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On first-order formalism in string theory

Andrei S. Losev [a], Andrei Marshakov [b,a], Anton M. Zeitlin [c,*]

[a] Institute of Theoretical and Experimental Physics, B. Cheremushkinskaya 25, Moscow 117259, Russia
[b] Department of Theoretical Physics, Lebedev Physics Institute, Leninsky pr. 53, Moscow 119991, Russia
[c] St. Petersburg Department of Steklov Mathematical Institute, Fontanka 27, St. Petersburg 191023, Russia

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Abstract

We consider the first-order formalism in string theory, providing a new off-shell description of the non-trivial backgrounds around an "infinite metric". The OPE of the vertex operators, corresponding to the background fields in some "twistor representation", and conditions of conformal invariance results in the quadratic equation for the background fields, which appears to be equivalent to the Einstein equations with a Kalb–Ramond B-field and a dilaton. Using a new representation for the Einstein equations with B-field and dilaton we find a new class of solutions including the plane waves for metric (graviton) and the B-field. We discuss the properties of these background equations and main features of the BRST operator in this approach.

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1. Introduction

String theory in non-trivial background is a very complicated problem. In contrast to the flat-space case, where the perturbative amplitudes can be computed by calculation of the Gaussian integrals, generally one has to study a non-trivial sigma-model, which is rarely equivalent to an exactly solvable two-dimensional conformal field theory [1]. The connection between the clear space–time sigma-model picture and axiomatically formulated two-dimensional conformal field theory is often hidden, and sometimes is not even clear on fundamental level—as in the case of $AdS_5 \times S^5$ [2] and pp-wave backgrounds [3] for ten-dimensional superstring. One of the possible ideas (with recently renewed interest) is that string theory of gravity can be more successful, being considered in vicinity of a background, which is singular from conventional point of view of classical gravity. For such backgrounds it is not even clear, whether a traditional sigma-model formalism can help in finding a two-dimensional conformal field theory, corresponding to their quantization.

String theory is usually formulated as a perturbative expansion of certain non-linear "string field theory" around some background classical solution to its equations of motion. Even suppressing all string loops and in the limit of vanishing string length $\alpha' \to 0$ it has to reproduce the highly non-linear Einstein equations on the background fields, containing all powers of perturbation, being expanded around the flat-space background. Since perturbative expansion generally depends on the background, it seems reasonable to start with studying some simple ones for this purpose. In this Letter we propose to start with a kind of "Gromov–Witten" background [4], with the infinite target space metric and the B-field $G_{i\bar{j}} = \pm B_{i\bar{j}}$: it turns out that theory drastically simplifies in this limit, and can be described in terms of a conformal *first*-order system.

Hence, we are going to study the first-order formalism in string theory, based on the $D/2$-tensor power of the free $c = 2$ conformal field theory, interacting with reparameteriza-

* Corresponding author.
E-mail addresses: losev@itep.ru (A.S. Losev), mars@lpi.ru, mars@itep.ru (A. Marshakov), zam@math.ipme.ru (A.M. Zeitlin).
URL: http://www.ipme.ru/zam.html.

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tion ghosts; during past years different two-dimensional first-order field theory models were extensively studied, see e.g. [5–10]. We demonstrate, that this formalism describes the string theory around the infinite-metric background (accomplished with the infinite B-field), which, being defined precisely, necessarily requires the target-space complex structure. One of the main advantages of this formalism is that the vertex operator perturbations of the first-order action correspond generally to the *off-shell* background fields in the "light sector" of the target-space theory. Due to absence of the self-contractions between the co-ordinate fields themselves, most anomalous dimensions vanish and there are no higher-spin states, causing complications in conventional formulation of the theory.

In order to get generic background, one has to perturb this theory by the set of all marginal fields, and the vertex operator perturbations of the first-order theory are adequately formulated in terms of certain "twistor" variables $g^{i\bar{j}}$, $\mu_i^{\bar{j}}$ (together with its complex conjugated $\mu_{\bar{i}}^{j}$) and $b_{i\bar{j}}$, which have clear algebraic origin and whose connection with physical background fields $G_{\mu\nu}$, $B_{\mu\nu}$ and dilaton field Φ is rather non-trivial. We study conditions for these fields to be marginal and exactly marginal (absence of the $1/|z|^2$-terms in the operator product expansions (OPE) of the vertex operators) and derive the field equations of motion. These constraints in the first-order formalism appear to have more rich structure, than in conventional sigma-model approach and we analyze the resulting equations of motion, in particular, the bilinear equation to the inverse metric $g^{i\bar{j}}$ (see Eq. (17)) from the point of view of target-space gravity and algebraic structure of the theory. Even restricted to a very special class of perturbations of the form $g^{i\bar{j}}p_i p_{\bar{j}}$ we obtain the set of linear and quadratic equations for the background fields, whose solutions (together with conditions that the background fields are primary) appear to be solutions of the full non-linear system of Einstein equations for the background physical fields.[1]

Finally we are going to discuss briefly some non-perturbative aspects of possible application of the first-order formalism. In particular, we note that disappearance of the higher-spin fields and conventional on-shell condition was observed recently in [11] where the infinite metric of the AdS-like backgrounds was generated by thick stack of D-branes. In this Letter we will not really go beyond the quadratic approximation and only briefly speculate on possible BRST structure of the model.

2. The first-order theory

Let us start with a two-dimensional conformal field theory (CFT) with the first-order action [4]:

$$S_0 = \frac{1}{2\pi\alpha'}\int_\Sigma d^2z\,\left(p_i\bar{\partial}X^i + p_{\bar{i}}\partial X^{\bar{i}}\right), \tag{1}$$

where the momentum $p, \bar{p}$-fields are the $(1,0)$- and $(0,1)$-forms correspondingly, while the co-ordinates $X, \bar{X}$ are scalars with the weights $(0,0)$, the volume element is $d^2z = i\,dz\wedge d\bar{z}$. Action (1) corresponds to the $D/2$-th tensor power of (holomorphic and anti-holomorphic) $c=2$ first-order conformal field theory, where the multipliers are labeled according to some choice of the target space complex structure, $i,\bar{i}=1,\dots,D/2$. The equations of motion following from the Lagrangian (1) provide the non-trivial operator product expansions (OPE):

$$\begin{aligned} X^i(z_1)p_j(z_2) &\sim \frac{\alpha'\delta^i_j}{z_1-z_2}+\cdots, \\ X^{\bar{i}}(\bar{z}_1)p_{\bar{j}}(\bar{z}_2) &\sim \frac{\alpha'\delta^{\bar{i}}_{\bar{j}}}{\bar{z}_1-\bar{z}_2}+\cdots \end{aligned} \tag{2}$$

(it is convenient to keep here explicit α'-dependence), and there are no singular contractions between the X- and p-fields themselves.

To study the theory with the action (1), let us, first, perturb it by the following vertex operator:

$$V_g = \frac{1}{2\pi\alpha'}g^{i\bar{j}}(X,\bar{X})p_i p_{\bar{j}} \tag{3}$$

so that the full action becomes:

$$S_g = \frac{1}{2\pi\alpha'}\int_\Sigma d^2z\,\left(p_i\bar{\partial}X^i + p_{\bar{i}}\partial X^{\bar{i}} - g^{i\bar{j}}p_i p_{\bar{j}}\right). \tag{4}$$

On classical level, solving equations of motion for $p, \bar{p}$, one immediately finds that the action (4) is equivalent to:

$$\begin{aligned} \mathcal{S} &= \frac{1}{2\pi\alpha'}\int_\Sigma d^2z\, g_{i\bar{j}}\bar{\partial}X^i\partial X^{\bar{j}} \\ &= \frac{1}{4\pi\alpha'}\int_\Sigma d^2z\,(G_{\mu\nu}+B_{\mu\nu})\partial X^\mu\bar{\partial}X^\nu, \end{aligned} \tag{5}$$

where μ, ν run now over both holomorphic and antiholomorphic indices, while G and B are the symmetric Riemann metric and antisymmetric Kalb–Ramond B-field correspondingly. The physical fields should obey the constraint $G_{i\bar{j}} = -B_{i\bar{j}}$, or

$$G_{i\bar{k}} = g_{i\bar{k}}, \qquad B_{i\bar{k}} = -g_{i\bar{k}}. \tag{6}$$

Note, that the operator (3) contains the inverse metric $g^{i\bar{j}}$, written in terms of the target-space holomorphic and anti-holomorphic co-ordinates, and, therefore, is a perturbation of (1) around the *infinite* metric background (with the infinite Kalb–Ramond field).

However, in quantum case the integration measure should be taken into account. For the first-order system (4) it is determined by the holomorphic $D/2$-form $\Omega = \Omega(X) = dX^1\wedge\cdots\wedge dX^{D/2}$. After integration over the p-fields

$$\int [dp]\,[d\bar{p}]e^{-S_g[X,\bar{X},p,\bar{p}]} \sim e^{-\mathcal{S}[X,\bar{X}]+\frac{1}{2\pi}\int_\Sigma d^2z\sqrt{h}R\log\sqrt{g}} \tag{7}$$

we arrive at the standard sigma-model (5), where the measure is determined with the help of non-degenerate target-space metric. The difference in two measures leads to appearance of the dilaton $\frac{1}{2\pi}\int d^2z\sqrt{h}R\log\sqrt{g}$ term in the action (7), related to determinant of the ultra-local operator.

[1] One could expect this generally, since cubic terms in these equations should be of higher order in α'.

Indeed, integration in (7) over the momenta $p, \bar{p}$ naively leads to the (infinite) factor $\prod_{z\in\Sigma} \det g_{i\bar{j}}(X(z))$, which plays a role of a factor, that turns the measure determined by holomorphic form Ω into the measure determined by non-degenerate metric g. However, the "number of factors" in the infinite product is the (infinite) number of one-forms, while the number of factors needed to complete the measure on the X-fields equals to the (infinite) number of functions (or zero-forms). It is well known, that difference between these two infinite numbers is finite and equals to the arithmetic genus $g-1$ of the world-sheet, or is proportional to the integral of the scalar curvature along the surface. If we regularize this anomaly, say, with the help of massive regulator fields, it becomes "locally distributed" along the world-sheet, and this is a shortcut to understanding the dilaton term (7).

Another way to test the validity of (7) is to consider the target-space holomorphic transformation $\delta X^i = \epsilon v^i(X)$, $\delta p_i = -\epsilon p_j \frac{\partial v^j}{\partial X^i}$. The corresponding current $p_i v^i(X)$ obeys the anomaly equation

$$\bar{\partial}\langle p_i v^i(X)\rangle = \frac{1}{2\pi} R\partial_i v^i(X) = \frac{1}{2\pi} R\mathcal{L}_v \log\Omega \tag{8}$$

computed at some fixed point of the world-sheet. That is in perfect agreement with (7): one has to take into account that $\det g$ is the ratio of two measures in the target-space, determined by metric and by the holomorphic top form Ω correspondingly. The anomalous current (8) naturally suggests considering the charges:

$$n_v = \frac{1}{2\pi i\alpha'}\oint_{S^1} dz\, v^i(X)p_i, \qquad r_\omega = \frac{1}{2\pi i\alpha'}\oint_{S^1} dz\, \omega_i(X)\partial X^i \tag{9}$$

together with their complex conjugated $\bar{n}_v$ and $\bar{r}_\omega$, generating the symmetries of the first-order action (1); their properties are studied in Appendix A.

Now one can perturb the free action (1) by all possible operators of dimension $(1,1)$, corresponding to more general deformation of metric, B-field as well as the deformation of the almost complex structure by the Beltrami differential $\mu^j_{\bar{i}}$ and $\bar{\mu}^{\bar{j}}_i$. The full perturbed action reads

$$S = \frac{1}{2\pi\alpha'}\int d^2z\,\big(p_i\bar{\partial}X^i + p_{\bar{i}}\partial X^{\bar{i}} - g^{i\bar{j}}p_i p_{\bar{j}} - \bar{\mu}^{\bar{j}}_i \partial X^i p_{\bar{j}} \\ - \mu^j_{\bar{i}}\bar{\partial}X^{\bar{i}} p_j - b_{i\bar{j}}\partial X^i\bar{\partial}X^{\bar{j}}\big). \tag{10}$$

These background fields (to be called the twistor variables) can be directly associated with the four independent terms in the expansion (see formula (A.5) in Appendix A) of the tensor product of representation spaces, corresponding to action of the world-sheet symmetries (9) of the model.

Again, on classical level, solving equations of motion for $p, \bar{p}$, one finds that this action is equivalent to the following sigma-model:

$$\mathcal{S} = \frac{1}{2\pi\alpha'}\int d^2z\,\big(g_{i\bar{j}}\big(\bar{\partial}X^i - \mu^i_{\bar{k}}\bar{\partial}X^{\bar{k}}\big)\big(\partial X^{\bar{j}} - \bar{\mu}^{\bar{j}}_k\partial X^k\big) \\ - b_{i\bar{j}}\partial X^i\bar{\partial}X^{\bar{j}}\big), \tag{11}$$

which can be rewritten in the conventional form (5) with an extra dilaton term [12]

$$\mathcal{S} = \frac{1}{4\pi\alpha'}\int d^2z\,(G_{\mu\nu}+B_{\mu\nu})\partial X^\mu\bar{\partial}X^\nu + \frac{1}{2\pi}\int d^2z\,\sqrt{h}R\Phi \tag{12}$$

with G, B and Φ now (compare to the previous section) defined as follows:

$$\begin{aligned}
&G_{s\bar{k}} = g_{\bar{i}j}\bar{\mu}^{\bar{i}}_s\mu^j_{\bar{k}} + g_{s\bar{k}} - b_{s\bar{k}}, \qquad B_{s\bar{k}} = g_{\bar{i}j}\bar{\mu}^{\bar{i}}_s\mu^j_{\bar{k}} - g_{s\bar{k}} - b_{s\bar{k}},\\
&G_{si} = -g_{i\bar{j}}\bar{\mu}^{\bar{j}}_s - g_{s\bar{j}}\bar{\mu}^{\bar{j}}_i, \qquad G_{\bar{s}\bar{i}} = -g_{\bar{s}j}\mu^j_{\bar{i}} - g_{\bar{i}j}\mu^j_{\bar{s}},\\
&B_{si} = g_{s\bar{j}}\bar{\mu}^{\bar{j}}_i - g_{i\bar{j}}\bar{\mu}^{\bar{j}}_s, \qquad B_{\bar{s}\bar{i}} = g_{\bar{i}j}\mu^j_{\bar{s}} - g_{\bar{s}j}\mu^j_{\bar{i}},\\
&\Phi = \log\sqrt{g}.
\end{aligned} \tag{13}$$

3. Main result

Let us now analyze the conformal invariance of the first-order theory, perturbed by a single vertex operator $g^{i\bar{j}}p_i p_{\bar{j}}$ (3). The OPE of (3) with the stress-energy tensor $T = -(\alpha')^{-1}p_i\partial X^i$ (and its counterpart of opposite chirality $\tilde{T} = -(\alpha')^{-1}p_{\bar{i}}\bar{\partial}X^{\bar{i}}$), corresponding to the first-order system (1) reads:

$$\begin{aligned}
&-(\alpha')^{-1}p_i\partial X^i(z)\cdot(\alpha')^{-1}g^{i\bar{j}}p_i p_{\bar{j}}(z')\\
&\quad = -\frac{1}{(z-z')^3}\partial_i g^{i\bar{j}}p_{\bar{j}}(z')\\
&\qquad + \frac{1}{\alpha'}\left(\frac{1}{(z-z')^2}g^{i\bar{j}}p_i p_{\bar{j}}(z') + \frac{1}{z-z'}\partial_{z'}g^{i\bar{j}}p_i p_{\bar{j}}(z')\right)\\
&\qquad + \cdots.
\end{aligned} \tag{14}$$

Two last terms in the r.h.s. of (14), proportional to $(\alpha')^{-1}$, are standard singular terms from the OPE of the stress-tensor with primary field of unit dimension, so that, being integrated over the world-sheet it becomes co-ordinate invariant, and they give no real constraints. However the first singular term in the r.h.s., proportional to $(\alpha')^0$, is the action of the L_1-Virasoro operator, which deviates it from the primary operator, unless

$$\partial_i g^{i\bar{j}} = 0, \qquad \partial_{\bar{j}}g^{i\bar{j}} = 0 \tag{15}$$

quite similarly to conventional "second-order" conformal field theory [1], but arising here *before* any mass-shell condition, the same condition comes from eliminating the contraction of p_i and X^j inside the operator (3) $g^{i\bar{j}}p_i p_{\bar{j}}$ itself. Moreover, in contrast to the second-order formalism, where transversality justifies itself as a gauge artefact, being proportional to the two-dimensional equations of motion and the total-derivative terms, here the constraint (15) appears as an independent requirement in the target-space description of the theory.

Consider now the OPE of two vertex operators (3) of the general structure $V(z_1)V(z_2) \sim \sum_{i,j}\frac{a^{(i,j)}(z_2)}{(z_1-z_2)^{2-i}(\bar{z}_1-\bar{z}_2)^{2-j}}$, and calculate some important coefficients $a^{(i,j)}$. For the vertex operators (3) the most singular term in OPE $a^{(0,0)} \propto (\alpha')^2\partial_k\partial_{\bar{l}}g^{i\bar{j}}\partial_i\partial_{\bar{j}}g^{k\bar{l}}$ does not contribute in the leading order in α' and we will not discuss it now. The next is (the only at the level $(\alpha')^0$) logarithmic divergence, coming from the double

p, X contractions, i.e.,

$$a^{(1,1)} = (2\pi^2)^{-1}\big(g^{i\bar{j}}\partial_i\partial_{\bar{j}}g^{k\bar{l}} - \partial_i g^{k\bar{j}}\partial_{\bar{j}}g^{i\bar{l}}\big)p_k p_{\bar{l}} + O(\alpha'). \tag{16}$$

To make the theory conformally invariant it should vanish, i.e., the background metric $g^{i\bar{j}}$ satisfies the "bilinear equation":

$$g^{i\bar{j}}\partial_i\partial_{\bar{j}}g^{k\bar{l}} - \partial_i g^{k\bar{j}}\partial_{\bar{j}}g^{i\bar{l}} = 0. \tag{17}$$

In the case of general Hermitian metric[2] we will show below that the conditions of conformal invariance for the first-order model (4) lead to the background Einstein equations with a dilaton, confirming the argument above.

This background equation is our new result, its algebraic properties are briefly discussed in Appendix A; it is quadratic since the first-order theory corresponds to expansion of the target-space theory around a singular background.

One can check (see Appendix B) that the quadratic system of equations (17), being supplied by the "gauge condition" (15), is indeed equivalent to the system of Einstein equations with a Kalb–Ramond field and a dilaton [12,13]:

$$R_{\mu\nu} = -\frac{1}{4}H_{\mu\lambda\rho}H_{\nu}^{\lambda\rho} + 2\nabla_\mu\nabla_\nu\Phi, \tag{19}$$

$$\nabla_\mu H^{\mu\nu\rho} - 2(\nabla_\lambda\Phi)H^{\lambda\nu\rho} = 0, \tag{20}$$

$$4(\nabla_\mu\Phi)^2 - 4\nabla_\mu\nabla^\mu\Phi + R + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} = 0, \tag{21}$$

where the change of variables from the "twistor variables" $g_{i\bar{j}}$ to the physical metric, B-field and dilaton G, B, Φ is given by the following expressions:

$$G_{i\bar{k}} = g_{i\bar{k}}, \qquad B_{i\bar{k}} = -g_{i\bar{k}}, \qquad \Phi = \log\sqrt{g}. \tag{22}$$

Note, that equivalence of the system (19)–(21) to Eqs. (17) and (15), coming directly from OPE of (3) in the first-order theory (4), confirms the preliminary conclusion of appearance of the dilaton from (7).

Let us stress again here, that the first-order theory corresponds to a singular-background expansion of the Einstein equations (19)–(21) and, therefore, in order to make equivalence with the bilinear equation (17) of the first-order theory one has to use explicitly the gauge condition (15), as required by conformal field theory (1). In the common sigma-model approach, corresponding to expansion of the action (12) around a non-singular background, say $G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the "gauge" terms $\partial_\mu h^{\mu\nu} = 0$ can be eliminated by a prescription that the terms proportional to the two-dimensional equations of motion and total derivatives are cut off. However, in the singular background of first-order theory (4) there is no linear approximation for the background field equations (17), i.e., the "reference point" of expansion is singular from the point of view of the target-space theory and this makes study of the symmetries in this point to be a delicate issue.

[2] For the Kähler target-space metric $g_{i\bar{j}}$ this condition leads to the vanishing Ricci tensor, and while gauge condition (15) is equivalent to the constant determinant, since

$$0 = \partial_i g^{i\bar{j}}g_{k\bar{j}} = -g^{i\bar{j}}\partial_i g_{k\bar{j}} = -g^{i\bar{j}}\partial_k g_{i\bar{j}} = -\partial_k \log g \quad \text{and c.c.} \tag{18}$$

We conjecture also that, as well as for the simplified model with the only perturbation (3), the conformal invariance of the model in general background (10) at the order $(\alpha')^0$ is provided by the Einstein equations with a B-field and a dilaton $\Phi = \log\sqrt{g}$.

4. Special solutions

An interesting question is to study the solutions of the system (17). A particular class of solutions was discussed in [14]. For example, one can show that Eq. (17) possess the following class of solutions:

$$g^{i\bar{j}} = \hat{g}^{i\bar{j}}(k_\mu X^\mu), \qquad \hat{g}^{i\bar{j}}k_i k_{\bar{j}} = 0, \quad k_i(\hat{g}^{i\bar{j}})' = 0 \quad \text{and c.c.}, \tag{23}$$

where prime means the derivative of functions $g^{i\bar{j}}(y)$ with respect to its argument $y = k_\mu X^\mu$. In the pure Kähler case one can write a solution

$$\hat{g}_{i\bar{j}} = \eta_{i\bar{j}} + k_i k_{\bar{j}} f(y), \qquad \eta^{i\bar{j}}k_i k_{\bar{j}} = 0, \tag{24}$$

where $f(y)$ is any function of the scalar product $y = k_\mu X^\mu$. Among these solutions one can find the plane waves, which are in physical variables:

$$G_{i\bar{j}} = \eta_{i\bar{j}} + e_{i\bar{j}}\big(A\cos(k_\mu X^\mu) + B\sin(k_\mu X^\mu)\big),$$
$$G_{i\bar{j}} = -B_{i\bar{j}}, \qquad k_{\bar{l}}\eta^{i\bar{l}}e_{i\bar{j}} = 0 \quad \text{and c.c.} \tag{25}$$

This means that the dilaton field, equal to $\log\sqrt{g}$, provides the plane waves for the G- and B-fields. One should also note that in our case B-field is pure imaginary (or the two-form $g_{i\bar{j}}dz^i \wedge dz^{\bar{j}}$ is anti-Hermitian). To make the B-field real, it is necessary to consider z^i and $z^{\bar{j}}$ not as complex conjugated variables, but as real ones with $i, \bar{j} = 1, \ldots, D/2$. Then the associated two-form becomes Hermitian, but by obvious reasons the metric $g_{i\bar{j}}dz^i dz^{\bar{j}}$ acquires the signature $(D/2, D/2)$. An interesting feature of these solutions is that they do not get additional α'-corrections in the world-sheet perturbation theory, since each loop diagram obviously contributes with the terms, vanishing due to (23).

5. Concluding remarks

5.1. Remarks on D-branes near the AdS throat

One of the attractive special features of the proposed first-order formalism is natural disappearance of the on-shell condition (as a linear equation on vertex operator) together with the simultaneous disappearance of the higher-spin fields or Regge descendants from the theory. Physically this phenomenon is a consequence of the infinite metric limit, when the co-ordinate fields do not have contractions with themselves. From the point of view of two-dimensional conformal theory this kills the anomalous dimensions of the plane waves and these anomalous dimensions cannot compensate therefore the dimensional polynomials of the derivatives of the co-ordinate fields.

Similar phenomenon has been already observed in [11], when D-brane is placed in the vicinity of the AdS throat.

Clearly, near the throat the metric tends to infinity, and that explains the observed in [11] effects in a rather similar way to how it happens the formalism we have discussed in the Letter.

5.2. Homotopic dreams

Our success in reproducing the solution to the Einstein equations (19)–(21) in expansion around the singular first-order background motivates the study of all perturbations, which should lead to the picture completely equivalent to the full set of Einstein equations. One can hope, that the Einstein equations in the language of string theory (in particular, of the proposed first-order theory) look like a kind of the Maurer–Cartan equation

$$Q\Psi + m_2(\Psi,\Psi) + m_3(\Psi,\Psi,\Psi) + \cdots = 0, \tag{26}$$

where Q is the BRST operator in given background (see e.g. (A.9)), Ψ is a (generalized!) vertex operator deforming the action, containing generally the polyvertex fields, and $m_n(\Psi,\ldots,\Psi)$ are some operations in conformal theory, corresponding to given background. Eqs. (17), (15) we have derived, correspond to

$$Q\Psi = 0, \qquad m_2(\Psi,\Psi) = 0 \tag{27}$$

for the deformation (3). We expect that the conjectured set of Eq. (26) would have a large symmetry group, promoting $\Psi \to \Psi + Q\epsilon$ to non-linear level, and that the operations $m_n(\Psi,\ldots,\Psi)$ would satisfy certain quadratic equations like for homotopic structures. We should also stress here that the conjectures higher operations are generally background dependent and we hope that within the proposed first-order formalism they could appear in the simplest possible form. We postpone the discussion of general deformation of the BRST operator and structure of Eq. (26) for a separate publication.

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Appendix A. Algebraic structure of the first-order theory

In this appendix we consider the properties of the symmetries, generated by the operators (9). The singularities coming from internal contractions (8) should be avoided by the vanishing of divergences $\partial_i v^i = \partial_{\bar i} v^{\bar i} = 0$. Application of (9) to the vertex operator V_g (3) generates the transformations of fields, which is, to the order α':

$$\begin{aligned}
&\delta_v g^{\bar i j} = -v^k \partial_k g^{\bar i j} - \bar v^{\bar k}\partial_{\bar k} g^{\bar i j} + \partial_{\bar k}\bar v^{\bar i} g^{\bar k j} + \partial_k v^j g^{\bar i k} + O(\alpha'),\\
&\delta_\omega g^{\bar i j} = O(\alpha'),\\
&\delta_\omega \mu_{\bar i}^{j} = \partial_{\bar i}\bar\omega_{\bar k} g^{\bar k j} - \partial_{\bar k}\bar\omega_{\bar i} g^{\bar k j} + O(\alpha'),\\
&\delta_\omega \bar\mu_i^{\bar j} = \partial_i \omega_k g^{k\bar j} - \partial_k\omega_i g^{k\bar j} + O(\alpha'),
\end{aligned} \tag{A.1}$$

i.e., n_v generates the holomorphic coordinate transformations, and r_ω is the generator of the gauge symmetry $B \to B + D\omega + \bar D\bar\omega$ (D and $\bar D$ are here the *target-space* Doulbeaux operators), which becomes clear, rewriting it for the G- and B-fields (13):

$$\delta_\omega B_{\mu\nu} = \partial_\nu\omega_\mu - \partial_\mu\omega_\nu + O(\alpha'), \qquad \delta_\omega G_{\mu\nu} = O(\alpha'). \tag{A.2}$$

The algebra of the charges (9) is

$$\begin{aligned}
&[n_{v_1}, n_{v_2}] = n_{[v_2,v_1]} + \alpha' r_{\omega(v_1,v_2)},\\
&[r_\omega, n_v] = r_{\mathcal{L}_v\omega}, \qquad [r_{\omega_1}, r_{\omega_2}] = 0,
\end{aligned} \tag{A.3}$$

where $\omega_n(v_1,v_2) = \frac{1}{2}(\partial_k v_2^l \partial_n \partial_l v_1^k - \partial_n\partial_k v_1^l \partial_l v_2^k)$, $\mathcal{L}_v\omega_k = \partial_i\omega_k v^i + \omega_i\partial_k v^i$ is a Lie derivative, and the same algebraic relations hold for the charges of the opposite chirality $\bar n_v$, $\bar r_\omega$. This is a deformation of the semidirect product of the algebra of holomorphic coordinate transformations and the B-field gauge transformations, where in the limit $\alpha' \to 0$ the extension disappears. One can also introduce a non-degenerate inner product, invariant under the adjoint action $\mathcal{L}_v$:

$$\begin{aligned}
&(n_{v_1}, n_{v_2}) = 0, \qquad (r_{\omega_1}, r_{\omega_2}) = 0,\\
&(n_v, r_\omega) = \int v^i(X)\omega_i(X)\Omega(X),
\end{aligned} \tag{A.4}$$

where Ω is a holomorphic volume form and the integral is taken along the half-dimensional target-space cycle. It means that vector fields v^i and one-forms ω_i correspond to the dual representations V and V^* of the algebra (A.3) (already studied in [15]) and these representations provide a natural algebraic structure of the background perturbation (10)

$$\begin{aligned}
&\left(V \oplus V^*\right) \otimes \left(\bar V \oplus \bar V^*\right)\\
&\quad = (V \otimes \bar V) \oplus \left(V \otimes \bar V^*\right) \oplus \left(V^* \otimes \bar V\right) \oplus \left(V^* \otimes \bar V^*\right),
\end{aligned} \tag{A.5}$$

where four terms in the r.h.s. literally correspond to the four background fields in (10). Thus, we see, that generic perturbation of the first-order theory by "light" fields has a natural algebraic origin.

The vertex operator V or (3) can be expanded

$$V(X,p,\bar X,\bar p) = \sum_I \mathcal{U}_I(X,p) \otimes \bar{\mathcal{U}}_I(\bar X,\bar p) \tag{A.6}$$

in the (generally infinite; I is some multi-index) bilinear combination of the left- and right-chiral parts $\mathcal{U}_I$ and $\bar{\mathcal{U}}_I$ with the conformal weights $(1,0)$ and $(0,1)$ correspondingly. In terms of (A.6) the transformation formulas (A.1) can be written in the form of adjoint action:

$$\delta_v V = \sum_I [n_v, \mathcal{U}_I] \otimes \bar{\mathcal{U}}_I + \sum_I \mathcal{U}_I \otimes [\bar{n}_{\bar{v}}, \bar{\mathcal{U}}_I],$$
$$\delta_\omega V = \sum_I [r_\omega, \mathcal{U}_I] \otimes \bar{\mathcal{U}}_I + \sum_I \mathcal{U}_I \otimes [\bar{r}_{\bar{\omega}}, \bar{\mathcal{U}}_I]. \tag{A.7}$$

Symmetries (A.1) are consistent with the conformal properties of the model and, in the most compact way, this can be written as

$$[Q, cn_v] = [Q, \tilde{c}\bar{n}_{\bar{v}}] = 0, \qquad [Q, cr_\omega] = [Q, \tilde{c}\bar{r}_{\bar{\omega}}] = 0 \tag{A.8}$$

commutativity with the BRST operator [16] for the free first-order theory (1)

$$Q = \oint_{S^1} \mathcal{J}, \quad \mathcal{J} = j\,dz - \tilde{j}\,d\bar{z}, \quad j = cT + {:}bc\partial c{:} + \frac{3}{2}\partial^2 c,$$
$$\tilde{j} = \tilde{c}\tilde{T} + {:}\tilde{b}\tilde{c}\bar{\partial}\tilde{c}{:} + \frac{3}{2}\bar{\partial}^2\tilde{c}, \tag{A.9}$$

where the $T = -(\alpha')^{-1} p_i \partial X^i$ and $\tilde{T} = -(\alpha')^{-1} p_{\bar{i}} \bar{\partial} X^{\bar{i}}$ are correspondingly holomorphic and antiholomorphic components of the energy–momentum tensor, and b, c (and $\tilde{b}$, $\tilde{c}$) are reparameterization ghosts.

The OPE (14) can be also encoded into the commutation relations of the BRST operator (A.9) with the fields, which are:

$$[Q, \phi_{h,\bar{h}}] = hc\partial\phi_{h,\bar{h}} + \bar{h}\tilde{c}\bar{\partial}\phi_{h,\bar{h}} + \partial c\phi_{h,\bar{h}} + \bar{\partial}\tilde{c}\phi_{h,\bar{h}},$$
$$[Q, c] = c\partial c, \qquad [Q, b] = T + T^{gh}, \quad \text{and c.c.}, \tag{A.10}$$

where $\phi_{h,\bar{h}}$ is a (primary) field with the conformal weights $(h,\bar{h})$. Denoting $c\tilde{c}V = \phi^{(0)}$, $V = \phi^{(2)}$, $cV - \tilde{c}V = \phi^{(1)}$, for the vertex operator V with conformal weights $(h,\bar{h}) = (1,1)$, one can easily obtain the simple relations $[Q, \phi^{(2)}] = d\phi^{(1)}$, $[Q, \phi^{(1)}] = d\phi^{(0)}$, $[Q, \phi^{(0)}] = 0$ and $[Q, \int_M \phi^{(2)}] = \int_{\partial M} \phi^{(1)}$, where M is some two-dimensional manifold with a boundary. Below we consider the deformation of the BRST operator for the non-trivial background in the first-order theory and interpret the equations of motion for the background in terms of the deformation theory for the BRST operator (A.9). We find that for generic perturbation this L_1-term should be supplemented by the singular contributions from OPE of two vertex operators, giving rise to (a linearized version of) some generalized Maurer–Cartan equation.

Eq. (17) can be also rewritten in the form:

$$\lim_{\epsilon,\alpha'\to 0} \oint_{C_{\epsilon,z}} \big(dz'\,\tilde{c}(\bar{z}')V(z') - d\bar{z}'\,c(z')V(z')\big)c(z)\tilde{c}(\bar{z})V(z) = 0, \tag{A.11}$$

where $C_{\epsilon,z}$ is a small contour around the point z, and this form is used below for studying connection with the BRST operator. From the point of view of Section 5 the bilinear structure and holomorphic properties of (A.11) lead to appearance of a *double*-commutator if one rewrites (17) in the algebraic form. Indeed, using (A.6) for the operator (3), one can write for (17)

$$g^{i\bar{j}}\partial_i\partial_{\bar{j}}g^{k\bar{l}} - \partial_i g^{k\bar{j}}\partial_{\bar{j}}g^{i\bar{l}}$$
$$= \sum_{I,I'} \big(\big(\mathcal{U}_I^i \partial_i \mathcal{U}_{I'}^k\big)\big(\mathcal{U}_I^{\bar{j}} \partial_{\bar{j}} \mathcal{U}_{I'}^{\bar{l}}\big) - \big(\mathcal{U}_{I'}^i \partial_i \mathcal{U}_I^k\big)\big(\mathcal{U}_I^{\bar{j}} \partial_{\bar{j}} \mathcal{U}_{I'}^{\bar{l}}\big)\big), \tag{A.12}$$

where $\mathcal{U}_I^i = \mathcal{U}_I^i(X)$ and $\mathcal{U}_I^{\bar{i}} = \mathcal{U}_I^{\bar{i}}(\bar{X})$ are holomorphic and antiholomorphic "blocks" for the background metric field. Multiplying (A.12) from the right by $\partial_k\partial_{\bar{l}}$ one can rewrite (A.12) as

$$\big(g^{i\bar{j}}\partial_i\partial_{\bar{j}}g^{k\bar{l}} - \partial_i g^{k\bar{j}}\partial_{\bar{j}}g^{i\bar{l}}\big)\partial_k\partial_{\bar{l}}$$
$$= \sum_{I,I'} [v_I, v_{I'}]\bar{v}_I\bar{v}_{I'} = \frac{1}{2}\sum_{I,I'} [v_I, v_{I'}][\bar{v}_I, \bar{v}_{I'}] = 0, \tag{A.13}$$

where we have introduced vector fields $v_I = \mathcal{U}_I^i\partial_i$ and $\bar{v}_I = \mathcal{U}_I^{\bar{i}}\partial_{\bar{i}}$. For the r.h.s. of (A.13) it is convenient to use the notation

$$\big[[V, \tilde{V}]\big](X, p, \bar{X}, \bar{p})$$
$$= \sum_{I,J} [\mathcal{U}_I, \tilde{\mathcal{U}}_J](X, p) \otimes [\bar{\mathcal{U}}_I, \bar{\tilde{\mathcal{U}}}_J](\bar{X}, \bar{p}), \tag{A.14}$$

so that Eq. (17) can be interpreted as vanishing of the double-commutator (A.13), (A.14) in some algebra, naturally acting in the tensor product of the holomorphic and antiholomorphic sectors of the first-order theory. We believe that this is an algebraic structure naturally related with the theory of target-space gravity.

Appendix B. Relation between twistor and physical variables

We use the formulas from Appendix Γ of the book [17]:

$$R^{\mu\nu} = -\frac{1}{2}G^{\alpha\beta}\partial_\alpha\partial_\beta G^{\mu\nu} - \Gamma^{\mu\nu} + \Gamma^{\mu,\alpha\beta}\Gamma^\nu_{\alpha\beta}, \tag{B.1}$$

where

$$\Gamma^{\mu\nu} = G^{\mu\rho}G^{\nu\sigma}\Gamma_{\rho\sigma}, \qquad \Gamma_{\rho\sigma} = \frac{1}{2}(\partial_\rho\Gamma_\sigma + \partial_\sigma\Gamma_\rho) - \Gamma^\nu_{\rho\sigma}\Gamma_\nu,$$
$$\Gamma_\nu = G^{\alpha\beta}\partial_\beta G_{\alpha\nu} - \frac{1}{2}\partial_\nu \log(G). \tag{B.2}$$

Remember that in our case $\partial_\mu G^{\mu\rho} = 0$, this leads to the simple relation: $\Gamma_\nu = -\frac{1}{2}\partial_\nu \log(G)$. Therefore $\Gamma_{\mu\nu} = -2\nabla_\mu\nabla_\nu\Phi$, for the $\Phi = \log\sqrt{g}$, where g is the determinant of matrix $g_{i\bar{j}}$. Now let us study the third term in (B.1): first, for the components of $\Gamma^\nu_{\alpha\beta}$, one has:

$$\Gamma^i_{rs} = \frac{1}{2}g^{i\bar{k}}(\partial_r g_{\bar{k}s} + \partial_s g_{\bar{k}r}),$$
$$\Gamma^i_{r\bar{s}} = \frac{1}{2}g^{i\bar{k}}(\partial_{\bar{s}} g_{r\bar{k}} - \partial_{\bar{k}} g_{r\bar{s}}) \quad \text{and c.c.}, \tag{B.3}$$

while all other components vanish. Therefore, one finds that $\Gamma_{\bar{i},r\bar{s}} = \frac{1}{2}H_{\bar{s}\bar{i}r}$, hence the third term in (B.1) provides contribution of the H^2-type, with an additional term in $\Gamma\Gamma$ for $\mu = \bar{i}$

and $\nu = j$:

$$\begin{aligned}\Gamma^{\bar{i},kl}\Gamma^{j}_{kl} &= -\frac{1}{4}\big(g^{k\bar{r}}\partial_{\bar{r}}g^{l\bar{i}} + g^{l\bar{r}}\partial_{\bar{r}}g^{k\bar{i}}\big)g^{j\bar{p}}(\partial_k g_{\bar{p}l} + \partial_l g_{\bar{p}k})\\ &= -\frac{1}{4}\big(g^{k\bar{r}}\partial_{\bar{r}}g^{l\bar{i}} - g^{l\bar{r}}\partial_{\bar{r}}g^{k\bar{i}}\big)g^{j\bar{p}}(\partial_k g_{\bar{p}l} - \partial_l g_{\bar{p}k})\\ &\quad - g^{k\bar{r}}\partial_{\bar{r}}g^{l\bar{i}}g^{j\bar{p}}\partial_l g_{\bar{p}k}\\ &= -\frac{1}{4}H^{\bar{i}kl}H^{j}_{kl} + \partial_{\bar{r}}g^{\bar{i}k}\partial_k g^{\bar{r}j},\end{aligned} \tag{B.4}$$

which, however, cancels with the first term in the r.h.s. of (B.1) due to Eq. (17). Thus, unifying all the information we have got the relation (B.1) can be rewritten as:

$$R^{\mu\nu} = -\frac{1}{4}H^{\mu\lambda\rho}H^{\nu}_{\lambda\rho} + 2\nabla^{\mu}\nabla^{\nu}\Phi. \tag{B.5}$$

Similarly, one can prove the following relation:

$$4(\nabla_\mu\Phi)^2 - 2\nabla_\mu\nabla^\mu\Phi - \frac{1}{6}H_{\mu\nu\rho}H^{\mu\nu\rho} = 0. \tag{B.6}$$

Namely, let us start with $\partial_\mu\Phi = \frac{1}{2}g^{\bar{i}k}\partial_\mu g_{\bar{i}k}$, i.e.,

$$2(\nabla_\mu\Phi)^2 = g^{\bar{l}k}\partial_i g_{\bar{l}k}g^{i\bar{j}}g^{\bar{s}r}\partial_{\bar{j}}g_{\bar{s}r} \tag{B.7}$$

and

$$\begin{aligned}-\nabla_\mu\nabla^\mu\Phi &= -g^{\bar{i}j}\partial_{\bar{i}}\big(g^{\bar{l}k}\partial_j g_{\bar{l}k}\big) + g^{\bar{i}j}\Gamma^{r}_{\bar{i}j}g^{\bar{l}k}\partial_r g_{\bar{l}k}\\ &\quad + g^{\bar{i}j}\Gamma^{\bar{r}}_{\bar{i}j}g^{\bar{l}k}\partial_{\bar{r}}g_{\bar{l}k}.\end{aligned} \tag{B.8}$$

Using (B.3) we arrive at

$$g^{\bar{i}j}\Gamma^{r}_{\bar{i}j} = -\frac{1}{2}g^{\bar{i}j}\partial_{\bar{l}}g_{\bar{i}j}g^{\bar{l}r}, \qquad g^{\bar{i}j}\Gamma^{\bar{r}}_{\bar{i}j} = -\frac{1}{2}g^{\bar{i}j}\partial_l g_{\bar{i}j}g^{\bar{r}l}. \tag{B.9}$$

The sum of (B.7) and (B.8) can be rewritten in the form:

$$(\nabla_\mu\Phi)^2 - \nabla_\mu\nabla^\mu\Phi = -g^{\bar{i}j}\partial_{\bar{i}}\big(g^{\bar{l}k}\partial_j g_{\bar{l}k}\big). \tag{B.10}$$

The H^2-term equals to:

$$\begin{aligned}\frac{1}{6}H_{\mu\nu\rho}H^{\mu\nu\rho} &= H_{i\bar{j}\bar{k}}H^{i\bar{j}\bar{k}}\\ &= (-\partial_{\bar{k}}g_{i\bar{j}} + \partial_{\bar{j}}g_{i\bar{k}})\big(-\partial_s g^{i\bar{j}}g^{s\bar{k}} + \partial_s g^{i\bar{k}}g^{s\bar{j}}\big)\\ &= 2g^{s\bar{k}}\partial_{\bar{k}}g_{i\bar{j}}\partial_s g^{i\bar{j}} - 2\partial_{\bar{j}}g_{i\bar{k}}g^{s\bar{k}}\partial_s g^{i\bar{j}}.\end{aligned} \tag{B.11}$$

One finds now, that (B.6) is satisfied due to (17). Combining (19) and (B.6) one obtains (21).

The third equation one can get by simple analysis of (17), i.e.:

$$\partial_{\bar{i}}\big(g^{\bar{i}j}\partial_j g^{\bar{k}l} - g^{\bar{k}r}\partial_r g^{\bar{i}l}\big) = 0 \quad \text{and c.c.} \tag{B.12}$$

leads to relation:

$$\partial_{\bar{i}}H^{\bar{i}\bar{k}l} = 0 \quad \text{and c.c.} \tag{B.13}$$

and identity

$$\partial_l\big(g^{\bar{i}j}\partial_j g^{\bar{k}l} - g^{\bar{k}r}\partial_r g^{\bar{i}l}\big) = 0 \quad \text{and c.c.,} \tag{B.14}$$

yields:

$$\partial_l H^{l\bar{i}\bar{k}} = 0 \quad \text{and c.c.} \tag{B.15}$$

These relations can be summarized as:

$$\nabla_\mu H^{\mu\nu\rho} - 2(\nabla_\lambda\Phi)H^{\lambda\nu\rho} = 0. \tag{B.16}$$

Note here, that in the case of Kähler metric g it is easy to show, that one does not need additional gauge constraint (15) to prove the coincidence of Eq. (17) with the vacuum Einstein equation $R_{i\bar{j}} = 0$.

References

[1] A.M. Polyakov, Gauge Fields and Strings, Harwood Academic, 1987.
[2] R.R. Metsaev, A.A. Tseytlin, Nucl. Phys. B 533 (1998) 109, hep-th/9805028.
[3] R.R. Metsaev, Nucl. Phys. B 625 (2002) 70, hep-th/0112044.
[4] E. Witten, Commun. Math. Phys. 118 (1988) 411.
[5] A.S. Cattaneo, G. Felder, Mod. Phys. Lett. A 16 (2001) 179, hep-th/0102208.
[6] L. Baulieu, A.S. Losev, N.A. Nekrasov, JHEP 0202 (2002) 021, hep-th/0106042.
[7] U. Lindstrom, Phys. Lett. B 587 (2004) 216, hep-th/0401100.
[8] M. Zabzine, hep-th/0502137.
[9] E. Witten, hep-th/0504078.
[10] E. Frenkel, A. Losev, hep-th/0505131.
[11] A.M. Polyakov, Nucl. Phys. B (Proc. Suppl.) 68 (1998) 1, hep-th/9711002;
A.M. Polyakov, V.S. Rychkov, Nucl. Phys. B 581 (2000) 116, hep-th/0002106;
A.M. Polyakov, V.S. Rychkov, Nucl. Phys. B 594 (2001) 272, hep-th/0005173;
V.S. Rychkov, JHEP 0212 (2002) 068, hep-th/0204250.
[12] E.S. Fradkin, A.A. Tseytlin, Phys. Lett. B 160 (1985) 69;
E.S. Fradkin, A.A. Tseytlin, Nucl. Phys. B 261 (1985) 1.
[13] C.G. Callan, D. Friedan, E.J. Martinec, M.J. Perry, Nucl. Phys. B 262 (1985) 593;
C.G. Callan, I.R. Klebanov, M.J. Perry, Nucl. Phys. B 278 (1986) 78.
[14] G.T. Horowitz, A.A. Tseytlin, Phys. Rev. D 50 (1994) 5204.
[15] F. Malikov, V. Schechtman, A. Vaintrob, math.AG/9803041.
[16] M. Kato, K. Ogawa, Nucl. Phys. B 212 (1983) 443;
D. Friedan, E.J. Martinec, S.H. Shenker, Phys. Lett. B 160 (1985) 55.
[17] V.A. Fock, Theory of Space, Time and Gravitation, Reader's Digest Young Families, 1964.