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Note

Note on the degree sequences of k-hypertournaments ☆

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Abstract

We obtain a criterion for determining whether or not a non-decreasing sequence of non-negative integers is a degree sequence of some k-hypertournament on n vertices. This result generalizes the corresponding theorems on tournaments proposed by Landau [H.G. Landau, On dominance relations and the structure of animal societies. III. The condition for a score structure, Bull. Math. Biophys. 15 (1953) 143–148] in 1953.

Keywords: k-hypertournament; Degree sequence; Consecutive path; Key vertex; Key arc

1. Introduction

Given two positive integers n and k with $n > k > 1$, a k-hypertournament H on n vertices is a pair (V, A), where V is a set of n vertices and A is a set of k-tuples of vertices, called arcs, such that for any k-subset W of V, A contains exactly one of the $k!$ possible k-tuples whose entries belong to W. Clearly, a 2-hypertournament is a tournament. If $e = (x_1, x_2, \ldots, x_k)$, then we call $\{x_1, x_2, \ldots, x_k\}$ the underlying vertex set of e, denoted by V_e.

Let $a = (x_1, \ldots, x_k)$ be an arc of H. We call x_i the ith entry of a; the $(i+1)$th entry of a, x_{i+1}, is called the successor of x_i, and x_i the predecessor of x_{i+1} in a, $1 \leqslant i \leqslant k-1$. It is obvious that x_k has no successor, and x_1 has no predecessor in a. Define a function ρ on a by

$$\rho(x, a) = \begin{cases} k - i & \text{if } x \in a \text{ and } x \text{ is the } i\text{th entry of } a, \\ 0 & \text{if } x \notin a. \end{cases}$$

For $v \in V(H)$, we denote $d_H^+(v) = \sum_{a \in H} \rho(v, a)$ (or simply $d^+(v)$) the degree of v in H. For $i < j$,

$$a(x_i, x_j) = (x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_k)$$

denotes a new arc obtained from a by exchanging x_i and x_j in a.

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A k-hypertournament $H(V, A)$ is said to be transitive if we can label $V(H)$ by $v_1, v_2, \ldots, v_n$ in such a order that: $i < j$ if and only if v_i precedes v_j in each arc containing v_i and v_j.

Let $S = (s_1, s_2, \ldots, s_n)$ be a non-decreasing sequence of non-negative integers. For $1 \leqslant i < j \leqslant n$, we denote $S(s_i^+, s_j^-) = (s_1, s_2, \ldots, s_i + 1, s_{i+1}, \ldots, s_{j-1}, s_j - 1, \ldots, s_n)$. And $S'(s_i^+, s_j^-) = (s_1', s_2', \ldots, s_n')$ will denote a permutation of $S(s_i^+, s_j^-)$ such that $s_1' \leqslant s_2' \leqslant \cdots \leqslant s_n'$.

The degree sequence of a k-hypertournament is a non-decreasing sequence of non-negative integers $(s_1, s_2, \ldots, s_n)$, where each s_i is the degree of some vertex in $V(H)$. When $k = 2$, the degree sequence is identical to the score-list in [3, Chapter 7]. In some papers, the score-list is also called score sequence. In 1953, Landau [7] proved that some rather obvious necessary conditions for a non-decreasing sequence of non-negative integers to be the score sequence for some tournament are also sufficient. Namely, the sequence $S = (s_1, s_2, \ldots, s_n)$ is a score sequence if and only if $\sum_{i=1}^{r} s_i \geqslant \binom{r}{2}$, $1 \leqslant r \leqslant n$, with equality for $r = n$. According to [5], there are now several proofs of this fundamental result in tournament theory. Many of these existing proofs are discussed in a 1996 survey by Reid [8]. In [9], Zhou et al. succeeded in generalizing the Landau's theorem to the hypertournaments under a different definition of vertex degree. In [10], Zhou and Zhang also raised the following conjecture and proved the case $k = 3$.

Conjecture 1. Given two positive integers n and k with $n > k > 1$, a non-decreasing sequence $S = (s_1, s_2, \ldots, s_n)$ of non-negative integers is a degree sequence of some k-hypertournament if and only if

$$\sum_{i=1}^{r} s_i \geqslant \binom{r}{2}\binom{n-2}{k-2} \quad \forall 1 \leqslant r \leqslant n,$$

with equality for $r = n$.

In this paper, we settle this conjecture in affirmative. Other references on k-hypertournaments can be found in [1,2,4,6].

2. Main result

The main result of this paper is the following theorem.

Theorem 1. *Given two positive integers n and k with $n > k > 1$, a non-decreasing sequence $S = (s_1, s_2, \ldots, s_n)$ of non-negative integers is a degree sequence of some k-hypertournament if and only if*

$$\sum_{i=1}^{r} s_i \geqslant \binom{r}{2}\binom{n-2}{k-2} \quad \forall 1 \leqslant r \leqslant n, \tag{$\otimes$}$$

with equality for $r = n$.

In order to prove Theorem 1, we need some lemmas and definitions as follows.

Lemma 1 (*Zhou and Zhang [10, Lemma 2.3]*). *If a non-decreasing sequence $S = (s_1, s_2, \ldots, s_n)$ of non-negative integers is a degree sequence of some k-hypertournament, then*

$$\sum_{i=1}^{r} s_i \geqslant \binom{r}{2}\binom{n-2}{k-2} \quad \forall 1 \leqslant r \leqslant n,$$

with equality for $r = n$.

Lemma 2 (*Zhou and Zhang [10, Lemma 2.3]*). *A non-decreasing sequence $S = (s_1, s_2, \ldots, s_n)$ of non-negative integers is a degree sequence of some transitive k-hypertournament if and only if $s_i = (i-1)\binom{n-2}{k-2}$, for all $1 \leqslant i \leqslant n$.*

Definition 1. Given a k-hypertournament $H(V, A)$, x and y being two distinct vertices in H. If we can choose t arcs $a_1, \ldots, a_t$ (repeating allowed) and $t-1$ distinct vertices $z_1, z_2, \ldots, z_{t-1}$ which are different with x and y, such that x is the predecessor of z_1 in a_1, z_i is the predecessor of z_{i+1} in a_{i+1}, $1 \leqslant i \leqslant t-2$, z_{t-1} is the predecessor of y in a_t, and $a_i \neq a_{i+1}$, $1 \leqslant i \leqslant t-1$, then we say that there is a consecutive path from x to y, denoted by

$$P(x, y) = x a_1 z_1 a_2 z_2 \cdots z_{t-2} a_{t-1} z_{t-1} a_t y.$$

y is called a reachable vertex from x, or simply reachable from x. $P(x, y)$ can be simply written as P_y if x is given. Denote all the consecutive paths from x to y in H by $\mathscr{P}_H(x, y)$.

Example 1. Let $n = 5$, $k = 4$. Consider a 4-hypertournament $H(V, A)$ on five vertices, where $V = \{1, 2, 3, 4, 5\}$, A consists of $a_1 = (1, 2, 3, 4)$, $a_2 = (1, 2, 3, 5)$, $a_3 = (1, 2, 4, 5)$, $a_4 = (2, 3, 4, 5)$, $a_5 = (4, 1, 5, 3)$. Then 1 is reachable from 5, and one consecutive path is $5a_5 3a_1 4a_5 1$.

Given a k-hypertournament $H(V, A)$ and a vertex x of V. We need to introduce some notations as follows:

$$\begin{aligned} \mathscr{R} = \{&v \in V\colon \exists e_v \in A \text{ such that all consecutive paths from } x \text{ to } v \text{ end in the arc } e_v. \\ &\text{Here we call } e_v \text{ the key arc of } v, \text{ and } v \text{ a key vertex of } e_v\}, \\ \mathscr{E} = \{&e \in A\colon e \text{ is the key arc of some vertex } r \in \mathscr{R}\}, \\ W = \{&v \in V\colon \text{there is no consecutive path from } x \text{ to } v, \text{ i.e., } v \text{ is not reachable from } x\}, \\ X = V &- \mathscr{R} - W. \end{aligned}$$

Lemma A. *Each arc $e \in A \backslash \mathscr{E}$ can be represented as $([W'][\mathscr{R}'][X'])$, where $W' = V_e \cap W$, $\mathscr{R}' = V_e \cap \mathscr{R}$, $X' = V_e \cap X$, $1 \leqslant i \leqslant m$, and $[\cdot]$ means optional. That is, all the vertices of W' precede the vertices of R' in e; and all the vertices of R' precede the vertices of X'.*

Proof. If some vertex from $\mathscr{R}' \cup X'$ is followed by a vertex from W' in e, then that vertex in W can be reached by a consecutive path. Contradiction! Furthermore if some vertex from X' is followed by a vertex in $\mathscr{R}'$ in e then that vertex in $\mathscr{R}$ can be reached by a consecutive path from x ending in e, a contradiction. This completes the proof. □

Lemma B. *Let $H(V, A)$ be a k-hypertournament with n vertices and let $x, y \in V$. When $n > k > 3$ and there is no consecutive path from x to y in A then $d_H^+(x) < d_H^+(y)$.*

Proof. First note that no edge in $e \in A \backslash \mathscr{E}$ contains two vertices in $\mathscr{R}$, since if there is such an edge, e, then by Lemma A we may assume that r_1 is the predecessor of r_2 in e and $r_1, r_2 \in \mathscr{R}$. Now r_2 is reachable by a consecutive path ending in e, a contraction.

Assume $|\mathscr{E}| \geqslant 2$ and let e_{r_1} and e_{r_2} be two distinct edges in $\mathscr{E}$(where $r_1, r_2 \in \mathscr{R}$). By the above all edges containing both r_1 and r_2 lie in $\mathscr{E}$, which imply that $|\mathscr{R}| \geqslant |\mathscr{E}| \geqslant \binom{n-2}{k-2} \geqslant n-2$. As $\mathscr{R} \subseteq V - \{x, y\}$, the above implies equality everywhere and $\mathscr{R} = V - \{x, y\}$. By the above, if $e \in A \backslash \mathscr{E}$ then e contains x, y and one vertex from $\mathscr{R}$, which implies that $k = 3$. A contradiction to $n > k > 3$. Therefore $|\mathscr{E}| \leqslant 1$.

Assume that $|\mathscr{E}| = 0$, i.e., $\mathscr{E} = \emptyset$. Let $Q \subseteq V - \{x, y\}$ be any set of $k-1$ vertices. Let e_y^Q be the edge in H with vertex set $\{y\} \cup Q$ and let e_x^Q be the edge in H with vertex set $\{x\} \cup Q$. By Lemma A (as $y \in W$ and $x \in X$) we have $\rho(e_y^Q, y) - \rho(e_x^Q, x) \geqslant 0$. As for every edge, e_{xy} containing both x and y we have $\rho(e_{xy}, y) - \rho(e_{xy}, x) \geqslant 1$, we note that $d_H^+(y) - d_H^+(x) \geqslant \binom{n-2}{k-2}$.

Now assume that $\mathscr{E} \neq \emptyset$ and $\mathscr{E} = \{e\}$. Let e' contain the same vertices as e but such that $e' = ([W'][\mathscr{R}'][X'])$, where $W' = V_e \cap W$, $\mathscr{R}' = V_e \cap \mathscr{R}$, $X' = V_e \cap X$. Let $H' = H \cup e' - e$ and $r \in \mathscr{R}$ be arbitrary. We know that $\rho(e_{xy}, y) - \rho(e_{xy}, x) \geqslant 1$ for any arc e_{xy} containing both x and y, and there are $\binom{n-2}{k-2}$ such arcs in H'. If $r \in Q$ (and e_y^Q and e_x^Q are defined as above, but in H') then we note that $\rho(e_y^Q, y) - \rho(e_x^Q, x) \geqslant 1$ in H'. As there are $\binom{n-3}{k-2}$ such sets Q, we get that $d_{H'}^+(y) - d_{H'}^+(x) \geqslant \binom{n-2}{k-2} + \binom{n-3}{k-2}$. In Hwe have to modify this bound by at most k (if $e = e_{xy}$ and $\rho(e_{xy}, y) - \rho(e_{xy}, x)) = 1 - k$ in H or if $e \in \{e_x^Q, e_y^Q\}$ and $\rho(e_y^Q, y) - \rho(e_x^Q, x) = 1 - k$ in H), which implies that $d_H^+(y) - d_H^+(x) \geqslant (n-2) + (n-3) - k \geqslant (n-k-1) + (n-4) > 0$.

In the above two cases, we have both $d_H^+(y) - d_H^+(x) > 0$, i.e., $d_H^+(x) < d_H^+(y)$. So Lemma B holds. □

Definition 2. If $xa_{j_1}z_1a_{j_2}z_2 \cdots z_{s-2}a_{j_{s-1}}z_{s-1}a_{j_s}y$ is a consecutive path from x to y in $H(V, A)$, then $H'(V', A')$ that consists of $V' = V$ and

$$A' = A - \{a_{j_1}, a_{j_2}, \ldots, a_{j_s}\} \cup \{a_{j_1}(x, z_1), a_{j_2}(z_1, z_2), \ldots, a_{j_s}(z_{s-1}, y)\},$$

is called a k-hypertournament obtained from H by reversing the consecutive path

$$xa_{j_1}z_1a_{j_2}z_2 \cdots z_{s-2}a_{j_{s-1}}z_{s-1}a_{j_s}y.$$

Lemma 3. *Suppose that $s = (s_1, s_2, \ldots, s_n)$ is a degree sequence of a k-hypertournament H. If the degree of the vertex x is s_j, the degree of the vertex y is $s_i (i < j)$, and there is a consecutive path from x to y, then $S'(s_i^+, s_j^-)$ is the degree sequence of another k-hypertournament.*

Proof. It is easy to see that the k-hypertournament obtained from H by reversing the consecutive path from x to y has the degree sequence $S'(s_i^+, s_j^-)$. □

Lemma 4. *Let $n > k > 3$ and $S = (s_1, s_2, \ldots, s_n)$ be the degree sequence of a k-hypertournament H. If $s_i \leqslant s_j - 2$, then $S'(s_i^+, s_j^-)$ is a degree sequence of another k-hypertournament.*

Proof. Suppose that the degree of y and x are s_i and s_j, respectively. By Lemma B, we know that there is a consecutive path from x to y. By Lemma 3, $S'(s_i^+, s_j^-)$ is a degree sequence of another k-hypertournament. This completes the proof. □

Proof of Theorem 1. The necessity can be obtained directly from Lemma 1. It suffices to prove the sufficiency. We may assume that $n > k > 3$ as the case when $k = 2$, $k = 3$ is already known and as the case when $n = k$ is trivial. Each non-decreasing sequence $S = (s_1, s_2, \ldots, s_n)$ of non-negative integers can be considered as a partition of $M = \binom{n}{2}\binom{n-1}{k-2}$ with n parts. Let Π_n^M denote the set of all partitions of M with n parts. Define $S = (s_1, s_2, \ldots, s_n) \preccurlyeq T = (t_1, t_2, \ldots, t_n)$ in Π_n^M if and only if $\sum_{i=1}^r s_i \leqslant \sum_{i=1}^r t_i$, $1 \leqslant r \leqslant n$, we know that $(\Pi_n^M, \preccurlyeq)$ is a poset. Let $\tilde{S} = (\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n)$ be the degree sequence of a transitive k-hypertournament. By Lemma 2, $\sum_{i=1}^r \tilde{s}_i = \binom{r}{2}\binom{n-2}{k-2}$, $\forall 1 \leqslant r \leqslant n$. So any non-decreasing sequence $S = (s_1, s_2, \ldots, s_n)$ of non-negative integers satisfying $(\otimes)$ if and only if $\tilde{S} \preccurlyeq S$, which means that all the non-decreasing sequences satisfying $(\otimes)$ compose a subposet of Π_n^M containing all partitions $\succcurlyeq \tilde{S}$. It is easy to see that in this subposet $T = (t_1, t_2, \ldots, t_n)$ covers $S = (s_1, s_2, \ldots, s_n)$ if and only if there exist i and j such that $i < j$, $s_i \leqslant s_j - 2$ and $T = S'(s_i^+, s_j^-)$. Then for any non-decreasing sequence S satisfying $(\otimes)$, there would be a sequence of partitions $S^1, S^2, \ldots, S^l$, such that $\tilde{S}$ is covered by S^1, S^1 is covered by $S^2, \ldots, S^l$ is covered by S. Since $\tilde{S}$ is the degree sequence of a transitive k-hypertournament, by using Lemma 4 recursively, we obtain that S is a degree sequence of a k-hypertournament. □

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