^{\mathbf{m},\mathbf{m}}$, A is also a weak block diagonally dominant matrix to partition π.*

Proof. Suppose that $x \in \mathbf{C}^{\mathbf{m}}$ with $x \neq 0$, $x^T = \left(x_1^T, x_2^T, \ldots, x_n^T\right)$, $x_i \in \mathbf{C}^{\mathbf{m_i}}$ satisfies

$$\sum_{j=1}^{n} A_{ij}x_j = 0 \tag{*}$$

and the block entries A_{ij} of $A \in \mathbf{C}_{\pi',\mathbf{n}'}^{\mathbf{m},\mathbf{m}}$ may be written as

$$A_{ij} = \begin{pmatrix} \overline{A}_{k_{i-1}+1,k_{j-1}+1} & \overline{A}_{k_{i-1}+1,k_{j-1}+2} & \cdots & \overline{A}_{k_{i-1}+1,k_j} \\ \overline{A}_{k_{i-1}+2,k_{j-1}+1} & \overline{A}_{k_{i-1}+2,k_{j-1}+2} & \cdots & \overline{A}_{k_{i-1}+2,k_j} \\ \vdots & \vdots & \cdots & \vdots \\ \overline{A}_{k_i,k_{j-1}+1} & \overline{A}_{k_i,k_{j-1}+2} & \cdots & \overline{A}_{k_i,k_j} \end{pmatrix}, \quad i, j = 1, 2, \ldots, n.$$

Similarly, $x_i \in \mathbf{C}^{\mathbf{m_i}}$ may be written as $x_i^T = \left(\bar{x}_{k_{i-1}+1}^T, \bar{x}_{k_{i-1}+2}^T, \ldots, \bar{x}_{k_i}^T\right)$. Since $A \in \mathbf{C}_{\pi',\mathbf{n}'}^{\mathbf{m},\mathbf{m}}$ is a weak block diagonally dominant matrix to partition π', by Eq. (*),

$$\sum_{j=1}^{k_n} \overline{A}_{k_{i-1}+s,j}\bar{x}_j = 0, \quad s = 1, 2, \ldots, k_i - k_{i-1},$$

then we have $\|\bar{x}_{k_{i-1}+s}\|_\infty < \|x\|_\infty$, $s = 1, 2, \ldots, k_i - k_{i-1}$ and

$$\|x_i\|_\infty = \max_{k_{i-1}+1 \leqslant t \leqslant k_i} \|\bar{x}_t\|_\infty < \|x\|_\infty.$$

So $A \in \mathbf{C}_{\pi,\mathbf{n}}^{\mathbf{m},\mathbf{m}}$ is also a weak block diagonally dominant matrix to partition π. $\square$

According to the definition of weak block H-matrix, it is easy to get that

Corollary 2.7. *Suppose that π' is a finer partition than π and $A \in \mathbf{C}_{\pi',\mathbf{n}'}^{\mathbf{m},\mathbf{m}}$ is a weak block H-matrix to partition π', then for partitioned block matrix $A \in \mathbf{C}_{\pi,\mathbf{n}}^{\mathbf{m},\mathbf{m}}$ is also a weak block H-matrix to partition π.*

Corollary 2.8. *If A is a pointwise strictly diagonally dominant matrix, then for any block partition (the diagonal blocks are square), A is a weak block diagonally dominant matrix.*

Lemma 2.9 (Hu [7]). *Let M be an $n \times n$ matrix and N be an $n \times m$ matrix. If M is a strictly diagonally dominant matrix, then*

$$\|M^{-1}N\|_\infty \leqslant \max_i \frac{\sum_{j=1}^m |n_{ij}|}{|m_{ii}| - \sum_{j \neq i} |m_{ij}|}.$$

Furthermore, if M is a strictly diagonally dominant M-matrix and N is nonnegative, then

$$\min_i \frac{\sum_{j=1}^m n_{ij}}{m_{ii}-\sum_{j\neq i}|m_{ij}|} \leqslant \min_i \sum_{j=1}^m (M^{-1}N)_{ij} \leqslant \rho(M^{-1}N) \leqslant \max_i \frac{\sum_{j=1}^m n_{ij}}{m_{ii}-\sum_{j\neq i}|m_{ij}|}.$$

Let the coefficient matrix $A \in \mathbf{C}_{\pi,\mathbf{n}}^{\mathbf{m},\mathbf{m}}$ be written as $A = D - L - U$ where D, L and U are defined in (1.2).

Theorem 2.10. *Let $A \in \mathbf{C}_{\pi,\mathbf{n}}^{\mathbf{m},\mathbf{m}}$ be a weak block H-matrix to partition π, then for $0 \leqslant r \leqslant \omega \leqslant \frac{2}{1+\rho(|B_J(A)|)}$,*

$$\rho(B_{\mathscr{L}_{r,\omega}}) \leqslant |1-\omega| + \omega\rho(|B_J(A)|), \quad \rho(B_{\mathscr{S}_{r,\omega}}) \leqslant \{|1-\omega| + \omega\rho(|B_J(A)|)\}^2. \tag{2.1}$$

Proof. Since for nonsingular block diagonal matrices $E_1, E_2 \in \mathbf{C}_{\pi,\mathbf{n}}^{\mathbf{m},\mathbf{m}}$, $\rho(B_{\mathscr{L}_{r,\omega}}(E_1AE_2)) = \rho(B_{\mathscr{L}_{r,\omega}}(A))$, without loss of generality, we can assume that $A \in \mathbf{C}_{\pi,\mathbf{n}}^{\mathbf{m},\mathbf{m}}$ is a weak block diagonally dominant matrix to partition π. Then $D^{-1}A = I - D^{-1}B$ is a pointwise strictly diagonally dominant matrix, $\|D^{-1}B\|_\infty < 1$ and $\||D^{-1}B|\|_\infty = \|D^{-1}B\|_\infty < 1$ where $A = D - L - U$ and $B = L + U$.

For nonnegative matrix $|D^{-1}B|$, by Perron–Frobenius theorem there is a positive vector $v = (v_1, v_2, \ldots, v_m)^T$ such that

$$|D^{-1}B|v = \rho(|D^{-1}B|)v = \rho(|B_J(A)|)v.$$

Define $E_0 = \mathrm{diag}(v_i)$ partitioned into form (1.1) and $G = D^{-1}AE_0 = (g_{ij})$. Then

$$\sum_{j\neq i}|g_{ij}| = \rho(|D^{-1}B|)|g_{ii}| = \rho(|B_J(A)|)|g_{ii}| \leqslant \||D^{-1}B|\|_\infty|g_{ii}| < |g_{ii}|.$$

G is a strictly diagonally dominant matrix and it is easy to verify that for $0 \leqslant r \leqslant \omega \leqslant \frac{2}{1+\rho(|B_J(A)|)}$,

$$|1-\omega||g_{ii}| + \omega\rho(|B_J(A)|)|g_{ii}| \leqslant |g_{ii}| \tag{2.2}$$

and notice that each element on the diagonal of $G \in \mathbf{C}_{\pi,\mathbf{n}}^{\mathbf{m},\mathbf{m}}$ is a diagonal matrix then

$$\begin{aligned}
|g_{ii}| - r\sum_{j<i}|g_{ij}| &\geqslant |g_{ii}| - \frac{2}{1+\rho(|B_J(A)|)}\sum_{j<i}|g_{ij}| \\
&= \frac{1}{1+\rho(|B_J(A)|)}\left\{|g_{ii}|(1+\rho(|B_J(A)|)) - 2\sum_{j<i}|g_{ij}|\right\} \\
&= \frac{1}{1+\rho(|B_J(A)|)}\left\{|g_{ii}| + \sum_{j\neq i}|g_{ij}| - 2\sum_{j<i}|g_{ij}|\right\} \\
&> \frac{1}{1+\rho(|B_J(A)|)}\left\{2\sum_{j>i}|g_{ij}|\right\} \geqslant 0.
\end{aligned} \tag{2.3}$$

By Lemma 2.9, we have

$$\begin{aligned}
\rho(B_{\mathscr{L}_{r,\omega}}(A)) &= \rho(B_{\mathscr{L}_{r,\omega}}(G)) \\
&\leqslant \|B_{\mathscr{L}_{r,\omega}}(G)\|_\infty
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \max_i \left\{ \frac{|1-\omega||g_{ii}| + (\omega - r)\sum_{j<i}|g_{ij}| + \omega\sum_{j>i}|g_{ij}|}{|g_{ii}| - r\sum_{j<i}|g_{ij}|} \right\} \\
&= \max_i \left\{ \frac{|1-\omega||g_{ii}| + \omega\sum_{j\neq i}|g_{ij}| - r\sum_{j<i}|g_{ij}|}{|g_{ii}| - r\sum_{j<i}|g_{ij}|} \right\} \\
&\leqslant \max_i \left\{ \frac{|1-\omega||g_{ii}| + \omega\rho(|B_J(A)|)|g_{ii}| - r\sum_{j<i}|g_{ij}|}{|g_{ii}| - r\sum_{j<i}|g_{ij}|} \right\}.
\end{aligned} \tag{2.4}
$$

By the inequality $\frac{c-b}{a-b} \leqslant \frac{c}{a}$ for $a \geqslant c \geqslant 0, \quad b \geqslant 0$, together with (2.2)–(2.4), one has

$$\rho(B_{\mathscr{L}_{r,\omega}}(A)) = \rho(B_{\mathscr{L}_{r,\omega}}(G)) \leqslant |1-\omega| + \omega\rho(|B_J(A)|). \tag{2.5}$$

Similarly, we can get the following estimations:

$$\rho(B_{\mathscr{U}_{r,\omega}}(A)) = \rho(B_{\mathscr{U}_{r,\omega}}(G)) \leqslant \|B_{\mathscr{U}_{r,\omega}}(G)\|_\infty \leqslant |1-\omega| + \omega\rho(|B_J(A)|) \tag{2.6}$$

and

$$\rho(B_{\mathscr{S}_{r,\omega}}(A)) \leqslant \|B_{\mathscr{U}_{r,\omega}}(G)\|_\infty \cdot \|B_{\mathscr{L}_{r,\omega}}(G)\|_\infty \leqslant \{|1-\omega| + \omega\rho(|B_J(A)|)\}^2. \qquad \square \tag{2.7}$$

Corollary 2.11. *Suppose that* A *is a pointwise* H*-matrix partitioned into* $\mathbf{C}^{\mathbf{m},\mathbf{m}}_{\pi,\mathbf{n}}$, *or* $A \in \mathbf{C}^{\mathbf{m},\mathbf{m}}_{\pi,\mathbf{n}}$ *is a strictly block diagonally dominant matrix to partition* π *defined as Feingold and Varga or by Robert, then the BJ and the BGS are convergent. Furthermore, the BAOR and the BSAOR iterations are convergent for* $0 < r \leqslant \omega < \frac{2}{1+\rho(|B_J(A)|)}$.

In the following, we will give a characterization for weak block H-matrices to partition π. Let $\Omega_b(A) = \{B = (B_{ij}) \in \mathbf{C}^{\mathbf{m},\mathbf{m}}_{\pi,\mathbf{n}} : |B_J(B)| = |B_J(A)|\}$, we have

Theorem 2.12. *For any irreducible matrix* $A \in \mathbf{C}^{\mathbf{m},\mathbf{m}}_{\pi,\mathbf{n}}$, $n \geqslant 2$, *the following are equivalent*:

(i) *A is a weak block H-matrix to partition π*;
(ii) *for each* $B \in \Omega_b(A)$, $\rho(B_J(B)) \leqslant \rho(|B_J(B)|) < 1$;
(iii) *for each* $B \in \Omega_b(A)$ *and for each* ω *satisfying*

$$0 < \omega < \frac{2}{1+\rho(|B_J(B)|)},$$

the associated BSSOR iteration matrix $B_{\mathscr{S}_\omega}(B)$ *for* B *satisfies*

$$\rho(B_{\mathscr{S}_\omega}(B)) < 1.$$

Proof. Suppose that A is a weak block H-matrix to partition π, then there are nonsingular block diagonal matrices E_1 and E_2 such that E_1AE_2 is a weak block diagonally dominant matrix to partition π. Note that by Definition 2.3, $\rho(B_J(A)) \leqslant \rho(|B_J(A)|) = \rho(|B_J(E_1AE_2)|) \leqslant \||B_J(E_1AE_2)|\|_\infty < 1$. Then for each $B \in \Omega_b(A)$, $\rho(B_J(B)) \leqslant \rho(|B_J(B)|) = \rho(|B_J(A)|) < 1$ and (i) implies (ii).

For each $B = D_B - L_B - U_B \in \Omega_b(A)$, since $\rho(|B_J(B)|) < 1$, then $I - |D_B^{-1}(L_B + U_B)|$ is an M-matrix and $I - D_B^{-1}(L_B + U_B)$ is an H-matrix. Thus there exists a diagonal matrix with positive diagonal entries denoted by D_0 partitioned into (1.1) such that $[I - D_B^{-1}(L_B + U_B)]D_0 = D_B^{-1}BD_0$ is strictly diagonally dominant. Then B is a weak block H-matrix to partition π. In Theorem 2.10, set $r = \omega$, then $\rho(B_{\mathscr{S}_\omega}(B)) \leqslant |1-\omega| + \omega\rho(|B_J(B)|) = |1-\omega| + \omega\rho(|B_J(A)|) < 1$ and (i) implies (iii).

Similarly to the above proof, from it follows that A has to be a weak block H-matrix to partition π, so (ii) implies (i).

Furthermore, define $C = I - |D^{-1}(L+U)| = I - L_C - U_C = (c_{ij})$ where L_C and U_C are nonnegative matrices, then $\rho(B_J(C)) = \rho(|B_J(A)|) \geqslant 1$. Since A is irreducible, $L_C + U_C$ is also irreducible. And by Perron-Frobenius theorem, there is a positive vector denoted by $v = (v_1, v_2, \ldots, v_m)^T$ such that

$$(L_C + U_C)v = \rho(L_C + U_C)v = \rho(|B_J(A)|)v \geqslant v.$$

Define $E_0 = \operatorname{diag}(v_i)$ partitioned into form (1.1) and $D_G = E_0$, $L_G = L_C E_0$ and $U_G = U_C E_0$. Then $G = (I - L_C - U_C)E_0 = (g_{ij}) \in \mathbf{C}_{\pi,\mathbf{n}}^{\mathbf{m},\mathbf{m}}$ can be written as $G = D_G - L_G - U_G$ and satisfies that

$$\sum_{j \neq i} |g_{ij}| \geqslant g_{ii} \quad \text{for all } i.$$

By Lemma 2.9, for sufficient small $\omega \in (0, 1)$ such that $D_G - \omega L_G$ and $D_G - \omega U_G$ are strictly diagonally dominant and $B_{\mathscr{L}_\omega}(G)$ is a nonnegative matrix, and notice that each element on the diagonal of $G \in \mathbf{C}_{\pi,\mathbf{n}}^{\mathbf{m},\mathbf{m}}$ is a diagonal matrix, then we have

$$\begin{aligned}\rho(B_{\mathscr{L}_\omega}(C)) = \rho(B_{\mathscr{L}_\omega}(G)) &= \rho[(D_G - \omega L_G)^{-1}((1-\omega)D_G + \omega U_G)] \\ &\geqslant \min_i \sum_j (B_{\mathscr{L}_\omega}(G))_{ij} \\ &\geqslant \min_i \left\{ \frac{g_{ii} - \omega g_{ii} + \omega \sum_{j>i} |g_{ij}|}{g_{ii} - \omega \sum_{j<i} |g_{ij}|} \right\} \\ &\geqslant 1.\end{aligned}$$

Similarly,

$$\rho(B_{\mathscr{U}_\omega}(C)) = \rho(B_{\mathscr{U}_\omega}(G)) \geqslant \min_i \left\{ \frac{g_{ii} - \omega g_{ii} + \omega \sum_{j>i} |g_{ij}|}{g_{ii} - \omega \sum_{j<i} |g_{ij}|} \right\} \geqslant 1.$$

Hence for nonnegative matrix $B_{\mathscr{L}_\omega}(G)$ and $B_{\mathscr{U}_\omega}(G)$, one has that

$$\begin{aligned}\rho(B_{\mathscr{S}_\omega}(C)) = \rho(B_{\mathscr{S}_\omega}(G)) &\geqslant \min_i \sum_j (B_{\mathscr{S}_\omega}(G))_{ij} \\ &\geqslant \min_i \sum_j (B_{\mathscr{U}_\omega}(G))_{ij} \cdot \min_i \sum_j (B_{\mathscr{L}_\omega}(G))_{ij} \\ &\geqslant 1.\end{aligned}$$

So the BSSOR of C is not convergent. Note that $C \in \Omega_b(A)$. Hence there is a $C \in \Omega_b(A)$ but $\rho(B_{\mathscr{S}_\omega}(C)) \geqslant 1$ and (iii) implies (i). $\square$

3. The exact domains for the convergence and divergence of the BSOR and BSSOR iterative methods

To consider the exact domains for the convergence and divergence of the BSOR and BSSOR iterative methods, we first recall the following results proved by Neumann and Varga (cf. [12]) and Neumaier and Varga (cf. [10]). Let

$$\mathscr{H}(v) = \left\{A \in \mathbf{C}^{\mathbf{m},\mathbf{m}}, m \text{ arbitrary} : A \text{ is an } H\text{-matrix with } \rho(|J(A)|) = v\right\}.$$

Theorem 3.1 [12]. *For each* $0 \leqslant v < 1$, $\rho(\mathscr{L}_\omega(A)) < 1$ *for all* $A \in \mathscr{H}(v)$ *if and only if* $0 < \omega < \frac{2}{1+\rho(|J(A)|)}$.

Set

$$\omega(v) = \begin{cases} 2, & 0 \leqslant v \leqslant \frac{1}{2}, \\ \frac{2}{1+\sqrt{2v-1}}, & \frac{1}{2} < v < 1. \end{cases}$$

Theorem 3.2 [10]. *For each matrix* $A \in \mathscr{H}(v)$, *and for each* ω *with* $0 < \omega < \omega(v)$, $\rho(\mathscr{S}_\omega(A)) < 1$. *On the other hand, for each* ω *with* $\omega \leqslant 0$ *or with* $\omega > \omega(v)$, *there is a matrix B in* $\mathscr{H}(v)$ *such that* $\rho(\mathscr{S}_\omega(A)) \geqslant 1$.

Let $\mathscr{H}_b(v) = \{A \in \mathbf{C}^{\mathbf{m},\mathbf{m}}_{\pi,\mathbf{n}}, m, n (m \geqslant n)$ arbitrary : A is a weak block H-matrix with $\rho(|B_J(A)|) = v\}$
and

$$\omega(v) = \begin{cases} 2, & 0 \leqslant v \leqslant \frac{1}{2}, \\ \frac{2}{1+\sqrt{2v-1}}, & \frac{1}{2} < v < 1. \end{cases}$$

For each v with $0 \leqslant v < 1$, and each matrix $A \in \mathscr{H}_b(v)$, notice that in the proof of Theorem 2.10, $\rho(|B_J(A)|) = \rho(|B_J(G)|) = \rho(|J(G)|) = v$ and G is a strictly diagonally dominant matrix satisfying

$$\rho(B_{\mathscr{L}_\omega}(G)) = \rho(B_{\mathscr{L}_\omega}(A)) = \rho(\mathscr{L}_\omega(G)), \quad \rho(B_{\mathscr{S}_\omega}(G)) = \rho(B_{\mathscr{S}_\omega}(A)) = \rho(\mathscr{S}_\omega(G)). \tag{3.1}$$

By Neumann and Varga [12] for each ω with $0 < \omega < \frac{2}{1+\rho(|B_J(A)|)}$, $\rho(B_{\mathscr{L}_\omega}(A)) = \rho(\mathscr{L}_\omega(G)) < 1$ and by Neumaier and Varga [10] for each ω with $0 < \omega < \omega(v)$, $\rho(B_{\mathscr{S}_\omega}(A)) = \rho(\mathscr{S}_\omega(G)) < 1$.

By considering the same counterexamples defined in [10,12], one obtains the following result.

Theorem 3.3. *For each v with* $0 \leqslant v < 1$, *for each matrix* $A \in \mathscr{H}_b(v)$, *for each* ω *with* $0 < \omega < \frac{2}{1+\rho(|B_J(A)|)}$, $\rho(B_{\mathscr{L}_\omega}(A)) < 1$ *and for each* ω *with* $0 < \omega < \omega(v)$, $\rho(B_{\mathscr{S}_\omega}(A)) < 1$. *On the other hand, for each* ω *with* $\omega \leqslant 0$ *or with* $\omega > \frac{2}{1+\rho(|B_J(A)|)}$, *there is a matrix B in* $\mathscr{H}_b(v)$ *such that* $\rho(B_{\mathscr{L}_\omega}(A)) \geqslant 1$, *and for each* ω *with* $\omega \leqslant 0$ *or with* $\omega > \omega(v)$, *there is a matrix B in* $\mathscr{H}_b(v)$ *such that* $\rho(B_{\mathscr{S}_\omega}(A)) \geqslant 1$.

4. Applications

We recall from [3] that for a matrix $A \in \mathbf{C}^{\mathbf{m},\mathbf{m}}_{\pi,\mathbf{n}}$, its block matrix multisplitting (BMM) denotes any collection of triples $(D - L_k, U_k, E_k)$ for which $A = D - L_k - U_k$, $D = \text{diag}(A_{11}, A_{22}$

$\dots, A_{nn})$ is nonsingular, $L_k \in \mathbf{C}_{\pi,\mathbf{n}}^{\mathbf{m},\mathbf{m}}$ is a strictly block lower triangular matrix, $-U_k \in \mathbf{C}_{\pi,\mathbf{n}}^{\mathbf{m},\mathbf{m}}$ is a submatrix of A with zero diagonal blocks, $k = 1, 2, \dots, \alpha$, and $\sum_{k=1}^{\alpha} E_k = I$ with $\sum_{k=1}^{\alpha}[E_k] \leqslant I$, where $E_k = (E_{ij}^{(k)})$ and $[E_k] = (\|E_{ij}^{(k)}\|)$.

Denote $L_k(r,\omega) = (D - rL_k)^{-1}[(1-\omega)D + (\omega - r)L_k + \omega U_k]$, $k = 1, 2, \dots, \alpha$, then the iterative method of BMM (Method I in [3]) can be written as: x^0 is an initial vector and for $p = 0, 1, 2, \dots$,

$$y^{p,k} = L_k(r,\omega)x^p + \omega(D - rL_k)^{-1}b, \quad k = 1, 2, \dots, \alpha, \quad \text{and} \quad x^{p+1} = \sum_{k=1}^{\alpha} E_k y^{p,k},$$

and the iterative matrix is $L_{\text{BMM}} = \sum_{k=1}^{\alpha} E_k L_k(r,\omega)$.

Method II in [3] can be written as

$$\begin{aligned} &y^{p,k} = L_k(r,\omega)x^p + \omega(D - rL_k)^{-1}b, \quad x^{p,k} = \mu y^{p,k} + (1-\mu)x^p, \\ &\quad k = 1, 2, \dots, \alpha, \quad 0 < \mu \leqslant 1, \end{aligned}$$

and

$$x^{p+1} = \sum_{k=1}^{\alpha} E_k x^{p,k}.$$

The iterative matrix is $\overline{L}_{\text{BMM}} = \mu \sum_{k=1}^{\alpha} E_k L_k(r,\omega) + (1-\mu)I$.

By arguing as in the proof of Theorem 2.10, one easily shows that if $A \in \mathbf{C}_{\pi,\mathbf{n}}^{\mathbf{m},\mathbf{m}}$ is a weak block H-matrix, then for $0 \leqslant r \leqslant \omega \leqslant \frac{2}{1+\rho(|B_J(A)|)}$, one has that $\rho(L_k(r,\omega)) \leqslant |1-\omega| + \omega\rho(|B_J(A)|)$. Theorem 4.1 and Theorem 4.2 in [3] can be improved together as follows:

Theorem 4.1. *Suppose that $A \in \mathbf{C}_{\pi,\mathbf{n}}^{\mathbf{m},\mathbf{m}}$ is a weak block H-matrix to partition π. Then for $0 < r \leqslant \omega < \frac{2}{1+\rho(|B_J(A)|)}$ and $0 < \mu \leqslant 1$,*

$$\begin{aligned} &\rho(L_{\text{BMM}}) \leqslant |1-\omega| + \omega\rho(|B_J(A)|), \\ &\rho(\overline{L}_{\text{BMM}}) \leqslant \mu\{|1-\omega| + \omega\rho(|B_J(A)|)\} + (1-\mu). \end{aligned} \tag{4.1}$$

And Method I and Method II are convergent.

Proof. First from the definition of E_k, it is not difficult to verify that E_k $(k = 1, 2, \dots, \alpha)$ are block diagonal matrices and $E_{ii}^{(k)} = \lambda_{i,k} I_{m_i}$ for some positive number $\lambda_{i,k}$, $0 \leqslant \lambda_{i,k} \leqslant 1$. Thus $E_k E = E E_k$ for arbitrary block diagonal matrix $E \in \mathbf{C}_{\pi,\mathbf{n}}^{\mathbf{m},\mathbf{m}}$. On the other hand, for nonnegative matrices $N_k \in \mathbf{C}_{\pi,\mathbf{n}}^{\mathbf{m},\mathbf{m}}$, $k = 1, 2, \dots, \alpha$, by Perron–Frobenius Theorem, one has that $\rho\left(\sum_{k=1}^{\alpha} E_k N_k\right) \leqslant \max_k \rho(N_k)$.

Since $A \in \mathbf{C}_{\pi,\mathbf{n}}^{\mathbf{m},\mathbf{m}}$ is a weak block H-matrix to partition π, there exist block diagonal matrices H_1, H_2 such that $A_0 = H_1 A H_2$ is weak block diagonally dominant to partition π. Similar to the proof of Theorem 2.10, there exists a block diagonal matrix $E_0 \in \mathbf{C}_{\pi,\mathbf{n}}^{\mathbf{m},\mathbf{m}}$ such that $G_0 = D_0^{-1} A_0 E_0 = (g_{ij}^{(0)})$ satisfies that $\sum_{j \neq i} |g_{ij}^{(0)}| = \rho(|D_0^{-1} B_0|)|g_{ii}^{(0)}|$, where $A_0 = D_0 - B_0$ and $D_0 = \operatorname{diag}(A_{11}^{(0)}, A_{22}^{(0)}, \dots, A_{n,n}^{(0)})$ and

$$\begin{aligned} \rho(\mathscr{L}_k(r,\omega)(G_0)) &\leqslant \rho(|\mathscr{L}_k(r,\omega)(G_0)|) \\ &\leqslant \rho(\mathscr{L}_k(r,\omega)\mathscr{U}(G_0))) \\ &\leqslant \|(|D_{G_0}| - r|L_{G_0}|)^{-1}[(1-\omega)|D_{G_0}| + (\omega - r)|L_{G_0}| + \omega|U_{G_0}|]\|_\infty \\ &\leqslant |1-\omega| + \omega\rho(|B_J(A)|). \end{aligned}$$

Note that for any nonsingular block diagonally matrix $E \in \mathbf{C}_{\pi,\mathbf{n}}^{\mathbf{m},\mathbf{m}}$, $L_k(r,\omega)(EA) = L_k(r,\omega)(A)$ and $A = H_1^{-1}D_0G_0E_0^{-1}H_2^{-1}$, thus

$$\begin{aligned}\rho\left(\sum_{k=1}^{\alpha} E_k L_k(r,\omega)(A)\right) &= \rho\left(\sum_{k=1}^{\alpha} E_k H_2 E_0 L_k(r,\omega)(G_0)E_0^{-1}H_2^{-1}\right)\\ &= \rho\left(\sum_{k=1}^{\alpha} E_k L_k(r,\omega)(G_0)\right)\\ &\leqslant \rho\left(\sum_{k=1}^{\alpha} E_k L_k(r,\omega)(\mathscr{U}(G_0))\right).\end{aligned}$$

Hence

$$\rho\left(\sum_{k=1}^{\alpha} E_k L_k(r,\omega)(A)\right) \leqslant \rho\left(\sum_{k=1}^{\alpha} E_k L_k(r,\omega)(\mathscr{U}(G_0))\right) \leqslant |1-\omega| + \omega\rho(|B_J(A)|).$$

Similarly, we can get

$$\rho(\overline{L}_{\mathrm{BMM}}) \leqslant \mu\{|1-\omega| + \omega\rho(|B_J(A)|)\} + (1-\mu). \qquad \square$$

Remark. The results in [4] can also be extended to weak block H-matrices to partition π.

Next, we consider the applications of weak block H-matrices in the numerical analysis of generalized ultrametric matrices. McDonald, Neumann, Schneider and Tsatsomeros [8], Nabben and Varga [9] introduced the new class of generalized ultrametric matrices in studying inverse M-matrices problem. Inverse M-matrices have a strong intrinsic value on inverse physical problems and on regularizing ill-posed problems.

Definition 4.2 [8, 9]. A matrix $A = (a_{ij}) \in \mathscr{R}^{n,n}$ is called a generalized ultrametric matrix if

(i) A has nonnegative entries;
(ii) $a_{ij} \geqslant \min\{a_{i,k}; a_{k,j}\}$ for all $i, k, j = 1, 2, \dots, n$;
(iii) $a_{i,i} \geqslant \max\{a_{i,k}; a_{k,i}\}$ for all $i = 1, 2, \dots, n$;
(iv) each triple $\{q, s, t\}$ can be reordered as a triple $\{i, j, k\}$ such that
 (iv.i) $a_{j,k} = a_{i,k}$ and $a_{k,j} = a_{k,i}$, (iv.ii) $\max\{a_{i,j}; a_{j,i}\} \geqslant \max\{a_{i,k}; a_{k,i}\}$,

where, if $n = 1$, (iii) is interpreted as $a_{1,1} \geqslant 0$. A matrix A is called a strictly generalized ultrametric matrix if the above conditions hold with strict inequality in (iii). Especially, if A is a symmetric generalized ultrametric matrix, A is a ultrametric matrix.

Let A be a strictly generalized ultrametric matrix and $\tau(A) = \min\{a_{ij} : i, j \in N\}$, $\omega(A) = \min\{a_{ji} : a_{ij} = \tau(A)\}$, $\delta(A) = \omega(A) - \tau(A)$, $\mu(A) = \max\{a_{ij}\}$, then there exists a permutate matrix P such

$$\overline{A} := P^TAP = \begin{bmatrix} C & 0 \\ \delta(A)\xi_{n-r}\xi_r^T & D \end{bmatrix} + \tau(A)\xi_n\xi_n{}^T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$C \in \mathbf{C}^{\mathbf{r},\mathbf{r}}$ and $D \in \mathbf{C}^{\mathbf{n-r},\mathbf{n-r}}$ are also strictly generalized ultrametric matrices and A^{-1}, C^{-1}, D^{-1} are strictly diagonally dominant M-matrices, where

$$A_{11} = C + \tau(A)\xi_r\xi_r{}^T, \quad A_{22} = D + \tau(A)\xi_{n-r}\xi_{n-r}^T, \quad \xi_r = (1, 1, \ldots, 1)^T \in \mathbf{C^r},$$

$\xi_{n-r} = (1, 1, \ldots, 1)^T \in \mathbf{C^{n-r}}$, $\xi_n = (1, 1, \ldots, 1)^T \in \mathbf{C^n}$ (cf. [9]). And $\overline{A}$ is a weak block H-matrix (cf. [17]). Hence the classic block iterative methods are convergent.

Example 4.1. Let A be a generalized ultrametric matrix written as

$$A = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ 1 & 3 & 2 & 2 & 2 \\ 1 & 1 & 5 & 4 & 3 \\ 1 & 1 & 1 & 5 & 3 \\ 1 & 1 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 4 & 3 & 2 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

$\rho(J(A)) > 1.2$. But for any block partition such that $A \in \mathbf{C}^{\mathbf{5,5}}_{\pi,\mathbf{2}}$, A is a weak block H-matrix to partition π. For example, let

$$A = \left[\begin{array}{cc|ccc} 2 & 2 & 2 & 2 & 2 \\ 1 & 3 & 2 & 2 & 2 \\ \hline 1 & 1 & 5 & 4 & 3 \\ 1 & 1 & 1 & 5 & 3 \\ 1 & 1 & 1 & 1 & 3 \end{array}\right],$$

$$D^{-1}A = \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & 1 \end{bmatrix}$$

is a pointwise H-matrix and $\rho(|B_J(A)|) \approx 0.5774$. So for $\omega \in \left(0, \frac{2}{1+\rho(|B_J(A)|)}\right)$, the BSOR is convergent. Furthermore, $D^{-1}A$ is 2-cyclic matrix, according to Young [18], $\omega_{\text{opt}} = \frac{2}{1+\sqrt{1-\rho(|B_J(A)|)^2}} \approx 1.1010$, $\rho(B_{\mathscr{L}_\omega}) \approx 0.1010$:

ω	0.2	0.4	0.6	0.8	1	ω_{opt}	1.2	1.4	1.6
$\rho(B_{\mathscr{L}_\omega})$	0.9102	0.8075	0.6176	0.5319	0.3333	0.1010	0.2000	0.4000	0.6000

Example 4.2. Let A be a ultrametric matrix written as

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 4 & 1 & 3 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

$\rho(J(A)) > 1.5$. But for any block partition such that $A \in \mathbf{C}^{\mathbf{5,5}}_{\pi,\mathbf{2}}$, A is a weak block H-matrix to partition π. For example, let

$$A = \left[\begin{array}{cc|ccc} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ \hline 1 & 1 & 4 & 1 & 3 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 3 & 1 & 4 \end{array}\right],$$

$$D^{-1}A = \begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{12} & \frac{1}{12} & 1 & 0 & 0 \\ \frac{5}{12} & \frac{5}{12} & 0 & 1 & 0 \\ \frac{1}{12} & \frac{1}{12} & 0 & 0 & 1 \end{bmatrix}$$

is a pointwise H-matrix and $\rho(|B_J(A)|) \approx 0.6236$. So for $\omega \in \left(0, \frac{2}{1+\rho(|B_J(A)|)}\right)$, the BSOR is convergent. Furthermore, $D^{-1}A$ is 2-cyclic matrix, according to Young [18], $\omega_{\text{opt}} = \frac{2}{1+\sqrt{1-\rho(|B_J(A)|)^2}} \approx 1.1225$, $\rho(B_{\mathscr{L}_\omega}) \approx 0.1225$:

ω	0.2	0.4	0.6	0.8	1	ω_{opt}	1.2	1.4	1.6	2
$\rho(B_{\mathscr{L}_\omega})$	0.8750	0.7744	0.7168	0.5799	0.3889	0.1225	0.2000	0.4000	0.6000	1

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