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Note

An eigenvalue bound for the Laplacian of a graph

J.P. Grossman

Department of Mathematics and Statistics, University of Dalhousie, Halifax, Canada B3H 3J5

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Abstract

We present a lower bound for the smallest non-zero eigenvalue of the Laplacian of an undirected graph. The bound is primarily useful for graphs with small diameter.

Keywords: Graph Laplacian; Graph eigenvalues; Eigenvalue bounds

1. Introduction

We use the following definition for the Laplacian matrix of a graph, consistent with [3]:

Definition. Let G be an undirected graph with adjacency matrix $\mathbf{A}$, and let $\mathbf{D}$ be the diagonal degree matrix defined by $d_{ii} = \deg(v_i)$ and $d_{ik} = 0$ for $i \neq k$. The *Laplacian* of G is the matrix $\mathscr{L} = \mathbf{I} - \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$.

The eigenvalues of $\mathscr{L}$ are in the range $[0, 2]$. Zero is always an eigenvalue with multiplicity equal to the number of connected components of G, and 2 occurs as an eigenvalue if and only if G is bipartite. The eigenvalues of $\mathscr{L}$ contain additional information regarding the structure of G. They can be used to establish bounds on the diameter of G as well as distances between subsets of G [1,4,2,5]. The magnitudes of the eigenvalues also determine the rate of convergence of various iterative computations such as those described in [6,7]; it is therefore desirable to find bounds on the eigenvalues themselves. One of the best lower bounds for

E-mail address: jpg@alum.mit.edu.

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the smallest non-zero eigenvalue λ_1 is established in [3]

$$\lambda_1 \geqslant \frac{1}{D\,\mathrm{vol}(G)}. \tag{1}$$

Here D is the diameter of G, and $\mathrm{vol}(G)$ is the sum of the degrees of all vertices. In this paper, we present a lower bound on λ_1 which is easy to compute and is tighter than (1) for certain graphs with low diameter.

2. A lower bound for λ_1

Consider the similar matrix $\mathbf{L}=\mathbf{D}^{-1/2}\mathscr{L}\mathbf{D}^{1/2}=\mathbf{I}-\mathbf{D}^{-1}\mathbf{A}$ which has the same eigenvalues as $\mathscr{L}$. If f is an eigenfunction of $\mathbf{L}$ corresponding to eigenvalue λ then for any vertex v

$$(1-\lambda)f(v) = \frac{1}{\deg(v)}\sum_{u\sim v} f(u), \tag{2}$$

where $u \sim v$ denotes that the vertices u, v are connected. Let $v_1, v_2, \ldots, v_{m+1}$ be a sequence of connected vertices such that $f(v_1)$ is maximal and $f(v_{m+1}) \leqslant 0$. For convenience, set $x_i = f(v_i)$. Let $\alpha = 1-\lambda$, and let d be the maximum degree of any vertex of G. Since $v_1 \sim v_2$ and $f(v_1)$ is maximal, Eq. (2) gives us

$$\alpha x_1 = \frac{1}{\deg(v_1)}\sum_{u\sim v_1} f(u) \leqslant \frac{x_2}{\deg(v_1)} + \frac{(\deg(v_1)-1)x_1}{\deg(v_1)} \leqslant \frac{x_2}{d} + \frac{(d-1)x_1}{d}. \tag{3}$$

Similarly, since $v_i \sim v_{i-1}$ and $v_i \sim v_{i+1}$ for $2 \leqslant i \leqslant m$, we have

$$\alpha x_i \leqslant \frac{x_{i-1}+x_{i+1}}{d} + \frac{(d-2)x_1}{d}. \tag{4}$$

Scaling f if necessary, we may assume that $x_1 = 1$ and rewrite inequalities (3) and (4) as

$$\begin{aligned} &x_2 \geqslant 1-\lambda d,\\ &x_{i+1} \geqslant \alpha d x_i - x_{i-1} - (d-2). \end{aligned} \tag{5}$$

We assume that $\lambda < 1$ and $\alpha d \geqslant 1$; otherwise we have the bound $\lambda > (d-1)/d$ which is much better than the one we will derive.

Lemma. *For $3 \leqslant k \leqslant m+1$ we have $x_k \geqslant 1 - \lambda\alpha^{k-3}d^{k-2} - \lambda\alpha^{k-2}d^{k-1}$.*

Proof. Our proof is by induction. Setting $i = 2$ in inequality (5) establishes the base case:

$$x_3 \geqslant \alpha d x_2 - 1 - (d-2) \geqslant \alpha d(1-\lambda d) + 1 - d = 1 - \lambda d - \lambda\alpha d^2.$$

Now suppose the inequality holds for $k \leqslant i$, where $i \geqslant 3$. If we add the inequalities (5) up to $i+1$ we obtain

$$
\begin{aligned}
&x_2 \geqslant 1 - \lambda d,\\
&x_3 \geqslant \alpha d x_2 - 1 - (d-2),\\
&x_4 \geqslant \alpha d x_3 - x_2 - (d-2),\\
&\vdots\\
&x_i \geqslant \alpha d x_{i-1} - x_{i-2} - (d-2),\\
&+\, x_{i+1} \geqslant \alpha d x_i - x_{i-1} - (d-2),
\end{aligned}
$$

$$x_{i+1} \geqslant (\alpha d - 2)(x_2 + x_3 + \cdots + x_{i-1} + x_i) + x_i - \lambda d - (i-1)(d-2),$$

so by induction

$$
\begin{aligned}
x_{i+1} &\geqslant (\alpha d - 2)((i-1) - 2\lambda d - 2\lambda\alpha d^2 - \cdots - 2\lambda a^{i-3} d^{i-2} - \lambda a^{i-2} d^{i-1})\\
&\quad + (1 - \lambda\alpha^{i-3} d^{i-2} - \lambda\alpha^{i-2} d^{i-1}) - \lambda d - (i-1)(d-2)\\
&= 1 - (i-4)\lambda d + 2\lambda\alpha d^2 + 2\lambda\alpha^2 d^3 + \cdots + 2\lambda\alpha^{i-4} d^{i-3} + \lambda\alpha^{i-3} d^{i-2}\\
&\quad - \lambda\alpha^{i-2} d^{i-1} - \lambda\alpha^{i-1} d^i\\
&\geqslant 1 - (i-4)\lambda d + 2\lambda d + 2\lambda d + \cdots + 2\lambda d + \lambda d - \lambda\alpha^{i-2} d^{i-1} - \lambda\alpha^{i-1} d^i\\
&= 1 + (i-3)\lambda d - \lambda\alpha^{i-2} d^{i-1} - \lambda\alpha^{i-1} d^i \geqslant -\lambda\alpha^{i-2} d^{i-1} - \lambda\alpha^{i-1} d^i. \qquad \square
\end{aligned}
$$

Theorem. *Let G be a graph with diameter D and maximum vertex degree d. Then*

$$\lambda_1 \geqslant \frac{1}{(d+1)d^{\lceil D/2 \rceil - 1}}. \tag{6}$$

Proof. Since the distance from a vertex which maximizes *f* to one which minimizes *f* is at most *D*, we can obtain the described sequence of connected vertices with $m \leqslant \lceil D/2 \rceil$, negating *f* if necessary. Then applying the lemma to x_{m+1} we have

$$0 \geqslant x_{m+1} \geqslant 1 - \lambda_1 \alpha^{m-2} d^{m-1} - \lambda_1 \alpha^{m-1} d^m \geqslant 1 - \lambda_1 d^{m-1} - \lambda_1 d^m$$

from which the result follows. $\square$

3. Discussion

Since the size of the denominator of (6) is exponential in *D*, the bound is primarily useful for graphs of low diameter. Note that in the special case $D = 2$ we have $0 \geqslant x_2 \geqslant 1 - \lambda_1 d$ which gives us the improved bound

$$\lambda_1 \geqslant \frac{1}{d}.$$

Table 1
Maximum value of d for which the bound (6) is necessarily tighter on d-regular graphs

D	5	6	7	8	9	10
d	10	18	7	5	3	3

In general, the bound (6) will be tighter than (1) when $(d+1)d^{\lceil D/2\rceil -1} \leqslant D\,\mathrm{vol}(G)$. For d-regular graphs in which all vertices have degree d, this condition becomes

$$(d+1)d^{\lceil D/2\rceil -2} \leqslant D|G|. \tag{7}$$

This is always true for $D=3,4$. For $D=5$, a d-regular graph has at least $2d+4$ vertices, so condition (7) will certainly be satisfied if $(d+1)d \leqslant 5(2d+4)$ which is true for $d \leqslant 10$; for $D=6$ a d-regular graph has at least $3d+3$ vertices, so (7) will be satisfied if $(d+1)d \leqslant 6(3d+3)$, i.e. if $d \leqslant 18$. Continuing in this manner we obtain Table 1 which lists, for $5 \leqslant D \leqslant 10$, the maximum value of d for which condition (7) is guaranteed to be satisfied.

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