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Scalar and Hermite subdivision schemes

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Abstract

A criterion of convergence for stationary nonuniform subdivision schemes is provided. For periodic subdivision schemes, this criterion is optimal and can be applied to Hermite subdivision schemes which are not necessarily interpolatory. For the Merrien family of Hermite subdivision schemes which involve two parameters, we are able to describe explicitly the values of the parameters for which the Hermite subdivision scheme is convergent.

Keywords: Subdivision; Hermite interpolation

1. Introduction

Many authors have introduced and studied subdivision schemes for modeling curves. The notion of subdivision schemes has been extended many times. However we will restrict ourselves to the family of stationary nonuniform schemes on regular grids where the subdivision rule is only allowed to vary in location but not in scale.

More precisely, given an initial function $f_0 : \mathbb{Z} \to \mathbb{R}$, we generate a sequence of *refinements* $f_n : \mathbb{Z} \to \mathbb{R}$, $n = 1, 2, 3, \ldots$, according to the recursive rule

$$f_{n+1}(i) = \sum_{j\in\mathbb{Z}} s_{i,j} f_n(j), \quad i \in \mathbb{Z}, \tag{1}$$

where $S = (s_{i,j})$, $i \in \mathbb{Z}$, $j \in \mathbb{Z}$, is a matrix with a finite number of nonzero entries on any row. The matrix S is called the *subdivision matrix*. It is important to note that in this paper the control value $f_n(i)$ is attached to the dyadic point $i/2^n$. A *subdivision scheme* $\mathcal{S}$ is essentially a way of generating refinements with places to which are attached the control values $f_n(i)$.

A subdivision scheme is *uniform* if the weights of the subdivision matrix satisfy the relation $s_{i+2,j+1} = s_{i,j}$ for every $i, j \in \mathbb{Z}$. In that case, there is unique set of coefficients $\{a_i\}$ for which $s_{i,j} = a_{i-2j}$. This set of coefficients a_i is called the *mask* of the uniform subdivision scheme.

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Stationary subdivision schemes with spatially varying masks have been investigated by Dyn et al. [5], D. Levin [11] and his son A. Levin [10]. In their papers, they extended various results coming from stationary uniform subdivision schemes. As it is the case for us, such extensions are needed in some applications.

For a given subdivision scheme $\mathcal{S}$, one of the first questions to arise is its convergence.

Definition 1. A subdivision scheme $\mathcal{S}$ is C^0 *convergent*, or simply C^0 if, for every sequence of refinements $f_n:\mathbb{Z}\to\mathbb{R}$, the sequence of polygonal lines $\{(i/2^n, f_n(i)):\ i\in\mathbb{Z}\}$ converges uniformly in any finite interval to the graph of $f:\mathbb{R}\to\mathbb{R}$. f is called the *limit* of the refinements f_n.

In the first part of the paper, we find a criterion for the C^0 convergence of a subdivision scheme. In the second part, we apply this criterion in order to obtain necessary and sufficient conditions for the C^1 convergence of Hermite subdivision schemes.

A *Hermite subdivision scheme* $\mathcal{H}$ is a recursive scheme for computing a function $\phi:\mathbb{R}\to\mathbb{R}$ and its derivative ϕ'. The initial state of the scheme is a vector function $\{f_0:\mathbb{Z}\to\mathbb{R}^2\}$. The first component of f_0 is a control value for ϕ and the second component is a control value for ϕ'. The sequence of *refinements* $\{f_n:\mathbb{Z}\to\mathbb{R}^2,\ n>0\}$ is recursively defined through a family of 2×2 matrices $\{A_i:\ i\in\mathbb{Z}\}$, a finite number of which are nonzero.

$$D^{n+1}f_{n+1}(i)=\sum_{j\in\mathbb{Z}}A_{i-2j}D^n f_n(j),\quad i\in\mathbb{Z},\ n=0,1,2,\ldots, \tag{2}$$

where D is the diagonal matrix whose diagonal elements are $1, 1/2$.

The analysis of interpolatory Hermite subdivision schemes has been well carried by Dyn and Levin [6], but to our knowledge, there are very few studies of noninterpolatory Hermite subdivision schemes, we cite one by Han, Yu and Xue [8] and another by Dubuc and Merrien [2]. For noninterpolatory schemes, it is important to specify the definition of C^1 convergence for a given Hermite subdivision scheme. We use the same definition as in [2].

Definition 2. A Hermite subdivision scheme $\mathcal{H}$ is C^1 if, for every sequence of refinements $f_n:\mathbb{Z}\to\mathbb{R}^2$, both sequences of polygonal lines $\{(i/2^n, f_n^{(0)}(i)):\ i\in\mathbb{Z}\}$ and $\{(i/2^{n+1}, u_n(i)):\ i\in\mathbb{Z}\}$, where $f_n(i)=(f_n^{(0)}(i), f_n^{(1)}(i))^T$, $u_n(2i)=f_n^{(1)}(i)$, $u_n(2i+1)=2^n[f_n^{(0)}(i+1)-f_n^{(0)}(i)]$, converge uniformly in any finite interval.

We inform the reader that the definition of C^1 convergence for noninterpolatory subdivision schemes in [8] is different.

The paper is organized as follows. We describe the basic terminology for subdivision schemes in Section 2. In Section 3, we obtain a difference subdivision scheme from an affine subdivision scheme as in Daubechies, Guskov and Sweldens [1]. In Section 4, we provide a criterion of convergence for nonuniform subdivision schemes which has been anticipated by Maxim and Mazure [13]. For periodic subdivision schemes ($s_{i+2p,j+p}=s_{i,j}$), this criterion is optimal. In Section 5, we study the basic properties of C^1 convergence and the reproduction of linear functions for Hermite subdivision schemes. In Section 6, we use an idea from Dyn and Levin [6] which associate a nonuniform scalar subdivision scheme to any interpolating Hermite subdivision scheme that reproduces constants. Thus we obtain a necessary and sufficient condition for C^1 convergence for a given Hermite subdivision scheme that reproduces linear functions.

In Section 7, we recall the Merrien family of Hermite subdivision schemes which involve two parameters λ,μ. The nonzero matrices of the mask of the Hermite subdivision scheme $\mathcal{H}(\lambda,\mu)$ are

$$A_{-1}=\begin{pmatrix}1/2 & \lambda\\ \mu/2 & (1-\mu)/4\end{pmatrix},\qquad A_0=\begin{pmatrix}1 & 0\\ 0 & 1/2\end{pmatrix},\qquad A_1=\begin{pmatrix}1/2 & -\lambda\\ -\mu/2 & (1-\mu)/4\end{pmatrix}.$$

The goal of the last sections is to characterize the convergence region of this family which is, by definition, the set $\Omega=\{(\lambda,\mu):\ \mathcal{H}(\lambda,\mu)\text{ is }C^1\}$. After drawing the first 20 approximations of the convergence region, we find necessary conditions for convergence in the Merrien family (Section 8). These conditions are in fact sufficient, and in Section 9, we obtain the precise definition of the convergence region, that is

$$\Omega=\bigl\{(\lambda,\mu):\ 0<-\lambda<1/2,\ 0<\mu<\min\bigl(-1/(2\lambda), 3/(1+2\lambda)\bigr)\bigr\}.$$

Before concluding, we provide examples in Section 10.

2. Local property of subdivision schemes

Let $\mathcal{S}$ be a subdivision scheme with its subdivision matrix S, the notation for the (i,j)-entry of S^n will be $s_n(i,j)$. Given this notation and the fact that $S^{m+n} = S^m S^n$, we obtain the relation

$$s_{m+n}(i,j) = \sum_{k\in\mathbb{Z}} s_m(i,k)s_n(k,j). \tag{3}$$

One property will be assumed from the matrix S, the so-called *local* property. There exist two numbers σ, σ' such that $s_{i,j} = 0$ whenever $i - 2j \notin [\sigma, \sigma']$. The interval $[\sigma, \sigma']$ will be called a support for the subdivision scheme. In a uniform subdivision scheme of mask a_i, $\sigma = \min\{i\colon a_i \neq 0\}$ and $\sigma' = \max\{i\colon a_i \neq 0\}$.

Lemma 1. *If $[\sigma, \sigma']$ is a support of the subdivision scheme, if $i \in \mathbb{Z}$, $j \in \mathbb{Z}$ and $n \in \mathbb{N}$ are such that $i - 2^n j \notin [(2^n-1)\sigma, (2^n-1)\sigma']$, then $s_n(i,j) = 0$.*

Proof. The lemma is obviously true for $n = 1$. We proceed by induction, assuming that there exists $n \in \mathbb{N}$ satisfying the property

$$(\forall i, j \in \mathbb{Z})\ i - 2^n j \notin \big[\big(2^n-1\big)\sigma, \big(2^n-1\big)\sigma'\big] \ \Rightarrow\ s_n(i,j) = 0.$$

We consider two integers i, j for which $s_{n+1}(i,j) \neq 0$. According to (3) with $m = 1$, there exists $k \in \mathbb{Z}$ such that $s_{i,k} \neq 0$ and $s_n(k,j) \neq 0$. So that $\sigma \leqslant i - 2k \leqslant \sigma'$ and $(2^n-1)\sigma \leqslant k - 2^n j \leqslant (2^n-1)\sigma'$. It follows that $(2^{n+1}-1)\sigma \leqslant i - 2^{n+1} j \leqslant (2^{n+1}-1)\sigma'$. We get the conclusion. □

3. Difference subdivision schemes

Definition 3. A subdivision scheme is *affine* if

$$\sum_j s_{i,j} = 1 \quad \forall i \in \mathbb{Z}. \tag{4}$$

Definition 4. A subdivision scheme is *nondegenerate* if there exists a set of initial values $\{f_0(i)\}$ such that $\lim_{n\to\infty} f_n(0) = 1$.

Theorem 2. *Any nondegenerate C^0 convergent subdivision scheme is affine.*

Proof. Let $[\sigma, \sigma']$ be a support for the subdivision scheme. From Definition 4, we may choose f_0 such that $f(0) = 1$, where f is the limit function of the sequence of the refinements $f_n = S^n f_0$. For any $j \in \mathbb{Z}$, $\lim_{n\to\infty} j/2^n = 0$ and by C^0 convergence, we get $\lim_{n\to\infty} f_n(j) = f(0) = 1$.

Now, for every $i \in \mathbb{Z}$, we have

$$f_{n+1}(i) = \sum_{j\in\mathbb{Z}} s_{i,j} f_n(j) = \sum_{\sigma \leqslant i-2j \leqslant \sigma'} s_{i,j} f_n(j).$$

The sum is finite. Thus by taking the limit as $n \to \infty$, we get (4) and the subdivision scheme is affine. □

If $f: \mathbb{Z} \to \mathbb{R}$, we recall the definition of the function $\Delta f(i) = f(i+1) - f(i)$, $i \in \mathbb{Z}$. As in [4], we now prove that condition (4) guarantees the existence of a related subdivision scheme for the finite differences of the original control points.

Theorem 3 *(Daubechies, Guskov and Sweldens). If S is the subdivision matrix of an affine subdivision scheme $\mathcal{S}$ with $[\sigma, \sigma']$ as support, if $S' = (s'_{i,j})$ is the matrix defined as*

$$s'_{i,j} = -\sum_{k=-\infty}^{j} (s_{i+1,k} - s_{i,k}), \tag{5}$$

then S' is the subdivision matrix of a subdivision scheme $\mathcal{S}'$ with $[\sigma+1,\sigma']$ as support. Moreover, the commutation formula $S'\Delta=\Delta S$ is satisfied and, for any sequence of refinements f_n according to $\mathcal{S}$, the sequence of refinements generated by the initial function Δf_0 according to $\mathcal{S}'$ is Δf_n.

Proof. It is obvious that $s'_{i,j}=0$ if $i-2j>\sigma'$. Moreover, if $i-2j<\sigma+1$ and $k\geqslant j+1$, then $i+1-2k<\sigma$, $s_{i+1,k}=s_{i,k}=0$ and $s'_{i,j}=-\sum_{k=-\infty}^{j}(s_{i+1,k}-s_{i,k})=-\sum_{k=-\infty}^{\infty}(s_{i+1,k}-s_{i,k})$. From (4), the last series is 0. We deduce that $[\sigma+1,\sigma']$ is a support for S'.

Now we show that $S'\Delta=\Delta S$.

$$S'\big(\Delta f(i)\big)=\sum_{j\in\mathbb{Z}}s'_{i,j}\big(f(j+1)-f(j)\big)=\sum_{j\in\mathbb{Z}}(s'_{i,j-1}-s'_{i,j})f(j)=\sum_{j\in\mathbb{Z}}(s_{i+1,j}-s_{i,j})f(j)=\Delta(Sf)(i).$$

If f_n is a sequence of refinements according to $\mathcal{S}$, from the previous commutation formula, we get $S'(\Delta f_n)=\Delta(Sf_n)=\Delta f_{n+1}$. Δf_n are the refinements of the initial function Δf_0 according to the scheme $\mathcal{S}'$. □

The last proposition has been written after Proposition 10 of [1], the only new fact being a more precise statement about the support of $\mathcal{S}'$. We will call $\mathcal{S}'$ the *difference subdivision scheme* and note $\Delta\mathcal{S}=\mathcal{S}'$.

4. Convergence criterion for subdivision schemes

In this section, we provide a criterion of convergence for nonuniform subdivision schemes. For periodic subdivision schemes, this criterion is optimal.

Theorem 4. *Let $\mathcal{S}$ be an affine subdivision scheme and let S' be the subdivision matrix of the difference subdivision scheme $\mathcal{S}'$. We assume that the entries of the subdivision matrix S of $\mathcal{S}$ are uniformly bounded and that there is an integer m such that*

$$\nu_m=\sup\left\{\sum_{j\in\mathbb{Z}}\big|s'_m(i,j)\big|:\ i\in\mathbb{Z}\right\}<1. \tag{6}$$

Then $\mathcal{S}$ is C^0 convergent.

Proof. We assume that $[\sigma,\sigma']$ is a support for $\mathcal{S}$. We consider a sequence of refinements f_n of $\mathcal{S}$ and the corresponding sequence of piecewise linear functions $\phi_n:\mathbb{R}\to\mathbb{R}$, $\phi_n(i/2^n)=f_n(i)$, $i\in\mathbb{Z}$. If $a<b$ are two integers, let $\|\cdot\|_\infty$ denote the uniform norm in $C[a,b]$. We will show that ϕ_n converge to a function f in the uniform norm.

The values of ϕ_n on $[a,b]$ are uniquely determined by the numbers $f_n(i)$, $i\in[a2^n,b2^n]$. Since $f_n(i)=\sum_{i\in\mathbb{Z}}s_n(i,j)f_0(j)$, it follows from Lemma 1 that the restriction of ϕ_n to $[a,b]$ is uniquely determined by the values of $f_0(j)$, $j\in J$, where $J=[a-|\sigma'|,b+|\sigma|]$. Hence, there is no loss of generality by assuming that $f_0(j)=0$, $j\notin J$.

We set

$$\omega_n=\sup\big\{\big|\Delta f_n(i)\big|:\ i\in\mathbb{Z}\big\}.$$

From the commutation formula of Theorem 3, $\Delta S=S'\Delta$, we get $\Delta(S^n)=(S')^n\Delta$ and $\Delta f_{n+m}=(S')^m\Delta f_n$. It then follows that $\omega_{n+m}\leqslant\nu_m\omega_n$, $n=0,1,\ldots,$ and

$$\sum_{n=0}^{\infty}\omega_n\leqslant\sum_{n=0}^{m-1}\omega_n/(1-\nu_m).$$

Since the maximum difference on $[a,b]$ between ϕ_{n+1} and ϕ_n is attained at a point on the $(n+1)$th mesh, then $\|\phi_{n+1}-\phi_n\|_\infty\leqslant\max(M_n,M'_n)$, where

$$\begin{cases} M_n=\max\{|f_{n+1}(2i)-f_n(i)|:\ i\in\mathbb{Z}\},\\ M'_n=\max\{|f_{n+1}(2i+1)-(f_n(i)+f_n(i+1))/2|:\ i\in\mathbb{Z}\}.\end{cases}$$

As $\mathcal{S}$ is affine, we obtain

$$f_{n+1}(2i)-f_n(i)=\sum_{-\sigma'/2\leqslant j-i\leqslant-\sigma/2}s_{2i,j}\big(f_n(j)-f_n(i)\big). \tag{7}$$

We make three remarks. Firstly, the number of terms on the right-hand side of (7) which are nonzero does not exceed $1+(\sigma'-\sigma)/2$; secondly, by hypothesis, the entries $s_{2i,j}$ are bounded; and thirdly, $|f_n(j)-f_n(i)|\leqslant|j-i|\omega_n$. From these remarks, it follows that

$$(\exists\beta)\ \bigl|f_{n+1}(2i)-f_n(i)\bigr|\leqslant\beta\omega_n.$$

The number $2f_{n+1}(2i+1)-f_n(i)-f_n(i+1)$ is the sum of the four terms $f_{n+1}(2i)-f_n(i)$, $f_{n+1}(2i+2)-f_n(i+1)$, $f_{n+1}(2i+1)-f_{n+1}(2i)$ and $f_{n+1}(2i+1)-f_{n+1}(2i+2)$. It follows that $M_n'\leqslant M_n+\omega_{n+1}$. Thus, we get

$$\sum_{n=0}^{\infty}\|\phi_{n+1}-\phi_n\|_\infty\leqslant(\beta+1)\sum_{n=0}^{\infty}\omega_n<\infty.$$

From the Weierstrass criterion, the sequence ϕ_n converges uniformly in $[a,b]$. The interval $[a,b]$ may be arbitrarily large, the scheme $\mathcal{S}$ is C^0 convergent. □

The foregoing proof is an adaptation of the proof of Lemma 3.1 given by Dyn, Gregory and Levin [3]. The number ν_m in (6) is sufficiently important to be called the *mth-norm factor*. Theorem 4 is a particular case of Theorem 2.4 of Levin [11]. Maxim and Mazure also got a more general result [13, Theorem 3.2].

Definition 5. A subdivision scheme whose subdivision matrix is S is *periodic of period p* if $s_{i+2p,j+p}=s_{i,j}$ for any $i,j\in\mathbb{Z}$.

A uniform subdivision scheme is by definition periodic of period 1.

Lemma 5. *Let $\mathcal{S}$ be a periodic subdivision scheme of period p, then $s_n(i+p2^n,j+p)=s_n(i,j)$ for every $i,j\in\mathbb{Z}$ and for every $n\in\mathbb{N}$.*

Proof. We proceed by induction on n. After using (3), the periodicity of $\mathcal{S}$, a change of variable, and again the periodicity of $\mathcal{S}$ and (3), we obtain

$$\begin{aligned}s_{n+1}\bigl(i+p2^{n+1},j+p\bigr)&=\sum_{k\in\mathbb{Z}}s_1\bigl(i+p2^{n+1},k\bigr)s_n(k,j+p)=\sum_{k\in\mathbb{Z}}s_1\bigl(i,k-p2^n\bigr)s_n(k,j+p)\\&=\sum_{k\in\mathbb{Z}}s_1(i,k)s_n\bigl(k+p2^n,j+p\bigr)=\sum_{k\in\mathbb{Z}}s_1(i,k)s_n(k,j)=s_{n+1}(i,j).\qquad\square\end{aligned}$$

Lemma 6. *Let $\mathcal{S}$ be a periodic C^0 subdivision scheme, then for every $\varepsilon>0$ there exists an integer N such that $|s_n(i+1,j)-s_n(i,j)|<\varepsilon$ for every $i,j\in\mathbb{Z}$ and for every $n>N$.*

Proof. Let $\mathcal{S}$ be a periodic C^0 subdivision scheme whose support is $[\sigma,\sigma']$ and whose period is p. Let $j\in\mathbb{Z}$. We consider the initial function $f_0:\mathbb{Z}\to\mathbb{R}$, $f_0(i)=\delta_{ij}$ and the corresponding sequence of refinements f_n generated by $\mathcal{S}$. By induction, from (1) and (3), $f_n(i)=s_n(i,j)$. The sequence of polygonal lines $\{(i/2^n,f_n(i))\colon i\in\mathbb{Z}\}$ converges to a continuous function $\phi:\mathbb{R}\to\mathbb{R}$. From Lemma 1, we obtain that $i/2^n\notin[\min(\sigma,0)+j,\max(\sigma',0)+j]\Rightarrow s_n(i,j)=0$. By C^0 convergence, the sequence of polygonal lines converges uniformly in $[\min(\sigma,0)+j,\max(\sigma',0)+j]$. It follows that there exists a number N_j' such that $(\forall i\in\mathbb{Z})$ $(\forall n>N_j')$ $|s_n(i,j)-\phi(i/2^n)|\leqslant\varepsilon$. The function ϕ is uniformly continuous in $\mathbb{R}$, so there exists a number N_j'' such that $(\forall i\in\mathbb{Z})$ $(\forall n>N_j'')$ $|\phi((i+1)/2^n)-\phi(i/2^n)|\leqslant\varepsilon$. We set N as the maximum number among $N_0',N_1',\ldots,N_{p-1}',N_0'',N_1'',\ldots,N_{p-1}''$. If $n>N$, then from Lemma 5, for every $i,j\in\mathbb{Z}$, $|s_n(i+1,j)-s_n(i,j)|<3\varepsilon$. □

Theorem 7. *Let $\mathcal{S}$ be a periodic C^0 affine subdivision scheme and let S' be the subdivision matrix of the difference subdivision scheme $\mathcal{S}'$, then the entries of the subdivision matrix S of $\mathcal{S}$ are uniformly bounded and there is an integer m such that*

$$\nu_m=\sup\Biggl\{\sum_{j\in\mathbb{Z}}\bigl|s_m'(i,j)\bigr|\colon i\in\mathbb{Z}\Biggr\}<1.$$

Proof. Let $\mathcal{S}$ be a periodic C^0 subdivision scheme whose period is p and whose support is $[\sigma,\sigma']$. Let $M = \max\{|s_{i,j}|\colon i-2j \in [\sigma,\sigma'], j=0,1,\dots,p-1\}$, then by the local property and by periodicity, every entry of S is between $-M$ and M.

Let $j \in \mathbb{Z}$. We consider the initial function $g_0:\mathbb{Z}\to\mathbb{R}$, $g_0(i)=\delta_{ij}$, and the corresponding sequence of refinements g_n generated by $\mathcal{S}'$. By induction, from (1) and (3), $g_n(i)=s'_n(i,j)$. We define the function $f_0:\mathbb{Z}\to\mathbb{R}$ such that $f_0(i)=-1$ if $i\leqslant j$ and $f_0(i)=0$ if $i>j$. This definition implies that $g_0=\Delta f_0$. Let f_n be the corresponding sequence of refinements generated by $\mathcal{S}$. By linearity, $f_n(i)=-\sum_{k=-\infty}^{j} s_n(i,k)$. From Theorem 3, we obtain

$$s'_n(i,j)=g_n(i)=\Delta f_n(i)=-\sum_{k=-\infty}^{j}\bigl[s_n(i+1,k)-s_n(i,k)\bigr]. \tag{8}$$

We set ℓ as the smallest integer larger or equal to $(i+\sigma')/2^n-\sigma'$ and ℓ' as the largest integer smaller or equal to $(i+1+\sigma)/2^n-\sigma$. From Lemma 1, we obtain that $s_n(i,k)=s_n(i+1,k)=0$ if $k<\ell$ or $k>\ell'$. It follows that $s'_n(i,j)=0$ if $j<\ell$ or $j>\ell'$ and for every $j\in[\ell,\ell']$,

$$\bigl|s'_n(i,j)\bigr|\leqslant\sum_{k=\ell}^{\ell'}\bigl|s_n(i+1,k)-s_n(i,k)\bigr|. \tag{9}$$

Let $\varepsilon>0$. From Lemma 6, there exists an integer N such that $|s_n(i+1,j)-s_n(i,j)|<\varepsilon$ for every $i,j\in\mathbb{Z}$ and for every $n>N$. It can be shown that the number $\ell'-\ell$ is bounded by $\sigma'-\sigma+3$. It follows that for every $n>N$ and for every i and $j\in\mathbb{Z}$, $|s'_n(i,j)|\leqslant(\sigma'-\sigma+4)\varepsilon$.

From Theorem 3, the support of the subdivision matrix S' is $[\sigma+1,\sigma']$. From Lemma 1, the number of integers j for which $s'_n(i,j)\neq 0$ will not exceed $\sigma'-\sigma$. It follows that $\sum_{j\in\mathbb{Z}}|s'_n(i,j)|<(\sigma'-\sigma)(\sigma'-\sigma+4)\varepsilon$ for $n>N$ and $i\in\mathbb{Z}$. The inequality $\sup_i\sum_{j\in\mathbb{Z}}|s'_n(i,j)|<1$ is satisfied if ε has been chosen sufficiently small. □

5. C^1 Hermite subdivision schemes

In this section, we study the basic properties of the C^1 convergence and the reproduction of linear functions for Hermite subdivision schemes. In a Hermite subdivision scheme $\mathcal{H}$ with mask $\{A_i\}$, the *support* of $\mathcal{H}$ is the smallest interval $[\sigma,\sigma']$ containing $\{i\colon A_i\neq 0\}$. If $\{A_i=(a_{k\ell}(i))_{k,\ell=0,1}\colon i\in\mathbb{Z}\}$, there is another way of writing the recurrence rule (2) of the refinements f_n of $\mathcal{H}$

$$f^{(0)}_{n+1}(i)=\sum_{j\in\mathbb{Z}}\sum_{k=0}^{1}a_{0k}(i-2j)f^{(k)}_n(j)/2^{kn}, \tag{10}$$

$$f^{(1)}_{n+1}(i)/2^{n+1}=\sum_{j\in\mathbb{Z}}\sum_{k=0}^{1}a_{1k}(i-2j)f^{(k)}_n(j)/2^{kn} \tag{11}$$

for $i\in\mathbb{Z}$, where $f^{(0)}_n(i)$, $f^{(1)}_n(i)$ are the 2 components of the vector $f_n(i)$.

Definition 6. If $f_n:\mathbb{Z}\to\mathbb{R}^2$ is a sequence of refinements of a Hermite subdivision scheme, the *first sequence of polygonal lines* is given by $\{(i/2^n, f^{(0)}_n(i))\colon i\in\mathbb{Z}\}$ and the *second sequence of polygonal lines* is given by $\{(i/2^{n+1}, u_n(i))\colon i\in\mathbb{Z}\}$, where $u_n(2i)=f^{(1)}_n(i)$, $u_n(2i+1)=2^n\Delta f^{(0)}_n(i)$, $i\in\mathbb{Z}$.

As mentioned in Definition 2, the Hermite subdivision scheme is C^1 if, for any sequence f_n of refinements, the two previous polygonal lines converge. If $f^{(0)}$ is the limit function of the first sequence of polygonal lines and $f^{(1)}$ is the limit function of the second sequence of polygonal lines, then $f^{(1)}$ is continuous, $f^{(0)}$ is C^1 and its derivative is $f^{(1)}$.

Let us consider an example of noninterpolatory Hermite subdivision scheme. Let $\mathcal{H}$ be the Hermite subdivision scheme whose nonzero matrices of its mask are

$$A_{-1}=\begin{pmatrix}1/2 & -1/8+c/2\\ 3/4 & -1/8\end{pmatrix},\qquad A_0=\begin{pmatrix}1 & c\\ 0 & 1/2\end{pmatrix},\qquad A_1=\begin{pmatrix}1/2 & 1/8+c/2\\ -3/4 & -1/8\end{pmatrix}.$$

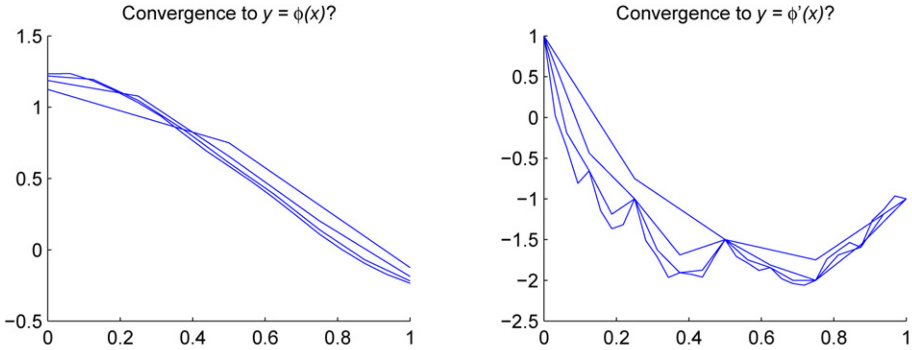

Fig. 1. The first four polygonal lines, first and second sequence.

c is a parameter. We set $c = 1/8$, $f_0(0) = (1, 1)^T$, $f_0(1) = (0, -1)^T$. In Fig. 1, the first four polygonal lines $\{(i/2^n, f_n^{(0)}(i)):\ i = 0, 1, \dots, 2^n\}$, $n = 1, 2, 3, 4$, are given at left and the first four polygonal lines $\{(i/2^{n+1}, u_n(i)):\ i = 0, 1, \dots, 2^n\}$, $n = 1, 2, 3, 4$, are given at right. A question arises, does these two sequences of polygonal lines converge? The answer will be given in Section 10.

The next lemma shows that the previous definition of C^1 convergence is consistent with the definition of C^1 convergence for interpolatory Hermite subdivision schemes as given by Dyn and Levin [6].

Lemma 8. *In a given interpolatory Hermite subdivision scheme, we assume that, for every initial vector function $f_0 : \mathbb{Z} \to \mathbb{R}^2$, there is a C^1-function $\phi : \mathbb{R} \to \mathbb{R}$ for which*

$$f_n^{(0)}(i) = \phi\big(i/2^n\big), \qquad f_n^{(1)}(i) = \phi'\big(i/2^n\big), \quad i \in \mathbb{Z},\ n \in \mathbb{N};$$

then the Hermite subdivision scheme is C^1.

Proof. It is a simple consequence of the uniform continuity of ϕ, ϕ' in bounded intervals and of the mean value theorem. □

Lemma 9. *Let $\mathcal{H}$ be a Hermite subdivision scheme. If for every initial vector function $f_0 : \mathbb{Z} \to \mathbb{R}^2$, the sequence $f_n^{(0)}(0)$ converges and the second sequence of polygonal lines converges uniformly in finite interval, then $\mathcal{H}$ is C^1.*

Proof. Let $f_n = (f_n^{(0)}, f_n^{(1)})^T$ be a sequence of refinements of $\mathcal{H}$. If $y_0 = \lim_{n\to\infty} f_n^{(0)}(0)$ and if $\phi_1 : \mathbb{R} \to \mathbb{R}$ is the limit of the second sequence of polygonal lines of $\mathcal{H}$, then we define $\phi_0(x) = y_0 + \int_0^x \phi_1(t)\,\mathrm{d}t$. We will show that the first sequence of polygonal lines of $\mathcal{H}$ converges to ϕ_0.

Let $L > 0$, $\varepsilon > 0$. There exists an integer N such that $\forall n > N$,

$$\begin{aligned}
&\big|f_n^{(0)}(0) - y_0\big| < \varepsilon,\\
&\forall i \in \big[0, L2^n\big],\ \big|2^n \Delta f_n^{(0)}(i) - \phi_1\big(i/2^n\big)\big| < \varepsilon,\\
&\forall i \in \big[0, L2^n\big],\ \forall t \in \big[i/2^n, (i+1)/2^n\big],\ \big|\phi_1(t) - \phi_1\big(i/2^n\big)\big| < \varepsilon.
\end{aligned}$$

For $n > N$ and $0 \leqslant i \leqslant L2^n$, we get

$$\left| f_n^{(0)}(i) - f_n^{(0)}(0) - \sum_{j=0}^{i-1} \phi_1\big(j/2^n\big)/2^n \right| \leqslant \sum_{j=0}^{i-1} \big|\Delta f_n^{(0)}(j) - \phi_1\big(j/2^n\big)/2^n\big| < L\varepsilon,$$

$$\left| \sum_{j=0}^{i-1} \phi_1\big(j/2^n\big)/2^n - \phi_0\big(i/2^n\big) + y_0 \right| < L\varepsilon$$

and

$$\big|f_n^{(0)}(i) - \phi_0\big(i/2^n\big)\big| < (2L+1)\varepsilon.$$

By a similar argument, the same inequality holds for $i \in [-L2^n, 0]$ and any large enough n. This shows that the first sequence of polygonal lines converges uniformly in any finite interval. $\mathcal{H}$ is C^1. □

Definition 7. A Hermite subdivision scheme of mask $A_i = \left(\begin{smallmatrix} a_{00}(i) & a_{01}(i) \\ a_{10}(i) & a_{11}(i) \end{smallmatrix}\right)$ *reproduces constants* if

$$\sum_{j\in\mathbb{Z}} a_{00}(i-2j) = 1, \qquad \sum_{j\in\mathbb{Z}} a_{10}(i-2j) = 0 \tag{12}$$

for every $i \in \mathbb{Z}$.

Definition 8. A Hermite subdivision scheme of mask $A_i = \left(\begin{smallmatrix} a_{00}(i) & a_{01}(i) \\ a_{10}(i) & a_{11}(i) \end{smallmatrix}\right)$ *reproduces linear functions* if it reproduces the constants and if there exists a number c such that

$$\begin{aligned} &\sum_{j\in\mathbb{Z}} \big[a_{00}(i-2j)j + a_{01}(i-2j)\big] = i/2 + c, \\ &\sum_{j\in\mathbb{Z}} \big[a_{10}(i-2j)j + a_{11}(i-2j)\big] = 1/2 \end{aligned} \tag{13}$$

for every $i \in \mathbb{Z}$.

Remark. If the Hermite subdivision scheme is interpolatory, the number c in (13) is equal to 0.

Definition 9. A Hermite subdivision scheme is *nondegenerate* if, for any vector y of $\mathbb{R}^2$, there exists a function $\{f_0 : \mathbb{Z} \to \mathbb{R}^2\}$ such that $\lim_{n\to\infty} f_n(0) = y$.

Theorem 10. *If a Hermite subdivision scheme is C^1 and nondegenerate, then the linear functions are reproduced.*

Proof. According to Definition 9, we may consider a set of initial vectors $\{f_0(i)\}$ for which $\lim_{n\to\infty} f_n(0) = (1, 0)^T$, where the refinements f_n are generated by f_0. By Definition 2, $(\forall i \in \mathbb{Z})$ $\lim_{n\to\infty} f_n^{(0)}(i) = 1$. By taking the limit as $n \to \infty$ in (10)–(11), we obtain, respectively, $\sum_{j\in\mathbb{Z}} a_{00}(i-2j) = 1$ and $\sum_{j\in\mathbb{Z}} a_{10}(i-2j) = 0$. The reproduction of constants (12) is satisfied.

We now consider another sequence of refinements f_n, one for which $\lim_{n\to\infty} f_n(0) = (0, 1)^T$. By Definition 2, $(\forall i \in \mathbb{Z})$ $\lim_{n\to\infty} 2^n \Delta f_n^{(0)}(i) = 1$ and $\lim_{n\to\infty} f_n^{(1)}(i) = 1$. From (10)–(11) and from the reproduction of constants, we obtain

$$\begin{aligned} 2^n\big[f_{n+1}^{(0)}(i) - f_n^{(0)}(0)\big] &= \sum_{j\in\mathbb{Z}} \big[a_{00}(i-2j)2^n\big(f_n^{(0)}(j) - f_n^{(0)}(0)\big) + a_{01}(i-2j) f_n^{(1)}(j)\big], \\ f_{n+1}^{(1)}(i)/2 &= \sum_{j\in\mathbb{Z}} \big[a_{10}(i-2j)2^n\big(f_n^{(0)}(j) - f_n^{(0)}(0)\big) + a_{11}(i-2j) f_n^{(1)}(j)\big]. \end{aligned}$$

We take the limit as $n \to \infty$ in the last two equations noting that $\lim_{n\to\infty} 2^n(f_n^{(0)}(i) - f_n^{(0)}(0)) = i$. We obtain

$$\begin{aligned} &\sum_{j\in\mathbb{Z}} \big[a_{00}(i-2j)j + a_{01}(i-2j)\big] = i/2 + c, \\ &\sum_{j\in\mathbb{Z}} \big[a_{10}(i-2j)j + a_{11}(i-2j)\big] = 1/2, \end{aligned}$$

where $c = \lim_{n\to\infty} 2^n(f_{n+1}^{(0)}(0) - f_n^{(0)}(0))$. □

Lemma 11. *Let $\mathcal{H}$ be a Hermite subdivision scheme which reproduces the constants and assuming that the second sequence of polygonal lines converges, then $\mathcal{H}$ is C^1.*

Proof. Let $A_i = \binom{a_{00}(i)\ a_{01}(i)}{a_{10}(i)\ a_{11}(i)}$ be the mask of $\mathcal{H}$. First, we prove the convergence of the sequence $f_n^{(0)}(0)$. By using the reproduction of constants of $\mathcal{H}$, we get

$$f_{n+1}^{(0)}(0) - f_n^{(0)}(0) = \sum_{j=-M}^{M} \big[a_{00}(-2j)\big(f_n^{(0)}(j) - f_n^{(0)}(0)\big) + a_{01}(-2j) f_n^{(1)}(j)/2^n\big],$$

where $M = \max\{|i|\colon\ A_{2i} \neq 0\}$. Since for $i = -M, \dots, M$, both sequences $2^n \Delta f_n^{(0)}(i)$, $f_n^{(1)}(i)$ are bounded, then $f_{n+1}^{(0)}(0) - f_n^{(0)}(0) = O(1/2^n)$ and $f_n^{(0)}(0)$ converge as $n \to \infty$. The hypotheses of Lemma 9 are fulfilled. It follows that $\mathcal{H}$ is C^1. $\square$

6. The associated subdivision scheme

In this section, we associate a subdivision scheme $\mathcal{S}$ to a Hermite subdivision scheme $\mathcal{H}$ which reproduces the constants. Furthermore, there is a strong relationship between C^1 convergence in $\mathcal{H}$ and C^0 convergence in $\mathcal{S}$. We obtain a necessary and sufficient condition for C^1 convergence for any Hermite subdivision scheme which reproduces linear functions.

Theorem 12. *Let $f_n = (f_n^{(0)}, f_n^{(1)})^T$, $n = 0, 1, 2, \dots$, be the refinements of a Hermite subdivision scheme of support $[\sigma, \sigma']$ and of mask $A_i = \binom{a_{00}(i)\ a_{01}(i)}{a_{10}(i)\ a_{11}(i)}$ which reproduces the constants. We set $u_n(2i) = f_n^{(1)}(i)$, $u_n(2i+1) = 2^n \Delta f_n^{(0)}(i)$,*

$$B_i = \begin{pmatrix} b_{00}(i) & b_{01}(i) \\ b_{10}(i) & b_{11}(i) \end{pmatrix} = 2\begin{pmatrix} \Delta \sum_{k=1}^{\infty} a_{00}(i-2k) & \Delta a_{01}(i) \\ \sum_{k=1}^{\infty} a_{10}(i-2k) & a_{11}(i) \end{pmatrix}, \tag{14}$$

and $S = (s_{i,j})$, where

$$\begin{aligned} s_{2i,2j} &= b_{11}(i-2j), & s_{2i,2j+1} &= b_{10}(i-2j), \\ s_{2i+1,2j} &= b_{01}(i-2j), & s_{2i+1,2j+1} &= b_{00}(i-2j). \end{aligned} \tag{15}$$

Then S is periodic of period 2, $B_i = 0$ if $i \notin [\sigma - 1, \sigma']$ and the sequence u_n is the sequence of refinements of the subdivision scheme $\mathcal{S}$ whose subdivision matrix is S.

Proof. The periodicity with period 2 of S, $s_{i+4,j+2} = s_{i,j}$, is easily verified. Let $[\sigma, \sigma']$ be the support of $\mathcal{H}$. We define 16 integers

$$\begin{aligned} \sigma_{k\ell} &= \min\big\{i\colon\ a_{k\ell}(i) \neq 0\big\}, & \sigma'_{k\ell} &= \max\big\{i\colon\ a_{k\ell}(i) \neq 0\big\}, \\ \tau_{k\ell} &= \min\big\{i\colon\ b_{k\ell}(i) \neq 0\big\}, & \tau'_{k\ell} &= \max\big\{i\colon\ b_{k\ell}(i) \neq 0\big\}, \end{aligned} \quad k, \ell = 0, 1.$$

The following equations hold: $\tau_{11} = \sigma_{11}$, $\tau_{01} = \sigma_{01} - 1$, $\tau_{10} = \sigma_{10} + 2$, $\tau_{00} = \sigma_{00} + 1$ and $\tau'_{k\ell} = \sigma'_{k\ell}$, $k, \ell = 0, 1$, as a consequence of the reproduction of constants. As $\sigma = \min\{\sigma_{k\ell}\colon\ k, \ell = 0, 1\}$ and $\sigma' = \max\{\sigma'_{k\ell}\colon\ k, \ell = 0, 1\}$, then $B_i = 0$ if $i \notin [\sigma - 1, \sigma']$.

For $i \in \mathbb{Z}$ and $n \geqslant 0$, (11) is equivalent to

$$f_{n+1}^{(1)}(i) = \sum_j 2^{n+1} a_{10}(i-2j) f_n^{(0)}(j) + \sum_j 2a_{11}(i-2j) f_n^{(1)}(j). \tag{16}$$

We note that $b_{10}(i+2) - b_{10}(i) = 2a_{10}(i)$, so that

$$\begin{aligned} \sum_j 2a_{10}(i-2j) 2^n f_n^{(0)}(j) &= \sum_j \big(b_{10}(i+2-2j) - b_{10}(i-2j)\big) 2^n f_n^{(0)}(j) \\ &= \sum_j b_{10}(i-2j) 2^n f_n^{(0)}(j+1) - \sum_j b_{10}(i-2j) 2^n f_n^{(0)}(j) \\ &= \sum_j b_{10}(i-2j) 2^n \Delta f_n^{(0)}(j). \end{aligned}$$

After substituting in (16), we obtain

$$f_{n+1}^{(1)}(i) = \sum_j \bigl[b_{10}(i-2j)2^n \Delta f_n^{(0)}(j) + b_{11}(i-2j) f_n^{(1)}(j)\bigr]. \tag{17}$$

By using (10) twice, we obtain

$$f_{n+1}^{(0)}(i+1) = \sum_j a_{00}(i+1-2j) f_n^{(0)}(j) + \sum_j a_{01}(i+1-2j) f_n^{(1)}(j)/2^n,$$

$$f_{n+1}^{(0)}(i) = \sum_j a_{00}(i-2j) f_n^{(0)}(j) + \sum_j a_{01}(i-2j) f_n^{(1)}(j)/2^n.$$

Since $b_{00}(i+2) - b_{00}(i) = 2\Delta a_{00}(i)$ and $b_{01}(i) = 2\Delta a_{01}(i)$, we deduce

$$\begin{aligned} 2^{n+1}\Delta f_{n+1}^{(0)}(i) &= \sum_j 2\bigl(a_{00}(i+1-2j) - a_{00}(i-2j)\bigr)2^n f_n^{(0)}(j) \\ &\quad + \sum_j 2\bigl(a_{01}(i+1-2j) - a_{01}(i-2j)\bigr) f_n^{(1)}(j) \\ &= \sum_j \bigl(b_{00}(i+2-2j) - b_{00}(i-2j)\bigr)2^n f_n^{(0)}(j) + \sum_j b_{01}(i-2j) f_n^{(1)}(j). \end{aligned}$$

This gives

$$2^{n+1}\Delta f_{n+1}^{(0)}(i) = \sum_j \bigl[b_{00}(i-2j)2^n \Delta f_n^{(0)}(j) + b_{01}(i-2j) f_n^{(1)}(j)\bigr]. \tag{18}$$

Formulae (17)–(18) are equivalent to the following equation:

$$u_{n+1}(i) = \sum_{j\in\mathbb{Z}} s_{i,j} u_n(j), \quad i \in \mathbb{Z},$$

with the subdivision matrix $S = (s_{i,j})$ given by (15). This means that u_n is the sequence of refinements of the corresponding subdivision scheme. □

Definition 10. The subdivision scheme whose subdivision matrix is S is called the subdivision scheme *associated* to $\mathcal{H}$.

Theorem 13. *Let $\mathcal{H}$ be a Hermite subdivision scheme which reproduces the linear functions, let $\mathcal{S}$ be the associated subdivision scheme, then $\mathcal{S}$ is affine.*

Proof. We continue with the same notation as in the previous theorem. From (15), we get

$$\begin{aligned} \sum_{j\in\mathbb{Z}} s_{2i,j} &= \sum_{j\in\mathbb{Z}} \bigl[b_{11}(i-2j) + b_{10}(i-2j)\bigr], \\ \sum_{j\in\mathbb{Z}} s_{2i+1,j} &= \sum_{j\in\mathbb{Z}} \bigl[b_{01}(i-2j) + b_{00}(i-2j)\bigr]. \end{aligned} \tag{19}$$

Let $i \in \mathbb{Z}$ and let $[\sigma, \sigma']$ be the support of $\mathcal{H}$. We set

$$J = \min\{j\colon\ i-2j \leqslant \sigma'\}, \qquad J' = \max\{j\colon\ i-2j \geqslant \sigma - 1\}.$$

It follows that

$$\sum_{j\in\mathbb{Z}} \bigl[b_{11}(i-2j) + b_{10}(i-2j)\bigr] = \sum_{j=J}^{J'} \bigl[b_{11}(i-2j) + b_{10}(i-2j)\bigr] \tag{20}$$

(from Theorem 12, $B_k = 0$ if $k \notin [\sigma - 1, \sigma')$. From (14), we get

$$\sum_{j=J}^{J'} \big[b_{11}(i-2j) + b_{10}(i-2j)\big] = 2\sum_{j=J}^{J'} \left[\sum_{k=1}^{\infty} a_{10}(i-2j-2k) + a_{11}(i-2j)\right]. \tag{21}$$

We simplify the following double sum:

$$\sum_{j=J}^{J'} \sum_{k=1}^{\infty} a_{10}(i-2j-2k) = \sum_{j=J}^{J'} \sum_{\ell=j+1}^{\infty} a_{10}(i-2\ell) = \sum_{j=J}^{J'} \sum_{\ell=j+1}^{J'} a_{10}(i-2\ell) = \sum_{\ell=J+1}^{J'} (\ell - J) a_{10}(i-2\ell),$$

$$\sum_{j=J}^{J'} \sum_{k=1}^{\infty} a_{10}(i-2j-2k) = \sum_{\ell=J}^{J'} (\ell - J) a_{10}(i-2\ell). \tag{22}$$

From (19)–(22), we obtain

$$\sum_{j\in\mathbb{Z}} s_{2i,j} = 2\sum_{j=J}^{J'} \big[(j-J)a_{10}(i-2j) + a_{11}(i-2j)\big] = 2\sum_{j=-\infty}^{\infty} \big[(j-J)a_{10}(i-2j) + a_{11}(i-2j)\big] = 1.$$

The last equation comes from the two equations $\sum_{j\in\mathbb{Z}}[a_{10}(i-2j)j + a_{11}(i-2j)] = 1/2$ and $\sum_{j\in\mathbb{Z}} a_{10}(i-2j) = 0$ (see (12) and (13)).

Similarly, from (14), we get

$$\sum_{j\in\mathbb{Z}} \big[b_{01}(i-2j) + b_{00}(i-2j)\big] = \sum_{j=J}^{J'} \big[b_{01}(i-2j) + b_{00}(i-2j)\big]$$
$$= 2\Delta \sum_{j=J}^{J'} \left[\sum_{k=1}^{\infty} a_{00}(i-2j-2k) + a_{01}(i-2j)\right].$$

In a similar way, we simplify the following double sum:

$$\sum_{j=J}^{J'} \sum_{k=1}^{\infty} a_{00}(i-2j-2k) = \sum_{j=J}^{J'} (j-J) a_{00}(i-2j)$$

and we obtain

$$\sum_{j\in\mathbb{Z}} s_{2i+1,j} = 2\Delta \sum_{j=J}^{J'} \big[(j-J)a_{00}(i-2j) + a_{01}(i-2j)\big]$$
$$= 2\Delta \sum_{j=-\infty}^{\infty} \big[(j-J)a_{00}(i-2j) + a_{01}(i-2j)\big]$$
$$= 1.$$

The last equation comes from the two equations $\sum_{j\in\mathbb{Z}}[a_{00}(i-2j)j + a_{01}(i-2j)] = i/2 + c$ and $\sum_{j\in\mathbb{Z}} a_{00}(i-2j) = 1$ (see (12) and (13)). That $\mathcal{S}$ is affine is proved. □

Theorem 14. *Let $\mathcal{H}$ be a Hermite subdivision scheme which reproduces the linear functions, let $\mathcal{S}$ be the associated subdivision scheme and let S' be the subdivision matrix of the difference subdivision scheme $\mathcal{S}'$, then $\mathcal{H}$ is C^1 iff there is an integer n such that*

$$\nu_n = \sup\left\{\sum_{j\in\mathbb{Z}} \big|s'_n(i,j)\big|:\ i\in\mathbb{Z}\right\} < 1.$$

Proof. Let us assume that there is an integer n such that $\nu_n < 1$. From Theorem 4, the subdivision scheme $\mathcal{S}$ is C^0. Then the second sequence of polygonal lines of the Hermite subdivision scheme $\mathcal{H}$ given by $\{(i/2^{n+1}, u_n(i)): i \in \mathbb{Z}\}$, where $u_n(2i) = f_n^{(1)}(i), u_n(2i+1) = 2^n \Delta f_n^{(0)}(i)$, $i \in \mathbb{Z}$, is convergent. From Lemma 11, the Hermite subdivision scheme $\mathcal{H}$ is C^1.

Conversely, let us assume that $\mathcal{H}$ is C^1. It follows that $\mathcal{S}$ is a C^0 subdivision scheme. From Theorem 7, there is an integer n such that $\nu_n < 1$. □

We note that the last theorem has been proved by Dyn and Levin [6] in the case of *interpolatory* Hermite subdivision schemes.

7. The Merrien family of Hermite subdivision schemes

As a model example, we recall a scheme treated by Merrien in his article [14]. The nonzero matrices of the mask of this Hermite subdivision scheme $\mathcal{H}$ are

$$A_{-1} = \begin{pmatrix} 1/2 & \lambda \\ \mu/2 & (1-\mu)/4 \end{pmatrix}, \qquad A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 1/2 & -\lambda \\ -\mu/2 & (1-\mu)/4 \end{pmatrix}.$$

This Hermite subdivision scheme $\mathcal{H} = \mathcal{H}(\lambda, \mu)$ involves two parameters of $\mathbb{R}$. For $\lambda = -1/8$ and $\mu = 3/2$, the scheme produces the Hermite-cubic interpolant to the given initial Hermite data $\{f_0(i): i \in \mathbb{Z}\}$ and thus is C^1. The goal of this section is to characterize the following subset of the plane $\Omega = \{(\lambda, \mu): \mathcal{H}(\lambda, \mu) \text{ is } C^1\}$. The set Ω will be called the *convergence* region for the Merrien family of Hermite subdivision schemes.

For every choice of λ and μ, the Hermite subdivision scheme reproduces linear functions. The associated subdivision scheme $\mathcal{S} = \mathcal{S}(\lambda, \mu)$ is defined through the sequence of matrices B_i, $i \in \mathbb{Z}$, where $B_{-2} = \left(\begin{smallmatrix} 0 & 2\lambda \\ 0 & 0 \end{smallmatrix}\right)$, $B_{-1} = \left(\begin{smallmatrix} 0 & -2\lambda \\ 0 & (1-\mu)/2 \end{smallmatrix}\right)$, $B_0 = \left(\begin{smallmatrix} 1 & -2\lambda \\ 0 & 1 \end{smallmatrix}\right)$, $B_1 = \left(\begin{smallmatrix} 1 & 2\lambda \\ \mu & (1-\mu)/2 \end{smallmatrix}\right)$ and the remaining matrices are all equal to the 2×2 zero matrix.

The subdivision scheme $\mathcal{S}$ is affine and periodic of period 2. We are able to characterize Ω by using the subdivision matrix $S' = (s'(i, j): i \in \mathbb{Z},\ j \in \mathbb{Z})$ of the difference subdivision scheme $\mathcal{S}' = \Delta\mathcal{S}$. From Theorem 14,

$$\Omega = \{(\lambda, \mu): (\exists n \in \mathbb{N})\ \nu_n(\lambda, \mu) < 1\}.$$

Let us describe the computation of the functions $\nu_n : \mathbb{R}^2 \to \mathbb{R}$. We begin with the matrix S'. On the four rows $i = 0, 1, 2, 3$, the nonzero entries of S' are

$$\big(s'(i, j)\big)_{i=0,1,2,3,\ j=0,1} = \begin{pmatrix} 1+2\lambda & 2\lambda \\ -1/2-2\lambda+\mu/2 & 1/2-2\lambda-\mu/2 \\ 1/2-2\lambda-\mu/2 & -1/2-2\lambda+\mu/2 \\ 2\lambda & 1+2\lambda \end{pmatrix}. \tag{23}$$

Every other row is a translation of one of these four rows

$$s'(i+4k, j+2k) = s'(i, j), \quad i = 0, 1, 2, 3,\ j = 0, 1,\ k \in \mathbb{Z}. \tag{24}$$

From Lemma 5, we obtain

$$s'_n\big(i + 2^{n+1}k, j + 2k\big) = s'_n(i, j), \quad i \in \big[0, 2^{n+1} - 1\big],\ j \in \mathbb{Z},\ k \in \mathbb{Z}. \tag{25}$$

Using the relation $(S')^{n+1} = (S')^n S'$ and by induction, we obtain

$$s'_n(i, j) \neq 0 \quad \text{and} \quad i \in \big[0, 2^{n+1} - 1\big] \ \Rightarrow\ j \in \{0, 1\}. \tag{26}$$

From (25)–(26), it follows that

$$\nu_n(\lambda, \mu) = \max\big\{\big|s'_n(i, 0)\big| + \big|s'_n(i, 1)\big|: 0 \leqslant i < 2^{n+1}\big\}.$$

We define the nth approximation of Ω as

$$\Omega_n = \big\{(\lambda, \mu): (\exists k \leqslant n)\ \nu_k(\lambda, \mu) < 1\big\}.$$

In Fig. 2, we provide the approximations of Ω up to order 20. Ω_1 is empty and Ω_2 is a connected open subset of the second quadrant. For every n between 2 and 20, the boundary of Ω_n is drawn. The points $A(0, 2)$, $D(-1/4, 2)$ and $G(-1/4, 0)$ belong to the boundary of every Ω_n. As we will show in Section 9, Ω is bounded by three segments OB, OF, EF and two arcs BC, CE. In Fig. 2, these arcs are shown as dashed lines. It follows that the convergence of Ω_n to Ω is slow.

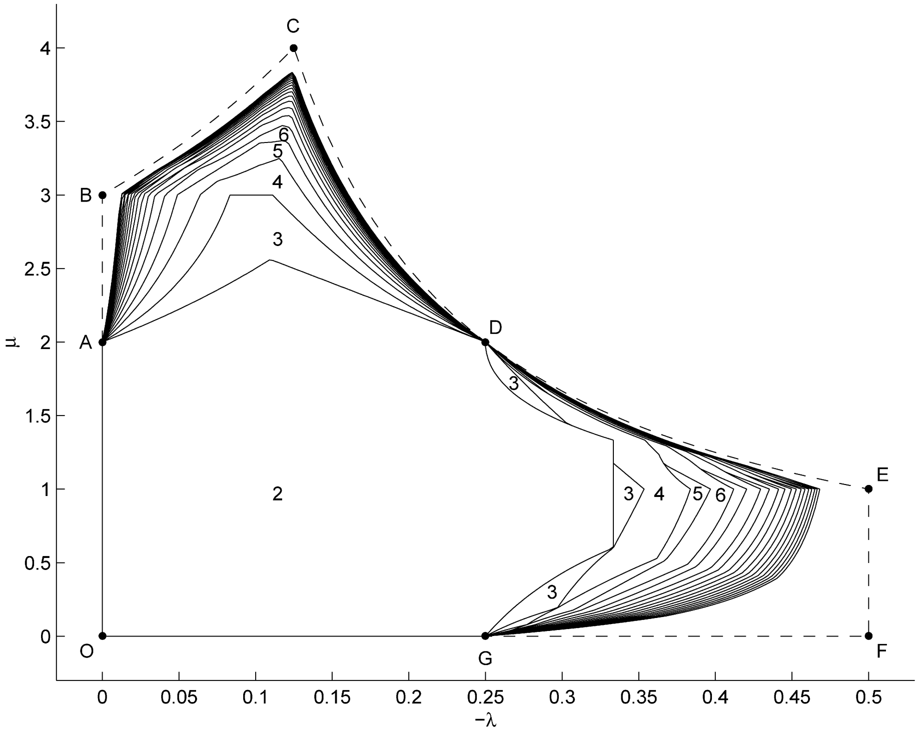

Fig. 2. The first 20 approximations of the convergence region for the Merrien family of Hermite subdivision schemes.

8. Necessary conditions of convergence

In order to study the convergence region Ω, we consider the two following matrices:

$$C_1=\begin{pmatrix}1+2\lambda & 2\lambda\\ -1/2-2\lambda+\mu/2 & 1/2-2\lambda-\mu/2\end{pmatrix},\qquad C_2=\begin{pmatrix}1/2-2\lambda-\mu/2 & -1/2-2\lambda+\mu/2\\ 2\lambda & 1+2\lambda\end{pmatrix}.$$

The main point to note for the analysis of the C^1 convergence of $\mathcal{H}(\lambda,\mu)$ is that any power $(S')^n$ is obtained from products of n matrices, each one being equal to C_1 or C_2. More precisely, we set $P_1=\{C_1,C_2\}$, $P_2=\{C_1C_1,C_2C_1,C_1C_2,C_2C_2\},\ldots,P_{n+1}=\{CC_1\colon\ C\in P_n\}\cup\{CC_2\colon\ C\in P_n\}$. The $2^{n+1}\times 2$ matrix $(s_n'(i,j)\colon\ i=0,1,\ldots,2^{n+1}-1,\ j=0,1)$ is the stacking of the successive 2×2 matrices of P_n. As a result, we obtain that

$$\nu_n(\lambda,\mu)=\max\bigl\{\|C\|_\infty\colon\ C\in P_n\bigr\},\tag{27}$$

where $\|(c_{i,j}\colon\ i=1,2,\ j=1,2)\|_\infty$ is the number $\max(|c_{11}|+|c_{12}|,|c_{21}|+|c_{22}|)$.

Lemma 15. (Merrien [14]) *Let C be a 2×2 matrix with real entries, then its spectral radius $\rho(C)$ is <1 iff $-1+|s|<p<1$, where s is the trace of C and p is the determinant of C.*

Proof. The characteristic polynomial of C is $\det C-zI=z^2-sz+p$. The eigenvalues of C are $z=(s\pm\sqrt{(s^2-4p)})/2$. If the discriminant $\Delta=s^2-4p$ is negative, both eigenvalues are in the unit disk iff $p<1$. If $\Delta\geqslant 0$, both eigenvalues are in $(-1,1)$ iff $|s|<1+p$. The conclusion is that $\rho(C)<1$ iff $-1+|s|<p<1$. □

Lemma 16. *The convergence region Ω is a subset of*

$$\bigl\{(\lambda,\mu)\colon\ 0<-\lambda<1/2,\ 0<\mu<\min\bigl(-1/(2\lambda),3/(1+2\lambda)\bigr)\bigr\}.\tag{28}$$

Proof. Let us assume that $(\lambda,\mu)\in\Omega$. There is an integer $n>1$ for which $\nu_n(\lambda,\mu)<1$. We first show that the spectral radius of C_1, $\rho(C_1)$ is <1. Indeed, if z is an eigenvalue of C_1, then z^n is an eigenvalue of C_1^n. From (27), we obtain $|z^n|\leqslant\|C_1^n\|_\infty\leqslant\nu_n(\lambda,\mu)<1$. The inequalities $|z|<1$ and $\rho(C_1)<1$ are satisfied.

Similarly, we show that $\rho(C_1C_2) < 1$. Indeed, if z is an eigenvalue of C_1C_2, then z^n is an eigenvalue of $(C_1C_2)^n$. From (27), we obtain $|z^n| \leqslant \|(C_1C_2)^n\|_\infty \leqslant \nu_{2n}(\lambda,\mu) \leqslant \nu_n^2(\lambda,\mu) < 1$. The inequality $\rho(C_1C_2) < 1$ is satisfied.

The trace of C_1 is $(3-\mu)/2$ and the determinant of C_1 is $(1-\mu-4\lambda\mu)/2$. Since $\rho(C_1) < 1$, from Lemma 15, the following inequality is satisfied:

$$|3-\mu| < 3-\mu-4\lambda\mu. \tag{29}$$

Let us use the other necessary condition $\rho(C_1C_2) < 1$. If we set $C_1 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $C_2 = \left(\begin{smallmatrix} d & c \\ b & a \end{smallmatrix}\right)$. We set $s' = \operatorname{trace}(C_1C_2)$ and $p' = \det(C_1C_2) = p^2$. A simple computation shows that $s' = 2p + (b+c)^2$ since

$$C_1C_2 = \begin{pmatrix} ad+b^2 & a(b+c) \\ d(b+c) & c^2+ad \end{pmatrix}.$$

From Lemma 15, we obtain the inequality $\pm s' < 1 + p'$ which is the inequality $\pm[2p + (b+c)^2] < 1 + p^2$. The inequality $-2p - (b+c)^2 < 1 + p^2$ is always true. The other inequality $2p + (b+c)^2 < 1 + p^2$ is equivalent to the statement $|b+c| < 1-p$. We obtain the inequality

$$|\mu-1| < 1+\mu+4\lambda\mu. \tag{30}$$

Inequalities (29)–(30) are equivalent to the inequalities prevailing in (28). □

9. Sufficient conditions of convergence

In this section, we will describe explicitly the values (λ,μ) for which the Hermite subdivision scheme $\mathcal{H}(\lambda,\mu)$ in the Merrien family is convergent. The main tools for this investigation are two matrices D_1 and D_2 which are similar to C_1 and C_2. These matrices (proposed by Merrien [14]) are $D_i = PC_iP^{-1}$, $i = 1, 2$, where $P = \left(\begin{smallmatrix} 1 & 1 \\ \mu & -\mu \end{smallmatrix}\right)$.

$$D_1 = 1/2\begin{pmatrix} 1 & 1 \\ a & b \end{pmatrix}, \qquad D_2 = 1/2\begin{pmatrix} 1 & -1 \\ -a & b \end{pmatrix},$$

where $a = (1+8\lambda)\mu$, $b = 2-\mu$. The region in the a, b-plane which corresponds to the set (28) according to the mapping $(\lambda,\mu) \mapsto (a,b)$ is a parallelogram.

Lemma 17. *Let (a,b) be in the open parallelogram $|a+b| < 2$, $|a-3b| < 6$ and let D_1, D_2 be the two linear mappings $D_1(x,y) = (x/2+y/2, ax/2+by/2)$ and $D_2(x,y) = (x/2-y/2, -ax/2+by/2)$, then there exists a compact balanced convex body K such that the four images of K, $D_i(D_j(K))$, $i = 1, 2$, $j = 1, 2$, are subsets of cK with $0 < c < 1$.*

Proof. Let (a,b) be in the parallelogram P defined by $|a+b| < 2$, $|a-3b| < 6$. The vertices of P are $\pm(0,2)$, $\pm(3,-1)$. Based on the location of (a,b) inside P we distinguish 5 cases.
Case 1. (a,b) is in the open triangle of vertices $(0,0)$, $(3,-1)$, $(0,-2)$ (see Fig. 3, inside the left parallelogram at the top).

We obtain that $a > 0$, $a-3b > 0$, $a-3b < 6$, we set $p = (a-3b)/6 \in (0,1)$, $a' = a/p$, $b' = b/p$. We consider the following three points $A(1+3/a', 0)$, $B(1,3)$, $C(1,-3)$ and we set K as the hexagon which is the convex hull of $\pm A, \pm B, \pm C$ (see the left hexagon in Fig. 3).

Let T be the linear mapping $T(x,y) = (x/2+y/2, a'x/2+b'y/2)$. Simple computations show that $T(A) = (1-\lambda)A + \lambda B$, where $\lambda = a'/6 + 1/2$, $T(B) = (1-\mu)A + \mu C$, where $\mu = 1 - a'/3$ and $T(C) = -C$. Since $0 < a' < 3$, both numbers λ, μ are between 0 and 1, and the image of any vertex under T belongs to K. We obtain that $T(K) \subset K$.

We define the other linear mapping $U(x,y) = (x/2+y/2, 0)$. It follows that $D_1 = pT + (1-p)U$. The image $U(K)$ is the segment $[-r,r] \times \{0\}$, where $r = \max(2, (1+3/a')/2)$. Both points $(2,0)$ and $((1+3/a')/2, 0)$ are in the interior of K, K°. If $(x,y) \in K$, then $T(x,y) \in K$, $U(x,y) \in K^\circ$. It follows that $D_1(x,y) = pT(x,y) + (1-p) \times U(x,y) \in K^\circ$, $D_1(K)$ is a compact subset of K° and there exists a number $t \in (0,1)$ such that $D_1(K) \subset tK$. If $S(x,y) = (-x,y)$ and $S(K) = K$, then $D_2 = S \circ D_1 \circ S$ and $D_2(K) \subset tK$. The four images of K, $D_i(D_j(K))$, $i = 1, 2$, $j = 1, 2$, are subsets of t^2K.

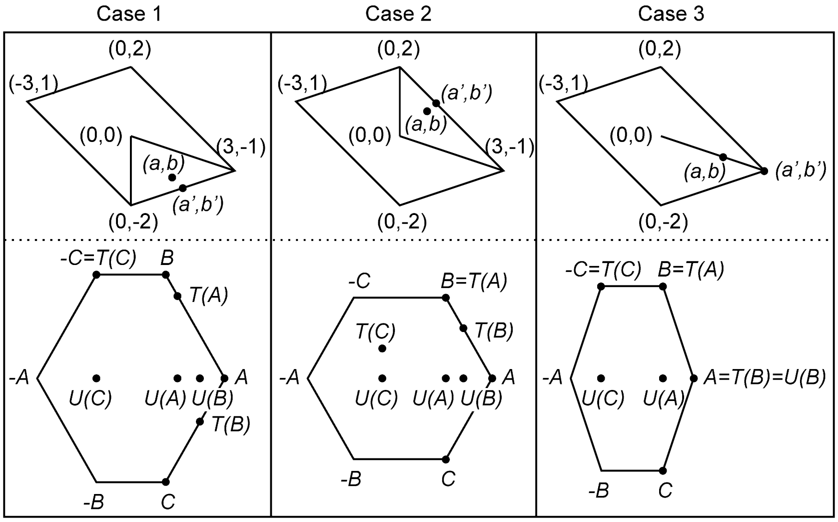

Fig. 3. Three first cases in the parallelogram.

Case 2. (a, b) is in the open triangle of vertices $(0, 0)$, $(3, -1)$, $(0, 2)$ (see Fig. 3, inside the middle parallelogram at the top).

We obtain that $a > 0$, $a + b > 0$, and we set $p = (a + b)/2 \in (0, 1)$, $a' = a/p$, $b' = b/p$. We consider the points $A(2, 0)$, $B(1, a')$, $C(-1, a')$ and we set K as the hexagon which is the convex hull of $\pm A, \pm B, \pm C$ (see the middle hexagon in Fig. 3).

Let T be the linear mapping $T(x, y) = (x/2 + y/2, a'x/2 + b'y/2)$. Simple computations show that $T(A) = B$, $T(B) = (1 - \lambda)A + \lambda C$, where $\lambda = (3 - a')/2 \in [0, 1/2]$ and $T(B) = \mu B$, where $\mu = (1 - a')/2 \in [-1, 1/2]$. Since $0 < a' < 3$, the image of any vertex under T belongs to K. We obtain that $T(K) \subset K$.

As before, we set $U(x, y) = (x/2 + y/2, 0)$. It follows that $D_1 = pT + (1 - p)U$. The image $U(K)$ is the segment $[-r, r] \times \{0\}$, where $r = \max(1, (1 + a')/2)$. Both points $(1, 0)$ and $((1 + a)/2, 0)$ are in the interior of K, K°. If $(x, y) \in K$, then $D_1(x, y) = pT(x, y) + (1 - p)U(x, y) \in K^\circ$. It follows that there exists a number $t < 1$ such that $D_1(K) \subset tK$. Similarly, $D_2(K) \subset tK$. The four images of K, $D_i(D_j(K))$, $i = 1, 2$, $j = 1, 2$, are subsets of t^2K.

Case 3. (a, b) is the open segment whose endpoints are $(0, 0)$, $(3, -1)$ (see Fig. 3, inside the right parallelogram at the top).

We consider the points $A(2, 0)$, $B(1, 3)$, $C(-1, 3)$ and we set K as the hexagon which is the convex hull of $\pm A, \pm B, \pm C$ (see the right hexagon in Fig. 3).

Let T be the linear mapping $T(x, y) = (x/2 + y/2, 3x/2 - y/2)$. It follows that $D_1 = pT + (1 - p)U$, where $p = (a - 3b)/6 \in (0, 1)$. Simple computations show that $T(A) = B$, $T(B) = A$ and $T(C) = -C$, $U(A) \in K^\circ$, $U(B) = A$ and $U(C) \in K^\circ$. It follows that $D_1(A) \in K^\circ$, $D_1(B) = A$ and $D_1(C) \in K^\circ$. If $S(x, y) = (-x, y)$ and $S(K) = K$, then $D_2 = S \circ D_1 \circ S$. Similarly $D_2(A) \in K^\circ$, $D_2(C) = A$ and $D_2(B) \in K^\circ$. Every vertex of the hexagon K is sent in K° under every map $D_i \circ D_j$, $i = 1, 2$, $j = 1, 2$.

$D_1(K)$ is a compact subset of K° and there exists a number $t \in (0, 1)$ such that $D_1(K) \subset tK$. If $S(x, y) = (-x, y)$ and $S(K) = K$, then $D_2 = S \circ D_1 \circ S$ and $D_2(K) \subset tK$. The four images of K, $D_i(D_j(K))$, $i = 1, 2$, $j = 1, 2$, are subsets of t^2K.

Case 4. $a = 0$.

We set K as the square $|x| \leqslant 1$, $|y| \leqslant 1$. The largest number among $\|D_iD_j\|_\infty$, $i = 1, 2$, $j = 1, 2$ is $c = 1/2 + |b|/4 < 1$. The four images of K, $D_i(D_j(K))$, $i = 1, 2$, $j = 1, 2$, are subsets of cK.

Case 5. $a < 0$.

By symmetry we are able to reduce this case to Cases 2, 3 or 4. If we consider the symmetry $S(x, y) = (x, -y)$, then $S \circ D_1(x, y) = (x/2 + y/2, -ax/2 - by/2)$ and $S \circ D_2(x, y) = (x/2 - y/2, ax/2 - by/2)$. If we replace a and b by $-a$ and $-b$, respectively, Case 5 is transformed into Cases 1, 2 or 3. For a suitable balanced compact convex body K, we obtain that $S \circ D_i \circ S \circ D_j(K)$, $i = 1, 2$, $j = 1, 2$, are subsets of cK with $c \in (0, 1)$. By using the relations

$S(K) = K$, $S \circ D_1 = D_2 \circ S$, $S \circ D_2 = D_1 \circ S$ we obtain that the four images of K, $D_i(D_j(K))$, $i = 1, 2$, $j = 1, 2$, are subsets of cK with $c \in (0, 1)$. □

Theorem 18. *Let $\mathcal{H}(\lambda, \mu)$ be the Hermite subdivision scheme whose nonzero matrices of the mask are*

$$A_{-1} = \begin{pmatrix} 1/2 & \lambda \\ \mu/2 & (1-\mu)/4 \end{pmatrix}, \qquad A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 1/2 & -\lambda \\ -\mu/2 & (1-\mu)/4 \end{pmatrix}.$$

Then $\mathcal{H}(\lambda, \mu)$ is C^1 iff $0 < -\lambda < 1/2$, $0 < \mu < \min(-1/(2\lambda), 3/(1+2\lambda))$.

Proof. From Lemma 16, the condition $0 < -\lambda < 1/2$, $0 < \mu < \min(-1/(2\lambda), 3/(1+2\lambda))$ is necessary for $\mathcal{H}(\lambda, \mu)$ to be a C^1 Hermite subdivision scheme. Let us assume that the necessary condition is satisfied. We set $a = (1+8\lambda)\mu$, $b = 2 - \mu$. The point (a, b) is in the interior of the parallelogram of vertices $(0, 2)$, $(3, -1)$, $(0, -2)$, $(-3, 1)$. We set $D_i = PC_iP^{-1}$, $i = 1, 2$ where $P = \begin{pmatrix} 1 & 1 \\ \mu & -\mu \end{pmatrix}$.

From Lemma 17, there exists a number $c \in (0, 1)$ and a compact balanced convex body K such that the four images of K, $D_i(D_j(K))$, $i = 1, 2$, $j = 1, 2$, are subsets of cK with $0 < c < 1$. Let Q be the unit square $[-1, 1]^2$, then there exist two positive numbers ε and M such that $P(\varepsilon Q) \subset K$ and $P^{-1}K \subset MQ$. In order to find a bound for the number ν_{2n}, we consider any sequence of $2n$ integers of value 1 or 2, $i_1, i_2, \ldots, i_{2n}$, and we obtain

$$\begin{aligned} C_{i_{2n}} \circ C_{i_{2n-1}} \circ \cdots \circ C_{i_1}(\varepsilon Q) &= P^{-1} \circ D_{i_{2n}} \circ \cdots \circ D_{i_{2n-1}} \circ \cdots \circ D_{i_1} \circ P(\varepsilon Q) \\ &\subset P^{-1} \circ D_{i_{2n}} \circ D_{i_{2n-1}} \circ \cdots \circ D_{i_1}(K) \\ &\subset P^{-1}(c^n K) \\ &\subset Mc^n Q. \end{aligned}$$

Thus $\nu_{2n} \leqslant Mc^n/\varepsilon$ and, if n is large enough, $\nu_{2n} < 1$ and the Hermite subdivision scheme is C^1. □

10. Examples and comparisons

In this section we provide examples and we compare Theorems 14 and 18 with other results in the topic of Hermite subdivision scheme.

Example. We consider the one-parameter family of Hermite subdivision schemes $\mathcal{H}(c)$ whose nonzero matrices of its mask are

$$A_{-1} = \begin{pmatrix} 1/2 & -1/8 + c/2 \\ 3/4 & -1/8 \end{pmatrix}, \qquad A_0 = \begin{pmatrix} 1 & c \\ 0 & 1/2 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 1/2 & 1/8 + c/2 \\ -3/4 & -1/8 \end{pmatrix}.$$

As it is easily checked, $\mathcal{H}$ reproduces linear functions. The nonzero matrices of the mask of the associated subdivision scheme are B_i, $-2 \leqslant i \leqslant 1$ which are respectively equal to

$$\begin{pmatrix} 0 & -1/4 + c \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1/4 + c \\ 0 & -1/4 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1/4 - c \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & -1/4 - c \\ 3/2 & -1/4 \end{pmatrix}.$$

Let us apply Theorem 14 to this example with two choices of c.

Example 1. $c = 0$. In this case, the Hermite subdivision scheme is interpolatory and converges to the Hermite cubic spline. If S' is the subdivision matrix of the difference subdivision scheme, the computation of the first norms $\nu_n = \|(S')^n\|_\infty$ shows that $\nu_1 = 1$, $\nu_2 = 5/8$. As $\nu_2 < 1$, we get the numerical confirmation that the scheme $\mathcal{H}(0)$ is C^1.

Example 2. $c = 1/8$. In this case, the Hermite subdivision scheme is not interpolatory. After computation, we get $\nu_1 = 1.1250$, $\nu_2 = 0.8125$. As $\nu_2 < 1$, the scheme $\mathcal{H}(1/8)$ is C^1.

Dubuc and Merrien [2] stated another criterion of convergence.

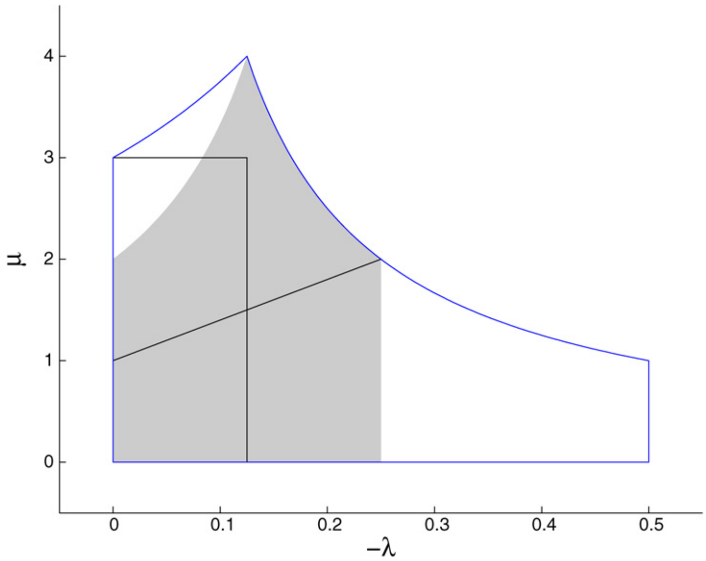

Fig. 4. Convergence of $\mathcal{H}(\lambda,\mu)$: Merrien region in gray, segment of Dyn and Levin, rectangle of Lyche and Merrien and the full region (Theorem 18).

Theorem 19. *Let $\mathcal{H}$ be a Hermite subdivision scheme of support $[\sigma,\sigma']$ which reproduces the linear functions and let S be the subdivision matrix of the associated subdivision scheme of $\mathcal{H}$, then $\mathcal{H}$ is C^1 iff there is an integer n such that*

$$\kappa_n = \sup\left\{\sum_{j\in\mathbb{Z}}\left|s_n(i,j)-s_n(i',j)\right|:\ |i-i'|\leqslant\sigma'-\sigma\right\}<2$$

where $(s_n(i,j)) = S^n$.

In Example 1, the 5 first values of κ_n, $n\geqslant 1$ are 3.5000, 3.6250, 3.0313, 2.0469, 1.1680. In Example 2, the 6 first values of κ_n, $n\geqslant 1$ are 3.7500, 4.1250, 3.6250, 2.6953, 2.3457, 1.8667. In the first case, $\kappa_5<2$, in the second case, $\kappa_6<2$, we get another confirmation that the schemes are convergent. In these examples, Theorem 14 is more efficient than Theorem 19.

Other authors published examples of noninterpolatory Hermite subdivision scheme. With this respect, we cite Jüttler and Schwanecke [9] and then Han, Yu and Xue [8].

We now compare Theorem 18 to results that have been found by other authors. First of all, the following sufficient condition for C^1 convergence of Hermite subdivision scheme $\mathcal{H}(\lambda,\mu)$ have been found by Merrien [14]: $|\mu(1+8\lambda)|+|2-\mu|<2$. This region is set in gray in Fig. 4.

Dyn and Levin [6] also considered the Merrien family of subdivision schemes. The authors justifies the C^1 convergence of the Hermite subdivision scheme if $\lambda\in(-1/4,0)$ and $4\lambda+\mu=1$ by asserting that $\|(S')^2\|_\infty<1$ (where S' is given by (23)–(24)). That the norm $\|(S')^2\|_\infty$ is smaller than 1 is corroborated by Fig. 2: the segment $\lambda\in(-1/4,0)$ and $4\lambda+\mu=1$ is contained in Ω_2.

More recently, Lyche and Merrien [12] have shown the C^1 convergence of $\mathcal{H}(\lambda,\mu)$ if (λ,μ) belongs to the rectangle $(-1/8,0)\times(0,3)$ (see Fig. 4). They used the region of convergence of $\mathcal{H}(\lambda,\mu)$ for constructing subdivision curves that preserve positivity, monotonicity and convexity.

11. Conclusion

We conclude with a question. What happens to a Hermite subdivision scheme $\mathcal{H}(\lambda,\mu)$ when (λ,μ) is on the boundary of the region of C^1 convergence? We do not know of a good answer other than that C^1 convergence does not occur. However we present a graphical investigation for the case where (λ,μ) are chosen as $(-1/8,4)$, $(-1/4,2)$ and $(-1/2,1)$. In each situation, we consider the Hermite subdivision scheme with two different initial function f_0.

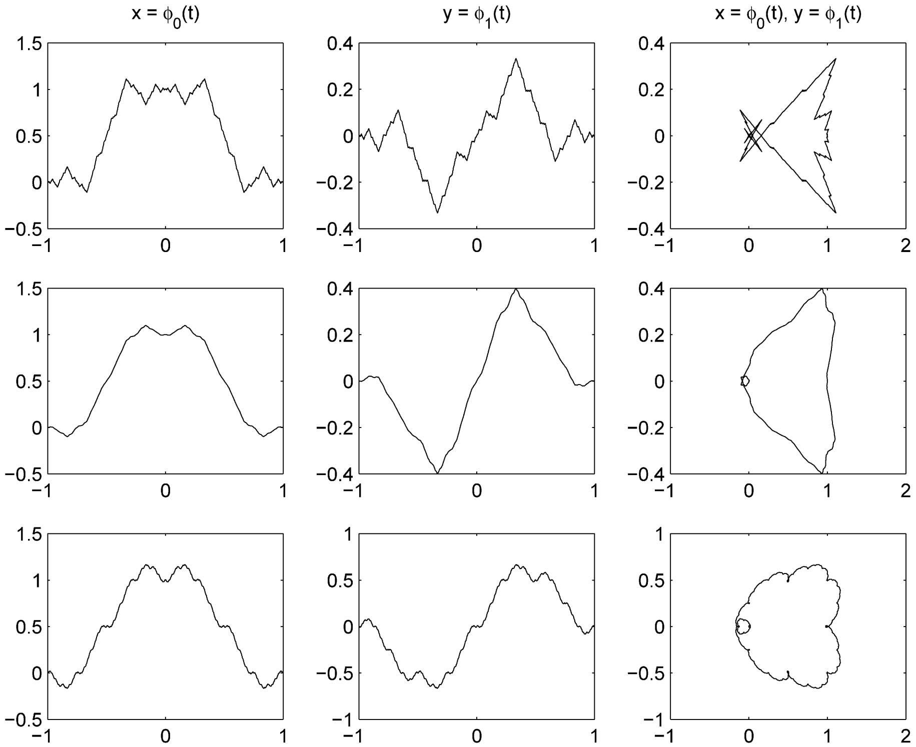

Fig. 5. Graphs of $x = \phi_0(t)$ and $y = \phi_1(t)$ and trace of the parametric curve $x = \phi_0(t)$, $y = \phi_1(t)$, $t \in [-1, 1]$. Three situations are considered: $(\lambda, \mu) = (-1/8, 4)$ (top), $(-1/4, 2)$ (middle) and $(-1/2, 1)$ (bottom).

In the first case, we set $f_0 : \mathbb{Z} \to \mathbb{R}^2$ as $f_0(0) = (1, 0)^T$ and $f_0(i) = (0, 0)^T$ if $i \neq 0$. The sequence of polygonal lines $\{(i/2^n, f_n^{(0)}(i)) \colon i \in \mathbb{Z}\}$, $n \in \mathbb{N}$, appears to converge to a function $x = \phi(t)$. In the second case, we set $f_0 : \mathbb{Z} \to \mathbb{R}^2$ as $f_0(0) = (0, 1)^T$ and $f_0(i) = (0, 0)^T$ if $i \neq 0$. The corresponding sequence of polygonal lines $\{(i/2^n, f_n^{(0)}(i)) \colon i \in \mathbb{Z}\}$, $n \in \mathbb{N}$, seems to converge to a function $y = \psi(t)$. In Fig. 5, we draw the graphs $x = \phi(t)$, $y = \psi(t)$ and the parametric curve $x = \phi(t)$, $y = \psi(t)$ for three choices of (λ, μ).

As we saw, the criterion of convergence for nonuniform subdivision schemes is useful for investigating the convergence of Hermite subdivision schemes. Other authors presented interesting results on the topic of convergence of Hermite subdivision schemes, as Merrien [14,15], Dyn and Levin [6], Zhou [16], Han [7], Jüttler and Schwanecke [9], Han, Yu and Xue [8].

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References

[1] I. Daubechies, I. Guskov, W. Sweldens, Commutation for irregular subdivision, Constr. Approx. 17 (2001) 479–514.
[2] S. Dubuc, J.-L. Merrien, Convergent vector and Hermite subdivision schemes, Constr. Approx. 23 (2006) 1–22.
[3] N. Dyn, J.A. Gregory, D. Levin, Analysis of linear binary subdivision schemes for curve design, Constr. Approx. 7 (1991) 127–147.
[4] N. Dyn, Subdivision schemes in computer-aided geometric design, in: W.A. Light (Ed.), Advances in Numerical Analysis, vol. II, Lancaster, 1990, Clarendon Press, Oxford, 1992, pp. 36–104.
[5] N. Dyn, J.A. Gregory, D. Levin, Piecewise uniform subdivision schemes, in: M. Dahlen, T. Lyche, L.L. Schumaker (Eds.), Mathematical Methods for Curves and Surfaces, Vanderbilt Univ. Press, Nashville, TN, 1995, pp. 111–119.
[6] N. Dyn, D. Levin, Analysis of Hermite-type subdivision schemes, in: C.K. Chui, L.L. Schumaker (Eds.), Approximation Theory, vol. VIII, Wavelets and Multilevel Approximation, vol. 2, College Station, TX, 1995, World Scientific, River Edge, NJ, 1995, pp. 117–124.

[7] B. Han, Approximation properties and construction of Hermite interpolants and biorthogonal multiwavelets, J. Approx. Theory 110 (2001) 18–53.
[8] B. Han, T.P.-Y. Yu, Y. Xue, Noninterpolatory Hermite subdivision schemes, Math. Comp. 74 (2005) 1345–1367.
[9] B. Jüttler, U. Schwanecke, Analysis and design of Hermite subdivision schemes, Visual Comput. 18 (2002) 326–342.
[10] A. Levin, Polynomial generation and quasi-interpolation in stationary non-uniform subdivision, Comput. Aided Geom. Design 20 (2003) 41–60.
[11] D. Levin, Using Laurent polynomial representation for the analysis of non-uniform binary subdivision schemes, Adv. Comput. Math. 11 (1999) 41–54.
[12] T. Lyche, J.-L. Merrien, C^1 interpolatory subdivision with shape constraints for curves, SIAM J. Numer. Anal., in press.
[13] V. Maxim, M.-L. Mazure, Subdivision schemes and irregular grids, Numer. Algorithms 35 (2004) 1–28.
[14] J.-L. Merrien, A family of Hermite interpolants by bisection algorithms, Numer. Algorithms 2 (1992) 187–200.
[15] J.-L. Merrien, Interpolants d'Hermite C^2 obtenus par subdivision, M2AN Math. Model. Numer. Anal. 33 (1999) 55–65.
[16] D.-X. Zhou, Multiple refinable Hermite interpolants, J. Approx. Theory 102 (2000) 46–71.