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ON THE RAMSEY NUMBERS $N(3,3,...,3;2)$

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Abstract. The main results of this paper are $N(3,3,3,3;2) > 50$ and $f(k+1) \geq 3f(k) + f(k-2)$, where $f(k) = N(\underbrace{3,3,...,3}_{k \text{ times}};2) - 1$ for $k \geq 3$.

1. Introduction

The theorem of Ramsey says: Given integers $S_1, S_2, S_3, ..., S_k$, where $S_1, S_2, ..., S_k \geq 2$, there exists a minimum integer $N(S_1, S_2, ..., S_k; 2)$ such that the following property is valid for all $n \geq N(S_1, S_2, ..., S_k; 2)$. Let the edges of a complete graph of n vertices be colored in k colors, then there exists a subset of S_i vertices with all its interconnecting segments of the i^{th} color for some $i \leq k$.

Now, consider the case of $S_1 = S_2 = ... = S_k = 3$. Let

$$f(k) = N(\underbrace{3,3,...,3}_{k \text{ times}};2) - 1 .$$

The problem reduces to the following: If the edges of K_n are colored in k colors and if $n > f(k)$, then there exists some triangle with all its sides in the same color. Find $f(k)$.

It is known [1] that $2^k \leq f(k) \leq [k!\,e]$. Particularly, $f(1) = 2$, $f(2) = 5$, $f(3) = 16$. Whitehead [3,4] has proved $f(4) \geq 49$. It will be shown here that $f(k+1) \geq 3f(k) + f(k-2)$ for $k \geq 3$ and, in particular, $f(4) \geq 50$, thus $N(3,3,3,3;2) > 50$.

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2. $N(3,3,3,3;2) > 50$ [1]

Consider the symmetric 16×16 matrix:

$$
T_3(x_0,x_1,x_2,x_3) = \left|\begin{array}{l}
x_0 \\
x_1x_0 \\
x_1x_2x_0 \\
x_1x_2x_3x_0 \\
x_1x_3x_3x_2x_0 \\
x_1x_3x_2x_3x_2x_0 \\
x_2x_3x_2x_2x_1x_1x_0 \\
x_2x_2x_3x_1x_1x_2x_3x_0 \\
x_2x_2x_1x_3x_2x_1x_3x_1x_0 \\
x_2x_1x_1x_2x_3x_2x_1x_1x_3x_0 \\
x_2x_1x_2x_1x_2x_3x_1x_3x_1x_3x_0 \\
x_3x_2x_1x_1x_3x_3x_2x_3x_3x_2x_2x_0 \\
x_3x_1x_2x_3x_3x_1x_3x_1x_3x_2x_3x_1x_0 \\
x_3x_1x_3x_2x_1x_3x_3x_2x_1x_3x_2x_1x_2x_0 \\
x_3x_3x_3x_1x_2x_1x_2x_2x_3x_1x_3x_2x_2x_1x_0 \\
x_3x_3x_1x_3x_1x_2x_2x_3x_2x_3x_1x_2x_1x_2x_1x_0
\end{array}\right|
$$

It is known [2] that $T_3(0,1,2,3)$ is the incidence matrix of one of the two non-isomorphic edge-coloring schemes of K_{16} without any one-color triangles.

Now construct the 50×50 incidence matrix in the following way:

$T_4(0,1,2,3,4) =$

A				
D	B			
E	F	C		
11 1	22 2	33 3	0	
11 1	22 2	33 3	4	0

[1] Dr. G.J. Porter proved 2 independently in Univ. of Pennsylvania.

where $A = T_3(0, 2, 3, 4)$,
$B = T_3(0, 3, 1, 4)$,
$C = T_3(0, 1, 2, 4)$,
$D = T_3(3, 2, 1, 4)$,
$E = T_3(2, 1, 3, 4)$,
$F = T_3(1, 3, 2, 4)$.

If there are some one-color triangles with vertices i, j, k, then $t_{i,j} = t_{k,j} = t_{k,i}$. We may assume $k > i > j$ without loss of generality.

Case 1: $t_{i,j} = t_{k,j} = t_{k,i} = 4$.

We notice that $t_{m,n} = t_{m',n'} = 4$ if $m \equiv m'$ (mod 16), $n \equiv n'$ (mod 16) for $m, m', n, n' \leq 48$. Hence we may pick i', j', k' such that $i \equiv i'$, $j \equiv j'$, $k \equiv k'$ (mod 16) and $i', j', k' \leq 16$; then $t_{i',j'} = t_{k',j'} = t_{k',i'} = 4$. This contradicts the fact that T_3 is the incidence matrix of a coloring without a one-color triangle. In case of $k = 50$, $i = 49$, we know that $t_{50,49} = 4$ and that $t_{j,49}$, $t_{j,50}$ do not have value 4 for any $j \neq 49, 50$.

Case 2: $t_{i,j} = t_{k,j} = t_{k,i} = 2$.

(1) $16 \geq j \geq 1$, $16 \geq i \geq 1$, $t_{i,j}$ is in part A.

(a) If $t_{k,j}$ is in part A, then $t_{k,i}$ is in part A. This contradicts the structure of T_3.

(b) If $t_{k,j}$ is in part D, then $t_{k,i}$ is in part D. We know that $t_{i+16,j} = t_{i,j} = 2$. Then $t_{i+16,j} = t_{k,j} = t_{k,i} = 2$. Impossible.

(c) If $t_{k,j}$ is in part E, then $t_{k,i}$ is in part E. But there is only one entry with value 2 in each row of E. Contradiction.

(2) $16 \geq j \geq 1$, $32 \geq i \geq 17$, $t_{i,j}$ is in part D.

(a) If $t_{k,j}$ is in part D, then $t_{k,i}$ is in part B. But there is no entry with value 2 in B. This is impossible.

(b) If $t_{k,j}$ is in part E, then $t_{k,i}$ is in part F. It is known that only the entries on the diagonal are of value 2 in E. Hence $k = 32+j$. We have $t_{i,j} = t_{32+j,j} = t_{32+j,i} = 2$. But $t_{32+j,i} = 3$ if $t_{i,j} = 2$. Contradiction.

(3) $16 \geq j \geq 1$, $50 \geq i \geq 33$, $t_{i,j}$ is in part E. There is only one entry with value 2 in part E. This is impossible.

(4) $32 \geq j \geq 17$, $32 \geq i \geq 17$, $t_{i,j}$ is in part B. This is impossible because there is no entry with value 2 in B.

(5) $32 \geq j \geq 17$, $48 \geq i \geq 33$, $t_{i,j}$ is in part F.

(a) $t_{k,j}$ is in part F and $t_{k,i}$ is in part C and $t_{k,i} = t_{k,i-16} = 2$. Then $t_{i,j}$, $t_{k,j}$, $t_{k,i-16}$ are all in F and all with value 2. This contradicts the structure of T_3.

(b) $k = 49$ or 50. In this case, $t_{k,i} = 3 \neq t_{i,j}$.

(6) $i = 49$, $32 \geq j \geq 17$, $k = 50$. Then $t_{50,49} = 4 \neq 2$. Impossible.

(7) $48 \geq j \geq 33$, $48 \geq i \geq 33$, $t_{i,j}$ is in part C. $t_{k,j}$, $t_{k,i}$ is in part C. This contradicts the structure of T_3.

Case 3: $t_{i,j} = t_{k,j} = t_{k,i} = 1$. This is impossible. The proof is similar to case 2.

Case 4: $t_{i,j} = t_{k,j} = t_{k,i} = 3$. Similarly impossible.

Hence we prove that $T_4(0,1,2,3,4)$ is the incidence matrix of the coloring of K_{50} without a one-color triangle.

Thus, $f(4) \geq 50$, i.e., $N(3,3,3,3;2) > 50$.

3. $f(k+1) \geqslant 3f(k) + f(k-2)$

The result in Section 2 can be generalized to any $k \geq 4$.

Let $T_k(x_0, x_1, ..., x_k)$ be the incidence matrix of the coloring of the complete graph of n_k vertices without a one-color triangle in k colors.

Similarly, we construct $T_{k+1}(0,1,2,...,k+1)$ as shown in Diagram 1.

$T_{k+1}(0,1,2,...,k+1) =$

A			
D	*B*		
E	*F*	*C*	
11 ....1 ⋮ ⋮ 1..........1	22 ....2 ⋮ ⋮ 2..........2	33 ....3 ⋮ ⋮ 3..........3	*G*

Diagram 1.

$A = T_k(0,2,3,4,5,...,k+1)$, $B = T_k(0,3,1,4,5,...,k+1)$,
$C = T_k(0,1,2,4,5,...,k+1)$, $D = T_k(3,2,1,4,5,...,k+1)$,
$E = T_k(2,1,3,4,5,...,k+1)$, $F = T_k(1,3,2,4,5,...,k+1)$,
$G = T_{k-2}(0,4,5,...,k+1)$.

The proof that such a coloring has no one-color triangle is quite similar to the proof in Section 2. Hence we have $f(k+1) \geq 3f(k) + f(k-2)$.

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