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Weak Monge arrays in higher dimensions [1]

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Abstract

An $n \times n$ matrix C is called a *weak Monge* matrix if $c_{ii} + c_{rs} \leqslant {}_{is} + c_{ri}$ for all $1 \leqslant i \leqslant r, s \leqslant n$. It is well known that the classical linear assignment problem is optimally solved by the identity permutation if the underlying cost-matrix fulfills the weak Monge property.

In this paper we introduce d-dimensional weak Monge arrays, ($d \geqslant 2$), and prove that d-dimensional axial assignment problems can be solved efficiently whenever the underlying cost-array fulfills the d-dimensional weak Monge property. Moreover, it is shown that all results also carry over into an abstract algebraic framework. Finally, the problem of testing whether or not a given array can be permuted to become a weak Monge array is investigated. © 1998 Elsevier Science B.V. All rights reserved

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1. Introduction

Let an $m \times n$ matrix C with real entries and two nonnegative vectors $a^1 = (a_1^1, \dots, a_m^1)$ and $a^2 = (a_1^2, \dots, a_n^2)$ such that $\sum_{i=1}^m a_i^1 = \sum_{j=1}^n a_j^2$ be given. Then the classical *Hitchcock* transportation problem (TP) can be stated as linear program in the following form:

$$\min \quad \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_{ij} = a_i^1 \quad \text{for all } i = 1, \dots, m,$$

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$$\sum_{i=1}^{m} x_{ij} = a_j^2 \quad \text{for all } j = 1,\dots,n,$$

$$x_{ij} \geqslant 0 \quad \text{for all } i,j.$$

Given a sequence $\mathscr{S} := ((i_1,j_1),\dots,(i_{mn},j_{mn}))$ of mn pairwise distinct pairs of indices corresponding to an instance of (TP), the subsequent greedy-algorithm G($\mathscr{S}$)

$$\begin{aligned} G(\mathrm{S}): \textbf{For } & k := 1 \textit{ to } nm \textbf{ do}, \\ & \mathit{Set} = x_{i_k j_k} := \min\{a_{i_k}^1, a_{j_k}^2\}, \\ & a_{i_k}^1 := a_{i_k}^1 - x_{i_k j_k}, \\ & a_{j_k}^2 := a_{j_k}^2 - x_{i_k j_k}, \end{aligned}$$

always produces a feasible solution of the given instance of (TP). Hoffman [7] identified a necessary and sufficient condition for the sequence $\mathscr{S}$ such that the greedy-algorithm G($\mathscr{S}$) always finds the optimal solution of (TP) no matter which supply and demand vectors a^1 and a^2 are considered. Such a sequence $\mathscr{S}$ (with respect to the given matrix C) must satisfy the following condition:

For every $1 \leqslant i, r \leqslant m, 1 \leqslant j, s \leqslant n$, whenever (i,j) precedes both (i,s) and (r,j) in S, the corresponding matrix entries in C are such that

$$c_{ij} + c_{rs} \leqslant c_{is} + c_{rj}. \tag{1}$$

Sequences fulfilling property (1) are called *Monge sequences* due to an ancient work of Monge [8]. A subclass of $m \times n$ matrices C having a Monge sequence is the class of *Monge matrices* which fulfill

$$c_{ij} + c_{rs} \leqslant c_{is} + c_{rj} \quad \text{for all } 1 \leqslant i < r \leqslant m, \ 1 \leqslant j < s \leqslant n. \tag{2}$$

To see this, take the *lexicographical* sequence $\mathscr{S}_{\text{lex}} := ((1,1),(1,2),\dots,(m,n))$. Note also that the greedy-algorithm G($\mathscr{S}_{\text{lex}}$) degenerates to the well-known north-west corner rule and–as a direct consequence of the result of Hoffman — always produces an optimal solution of (TP) for all feasible supply and demand vectors a^1 and a^2, whenever the cost matrix satisfies property (2).

A special case of (TP) is the *linear assignment* problem, (AP), where $m = n$, $a_i^1 = a_j^2 = 1$ for all i,j and the variables x_{ij} are forced to be either 0 or 1. Interpreting the above considerations it follows that the (AP) restricted to Monge matrices is always solved to optimality by the identity permutation. However, Derigs et al. [6] proved a more general result showing that the identity permutation is always optimal whenever the underlying cost-matrix C satisfies the following weaker condition:

$$c_{ii} + c_{rs} \leqslant c_{is} + c_{ri} \quad \text{for all } 1 \leqslant i < r \leqslant n, \ 1 \leqslant i < s \leqslant n. \tag{3}$$

Matrices of this type are called *weak* Monge matrices (cf. [4]).

More interesting is the step to d dimensions, $d \geqslant 2$. The Monge property (2) was defined by Aggarwal and Park [2] for d-dimensional arrays. In [9] the concept of Monge sequences is generalized to d dimensions (cf. Section 2 for definitions). The main applications of these higher-dimensional Monge structures are the d-dimensional transportation problem, (d TP), and the closely related NP-hard d-dimensional axial assignment problem, (d AP). The (d TP) is formulated as follows:

$$\min \quad \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_d=1}^{n_d} c[i_1, i_2, \ldots, i_d] x_{i_1 i_2 \ldots i_d}$$

$$\text{s.t.} \quad \sum_{\substack{i_1, i_2, \ldots, i_d \\ i_k = q}} x_{i_1 i_2 \ldots i_d} = a_q^k \quad \text{for all } k = 1, \ldots, d \quad \text{and} \quad q = 1, \ldots, n_k,$$

$$x_{i_1 i_2 \cdots i_d} \geqslant 0 \quad \text{for all } i_1, i_2, \ldots, i_d.$$

In this formulation C is a given $n_1 \times \cdots \times n_d$ cost array with real entries and $a^1, \ldots, a^d$ are given nonnegative supply (demand) vectors where a^k is a n_k-dimensional real vector, $k = 1, \ldots, d$, such that $\sum_{p=1}^{n_k} a_p^k = \sum_{q=1}^{n_l} a_q^l$ for all $1 \leqslant k < l \leqslant d$. Setting $n_1 = \cdots = n_d$, $a_q^k = 1$ for $1 \leqslant k \leqslant d$, $1 \leqslant q \leqslant n_k$ and requiring that all variables must be integer we arrive at (d AP).

Bein et al. [3] showed that, if the cost array is a Monge array, a simple greedy approach determines an optimal solution of (d TP) as well as for (d AP). This greedy approach is a natural extension of $G(\mathscr{S}_{\text{lex}})$ to d-dimensions. Rudolf [9] proved that (d TP) is still greedily solvable whenever the underlying cost-array possesses a d-dimensional Monge sequence. For further applications of Monge and Monge-like properties the interested reader is referred to the survey by Burkard et al. [4].

The main purpose of this paper is to introduce d-dimensional weak Monge arrays and to show that the result of Derigs et al. [6] also carries over to d dimensions. Moreover, some relationships between higher-dimensional Monge structures are investigated. Additionally, the whole concept is embedded into an algebraic framework.

The paper is organized as follows: Starting with the definitions in Section 2 we investigate the relationships and differences of higher-dimensional Monge structures in Section 3. Section 4 contains the generalization of the result of Derigs et al. [6] and Section 5 finally deals with the problem whether or not a given array can be permuted to become a weak Monge array.

2. Definitions

In this section we will briefly recall the definitions of Monge arrays and higher-dimensional Monge sequences and introduce the concept of weak Monge arrays.

Let C be an $n_1 \times n_2 \times \cdots \times n_d$ array with $d \geqslant 2$, $N_k = \{1, \ldots, n_k\}$ for all $1 \leqslant k \leqslant d$ and $n := \min\{n_i | i = 1, \ldots, d\}$. We call a d-tuple of indices $(i^1, \ldots, i^d)$ *feasible*, if $i^k \in N_k$ for all $1 \leqslant k \leqslant d$.

Definition 1 (*Aggarwal and Park* [2]). An $n_1 \times n_2 \times \cdots \times n_d$ array C is called a Monge array, if it fulfills

$$c[s^1,\ldots,s^d]+c[t^1,\ldots,t^d]\leqslant c[i^1,\ldots,i^d]+c[j^1,\ldots,j^d] \tag{4}$$

for all feasible tuples $(i^1,\ldots,i^d)$ and $(j^1,\ldots,j^d)$ where $s^k := \min\{i^k,j^k\}$ and $t^k := \max\{i^k,j^k\}$ for $k = 1,\ldots,d$.

To define d-dimensional Monge sequences as well as weak Monge arrays some definitions and notations are needed. Given a set $\mathscr{F} := \{(j_1^1,\ldots,j_1^d), \ldots, (j_q^1,\ldots,j_q^d)\}$ of q pairwise distinct feasible d-tuples of indices we denote by $J^k(\mathscr{F}) := \{j_l^k | (j_l^1,\ldots,j_l^d) \in \mathscr{F}\}$ the associated multisets of indices occurring on the *kth* position of the tuples in the set $\mathscr{F}$.

Let $x = (i^1,\ldots,i^d)$ be a fixed feasible tuple of indices and let $\mathscr{F}$ be given as before. Then $\mathscr{F}$ is called a *covering* with respect to x, if (i) $x \notin \mathscr{F}$ and (ii) if $i^k \in J^k(\mathscr{F})$ for all $1 \leqslant k \leqslant d$, i.e. x is 'covered' by $\mathscr{F}$. If additionally no proper subset of a covering $\mathscr{F}$ is a covering w.r.t. x then $\mathscr{F}$ is said to be *minimal*. And $\mathscr{F}$ is called an *upper* covering w.r.t. x whenever it is minimal and for all $k = 1,\ldots,d$ and $j_l^k \in J^k(\mathscr{F})$ with $j_l^k \neq i^k$ we have that $j_l^k > i^k$. Moreover, we call an upper covering w.r.t. x *simple*, whenever each element $j_l^k \neq i^k$ occurs only once in the multiset $J^k(\mathscr{F})$, i.e. the set $J^k(\mathscr{F}) \setminus \{i^k\}$ collapses to a set.

Moreover, whenever $\mathscr{F}$ is a covering w.r.t. x we define for all $1 \leqslant k \leqslant d$ the multisets $I^k(\mathscr{F}) := J^k(\mathscr{F}) \setminus \{i^k\}$ obtained by deleting i^k exactly once from the multiset $J^k(\mathscr{F})$ and by $M(\mathscr{F}) := \{(s^1,\ldots,s^d) | s^k \in J^k(\mathscr{F}), 1 \leqslant k \leqslant d\}$ the set of all feasible d-tuples which are covered by $\mathscr{F}$.

Now, we are prepared to define d-dimensional Monge sequences and to introduce weak Monge arrays.

Definition 2 (*Rudolf* [9]). Given an array C and a sequence $\mathscr{S}$ of all elements of the Cartesian product $N_1 \times \cdots \times N_d$. Then $\mathscr{S}$ is called a d-dimensional Monge sequence, if the subsequent condition holds for all $(i^1,\ldots,i^d) \in \mathscr{S}$ and all corresponding minimal coverings $\mathscr{F} := \{(j_l^1,\ldots,j_l^d) | 1 \leqslant l \leqslant q\}$ with respect to $(i^1,\ldots,i^d)$.

Whenever $(i^1,\ldots,i^d)$ is the element which occurs first in $\mathscr{S}$ among all elements contained in $M(\mathscr{F})$, then there exist permutations $\phi_1,\ldots,\phi_d$ on $\{1,\ldots,q-1\}$ such that

$$c[i^1,\ldots,i^d]+\sum_{l=1}^{q-1} c[i^1_{\phi_1(l)},i^2_{\phi_2(l)},\ldots,i^d_{\phi_d(l)}]\leqslant \sum_{(j_l^1,\ldots,j_l^d)\in\mathscr{F}} c[j_l^1,\ldots,j_l^d], \tag{5}$$

where $i_l^k \in I^k(\mathscr{F})$ for all $1 \leqslant l \leqslant q-1$, $k = 1,\ldots,d$.

Definition 3. An $n_1 \times \cdots \times n_d$ array C is called a d-dimensional weak Monge array, if for all $1 \leqslant i \leqslant n$ and for all simple upper coverings $\mathscr{F} := \{(j_l^1,\ldots,j_l^d) | 1 \leqslant l \leqslant q\}$

with respect to the d-tuple $(i, i, \ldots, i)$ there exist permutations $\phi_1, \ldots, \phi_d$ on the set $\{1, \ldots, q-1\}$ such that

$$c[i, i, \ldots, i] + \sum_{l=1}^{q-1} c[i^1_{\phi_1(l)}, i^2_{\phi_2(l)}, \ldots, i^d_{\phi_d(l)}] \leqslant \sum_{(j^1_l, \ldots, j^d_l) \in \mathscr{F}} c[j^1_l, \ldots, j^d_l], \tag{6}$$

where $i^k_l \in I^k(\mathscr{F})$ for all $1 \leqslant l \leqslant q-1$, $k = 1, \ldots, d$.

To make the above definition more transparent, we investigate the cases for $d = 2$ and $d = 3$ explicitly.

Given an $n \times n$ matrix C. Then for each $(i, i), 1 \leqslant i \leqslant n$, all possible simple upper coverings w.r.t. (i, i) are characterized by $\mathscr{F} := \{(i, s), (r, i)\}$ with $i < r \leqslant n$ and $i < s \leqslant n$. Note that $J^1(\mathscr{F}) = \{i, r\}, J^2(\mathscr{F}) = \{i, s\}$, $I^1(\mathscr{F}) = \{r\}$ and $I^2(\mathscr{F}) = \{s\}$. Thus, (6) reduces to $c_{i,i} + c_{r,s} \leqslant c_{i,s} + c_{r,i}$, equivalent to (3).

The situation is more complicated in the case $d = 3$. Let C be an $n_1 \times n_2 \times n_3$ array and let (i, i, i) be a fixed triple of indices with $1 \leqslant i \leqslant n = \min\{n_1, n_2, n_3\}$. Then we have seven different types of simple upper coverings with respect to (i, i, i): $\mathscr{F}_1 = \{(i, i, t), (i, s, i)\}, \mathscr{F}_2 = \{(i, i, t), (r, i, i)\}, \mathscr{F}_3 = \{(i, s, i), (r, i, i)\}, \mathscr{F}_4 = \{(i, i, t), (r, s, i)\}$, $\mathscr{F}_5 = \{(i, s, i), (r, i, t)\}, \mathscr{F}_6 = \{(r, i, i), (i, s, t)\}$ and $\mathscr{F}_7 = \{(i, s_1, t_1), (r_1, i, t_2), (r_2, s_2, i)\}$ where $i < r, r_1, r_2 \leqslant n_1$, $i < s, s_1, s_2 \leqslant n_2$, $i < t, t_1, t_2 \leqslant n_3$ and $r_1 \neq r_2$, $s_1 \neq s_2$ and $t_1 \neq t_2$ (note that this restriction is necessary, since we restrict ourselves only to simple upper coverings).

Exploiting (6) for each covering from above we obtain the following conditions for $1 \leqslant i < r, r_1, r_2 \leqslant n_1, 1 \leqslant i < s, s_1, s_2 \leqslant n_2, 1 \leqslant i < t, t_1, t_2 \leqslant n_3$ and $r_1 \neq r_2, s_1 \neq s_2$ and $t_1 \neq t_2$:

(i) $c[i, i, i] + c[i, s, t] \leqslant c[i, i, t] + c[i, s, i]$,
(ii) $c[i, i, i] + c[r, i, t] \leqslant c[i, i, t] + c[r, i, i]$,
(iii) $c[i, i, i] + c[r, s, i] \leqslant c[i, s, i] + c[r, i, i]$,
(iv) $c[i, i, i] + c[r, s, t] \leqslant c[i, i, t] + c[r, s, i]$,
(v) $c[i, i, i] + c[r, s, t] \leqslant c[i, s, i] + c[r, i, t]$,
(vi) $c[i, i, i] + c[r, s, t] \leqslant c[r, i, i] + c[i, s, t]$ and
(vii) $c[i, i, i] + \min_{\phi, \psi} \{c[r_1, s_{\phi(1)}, t_{\psi(1)}] + c[r_2, s_{\phi(2)}, t_{\psi(2)}]\} \leqslant c[i, s_1, t_1] + c[r_1, i, t_2] + c[r_2, s_2, i]$, where ϕ and ψ are arbitrary permutations of the set $\{1, 2\}$.

Note that conditions (i)–(iii) coincide with the conditions for a two-dimensional matrix (one dimension is fixed) and that conditions (iv)–(vi) relate to condition (4), whereas condition (vii) is new.

We close this section by mentioning that the definition of weak Monge arrays can be extended into an algebraic framework.

Therefore, let $(H, \oplus, \preceq)$ be a totally ordered commutative semigroup such that $\oplus$ is compatible with $\preceq$, i.e.

$$a \preceq b \quad \Longrightarrow \quad a \oplus c \preceq b \oplus c \quad \text{for all} \quad a, b, c \in H. \tag{7}$$

Let an $n_1 \times \cdots \times n_d$ array C be given whose entries are taken from H. Then we obtain (d-dimensional) weak *algebraic* Monge arrays in a straightforward way by simply replacing $+$ and $\leqslant$ with $\oplus$ and $\preceq$ in Definition 3. With $\oplus := \max$ and $\preceq := \leqslant$ we arrive at *bottleneck* weak Monge arrays. Further examples for $(H, \oplus, \preceq)$ can e.g. be found in Burkard et al. [4].

3. Relationships of higher-dimensional Monge structures

In this section we try to relate arrays which are weak Monge arrays to Monge arrays and arrays possessing a higher-dimensional Monge sequence. In particular, we will show that Monge arrays form a proper subclass of weak Monge arrays and that each array possessing a Monge sequence can be permuted in such a way that it becomes a weak Monge array.

To that end, we first present an equivalent characterization of Monge arrays using the notation introduced in the previous section.

Lemma 4. *A d-dimensional array C is a Monge array iff for all feasible tuples $(i^1,\dots,i^d)$ and for all upper coverings $\mathscr{F} := \{(j_l^1,\dots,j_l^d) | 1 \leqslant l \leqslant q\}$ with respect to $(i^1,\dots,i^d)$ there exist permutations $\phi_1,\dots,\phi_d$ on the set $\{1,\dots,q-1\}$ such that*

$$c[i^1,\dots,i^d] + \sum_{l=1}^{q-1} c[i^1_{\phi_1(l)}, i^2_{\phi_2(l)}, \dots, i^d_{\phi_d(l)}] \leqslant \sum_{(j_l^1,\dots,j_l^d)\in\mathscr{F}} c[j_l^1,\dots,j_l^d], \tag{8}$$

where $i_l^k \in I^k(\mathscr{F})$ for all $1 \leqslant l \leqslant q-1$, $k = 1,\dots,d$.

Proof. '$\Longleftarrow$': Let $(i^1,\dots,i^d)$ be a feasible d-tuple of indices. Since for each upper covering $\mathscr{F}$ with respect to $(i^1,\dots,i^d)$ condition (8) holds, it holds for all upper coverings $\mathscr{F} := \{(s^1,\dots,s^d),(t^1,\dots,t^d)\}$. Since $\mathscr{F}$ is an upper covering with respect to $(i^1,\dots,i^d)$ and $q = 2$, each multiset $I^k(\mathscr{F})$ contains exactly one entry, say j^k, such that $j^k \geqslant i^k$, or in other words $i^k = \min\{s^k, t^k\}$ and $j^k = \max\{s^k, t^k\}$. But now, condition (8) turns into

$$c[i^1,\dots,i^d] + c[j^1,\dots,j^d] \leqslant c[s^1,\dots,s^d] + c[t^1,\dots,t^d],$$

for $i^k = \min\{s^k,t^k\}$ and $j^k = \max\{s^k,t^k\}$. Since $\mathscr{F}$ is an arbitrary upper covering w.r.t. an arbitrary tuple $(i^1,\dots,i^d)$, we arrive exactly at the Monge property in d dimensions.

'$\Longrightarrow$': Assume that C is a Monge array. Let $(i^1,\dots,i^d)$ be feasible, $\mathscr{F} = \{(j_l^1,\dots,j_l^d) | 1 \leqslant l \leqslant q\}$ be an arbitrary upper covering with respect to $(i^1,\dots,i^d)$ and let $J^k(\mathscr{F})$ be the multisets associated with $\mathscr{F}$. We will show that (8) holds. First construct an ordered d-dimensional subarray of C which we get by deleting all entries $c[s^1,\dots,s^d]$ in C for which there exists a $k, 1 \leqslant k \leqslant d$, s.t. $s^k \notin J^k(\mathscr{F})$ and denote it with $B(\mathscr{F})$. Note that $B(\mathscr{F})$ is a Monge array itself, since each ordered subarray of a Monge array is Monge itself. In a next step, $B(\mathscr{F})$ is expanded to a $q \times \cdots \times q$ array D. For each dimension k,

$1 \leqslant k \leqslant d$, investigate the multiset $J^k(\mathscr{F})$. Whenever an element p occurs α_p times in $J^k(\mathscr{F})$, the induced $(d-1)$-dimensional subarray (all elements in $B(\mathscr{F})$ whose index at the kth position is equal to p) is replaced by exactly α_p copies of this subarray. Since $|J^k(\mathscr{F})| = q$ for all k we finally obtain a $q \times q \times \cdots \times q$ array D. It is evident that D obtained this way is a Monge array (duplicating subarrays does not violate the Monge condition (4)). Therefore — since D is a Monge array–it follows from the result of Bein et al. [3] applied to d-dimensional assignment problems that the sum of the main diagonal in D is less or equal than the value of any other assignment. Thus, it follows that there exist appropriate permutations $\phi_1, \ldots, \phi_d$ acting on $\{1, \ldots, q-1\}$ such that

$$c[i^1, \ldots, i^d] + \sum_{l=1}^{q-1} c[i^1_{\phi_1(l)}, i^2_{\phi_2(l)}, \ldots, i^d_{\phi_d(l)}] \leqslant \sum_{(j^1_l, \ldots, j^d_l) \in \mathscr{F}} c[j^1_l, \ldots, j^d_l],$$

where $i^k_l \in I^k(\mathscr{F})$ for all $1 \leqslant l \leqslant q-1$, $k = 1, \ldots, d$. (Note that since $\mathscr{F}$ is an upper covering, $(i^1, \ldots, i^d)$ is the lexicographically smallest tuple of indices occurring in D and therefore always part of the main diagonal in D.) □

As a direct consequence of Definition 3 and Lemma 4 we get the following corollary.

Corollary 5. *Each Monge array is a weak Monge array.*

Next, the relationship between *permuted* weak Monge arrays and arrays having a d-dimensional Monge sequence is established. An $n_1 \times \cdots \times n_d$ array C is a *permuted* (weak) Monge array whenever there exist permutations $\psi_1, \ldots, \psi_d$ acting on the sets $N_k, 1 \leqslant k \leqslant d$ such that the permuted array $C_{\psi_1, \ldots, \psi_d} := \left(c[\psi_1(i^1), \ldots, \psi_d(i^d)]\right)$ is a (weak) Monge array.

Burkard et al. [4] show that if a given $n \times n$ matrix C possesses a Monge sequence, then C is a permuted weak Monge matrix. The situation in d dimensions is similar.

Lemma 6. *Let C be an $n_1 \times \cdots \times n_d$ array. If C possesses a Monge sequence $\mathscr{S}$, then C is a permuted weak Monge array, too.*

Proof. We show in the following that, given a d-dimensional Monge sequence $\mathscr{S}$, it is always possible to define permutations $\psi_1, \ldots, \psi_d$ such that $B := C_{\psi_1, \ldots, \psi_d}$ is weak Monge. Let $\mathscr{S} = \{(i^1_1, \ldots, i^d_1), \ldots, (i^1_v, \ldots, i^d_v)\}$ with $v = \prod_{k=1}^d n_k$. Define $\psi_k(1) = i^k_1$ for $k = 1, \ldots, d$, i.e. $b[1, \ldots, 1] := c[i^1_1, \ldots, i^d_1]$. Since $\mathscr{S}$ is a Monge sequence and $(i^1_1, \ldots, i^d_1)$ is the first element in $\mathscr{S}$, condition (5) holds for each simple and minimal covering $\mathscr{F}$ with respect to $(i^1_1, \ldots, i^d_1)$. No matter how the permutations $\psi_1, \ldots, \psi_d$ are completed, any such covering $\mathscr{F}$ turns into a simple upper covering with respect to $(1, \ldots, 1)$ in B. Therefore condition (6) is fulfilled for $(1, \ldots, 1)$ and the array B.

Now, we are almost done. Construct a new array C' by deleting all entries $c[s^1, \ldots, s^d]$ such that there $\exists k$ s.t. $s^k = i^k$, i.e. we remove all $(d-1)$-dimensional subarrays from

C which contain the entry $c[i^1,\dots,i^d]$. At the same time we also remove the tuples of indices from the Monge sequence $\mathscr{S}$ corresponding to these entries and get a smaller sequence $\mathscr{S}'$. After this deletion an $(n_1-1)\times\cdots\times(n_d-1)$ array C' and a Monge sequence $\mathscr{S}'$ for C' remain (since $\mathscr{S}$ was a Monge sequence for C, $\mathscr{S}'$ is also a Monge sequence for C').

Thus it is possible to recursively define the permutations $\psi_k(i)$ using this procedure for all $1\leqslant k\leqslant d$ and $2\leqslant i\leqslant n$ where $n:=\min\{n_k|1\leqslant k\leqslant d\}$. Finally, after n steps all conditions (6) are verified for B. In the last step the permutations are completed in an arbitrary way. So B is a weak Monge array and the lemma is proven. □

4. An application of weak Monge arrays

In this section the main result of this paper is given, namely that each instance of an (algebraic) d-dimensional axial assignment problem can be solved in polynomial time, if the underlying cost structure fulfills the weak (algebraic) Monge property in d dimensions. This result generalizes the result obtained by Derigs et al. [6] for two dimensions.

Theorem 7. *If the $n\times\cdots\times n$ cost array C of a d-dimensional axial assignment problem is a d-dimensional weak Monge array, then the optimal value is obtained by*

$$\sum_{i=1}^{n} c[i,i,\dots,i].$$

Proof. Let X denote the solution obtained by setting $x[i,\dots,i]=1$ for all $i=1,\dots,n$ and zero otherwise and assume that X is not optimal. Consider all optimal solutions of (dAP) with respect to C and denote with Y an optimal solution where $i:=\min\{j|y[j,\dots,j]\neq 1\}$ is maximum. Since $Y\neq X$, $i<n-1$. Now, investigate the set $\mathscr{F}_1$ defined as $\mathscr{F}_1:=\{(j_l^1,j_l^2,\dots,j_l^d)|y[j_l^1,j_l^2,\dots,j_l^d]=1\}\setminus\{(j,j,\dots,j)|1\leqslant j<i\}$. Now, let $\mathscr{F}\subseteq\mathscr{F}_1$ be a minimal covering with respect to $(i,\dots,i)$ and let $q:=|\mathscr{F}|$. Due to the fact that Y is a d-dimensional assignment and since $y[j,\dots,j]=1$ for all $1\leqslant j<i$, the set $\mathscr{F}_1$ is a simple upper covering with respect to $(i,\dots,i)$, since $j_l^k\geqslant i$ for all $j_l^k\in J^k(\mathscr{F})$ and $J^k(\mathscr{F})\setminus\{i\}$ collapses to a set. Since C is a weak Monge array, due to condition (6), there exist permutations $\phi_1,\dots,\phi_d$ such that

$$c[i,i,\dots,i]+\sum_{l=1}^{q-1}c[i^1_{\phi_1(l)},\dots,i^d_{\phi_d(l)}]\leqslant\sum_{(j_l^1,\dots,j_l^d)\in\mathscr{F}}c[j_l^1,\dots,j_l^d],$$

where $i_l^k\in I^k(\mathscr{F})$ for all $1\leqslant l\leqslant q-1$, $k=1,\dots,d$. But now, we are able to construct a new feasible solution $Y^\star$ in the following way: set $y^\star[i,i,\dots,i]=1$, $y^\star[j_l^1,\dots,j_l^d]=0$ for all $(j_l^1,\dots,j_l^d)\in\mathscr{F}$, $y^\star[i^1_{\phi_1(l)},\dots,i^d_{\phi_d(l)}]=1$ for all $l=1,\dots,q-1$ and leave all other elements of Y unchanged. Due to the above inequality, it is easy to see that

$Y^\star$ is also an optimal assignment and since $y^\star[j,\dots,j] = 1$ for all $j = 1,\dots,i$ we have achieved a contradiction to the choice of Y w.r.t. the maximality of i. Thus, the theorem is proven. □

It is evident that Theorem 7 also holds in the algebraic case. For example, we obtain a polynomial-time solvable special case of the bottleneck d-dimensional assignment problem, if the underlying cost-structure is a bottleneck weak Monge array.

5. Recognizing permuted algebraic weak Monge arrays

Before we investigate the problem of recognizing permuted weak (algebraic) Monge arrays we briefly mention the results and algorithms derived for the two-dimensional case. Given an $n \times n$ matrix C we ask for permutations ϕ and ψ such that $C_{\phi,\psi}$ is a permuted weak (algebraic) Monge matrix.

An efficient algorithm with running time $O(n^4)$ which can be improved to $O(n^3 \log n)$ time is due to Cechlárová and Szabó [5]. However, we will follow another approach due to Burkard et al. [4]. Although it does not improve the time complexity, it is much simpler and also works in an abstract algebraic setting. This algorithm is based on the fact that matrices having a Monge sequence are permuted weak Monge. Therefore their algorithm can be seen as a minor adjustment of the algorithm of Alon et al. [1] for the detection and construction of Monge sequences. It works as follows:

Start with an empty graph consisting of n^2 nodes corresponding to all pairs of indices. Consider all pairs (i,j) and (r,s) with $i \neq r$ and $j \neq s$ and add an edge from (i,j) to (r,s) whenever $c_{ij} + c_{rs} > c_{is} + c_{jr}$. Set $k = 1$. As long as there exists an isolated node (i,j) in G, set $\phi(k) = i$ and $\psi(k) = j$, increase k by one and update G as follows: Delete the node (i,j), all nodes (i,s) and (r,j) and all edges incident to those nodes. The algorithm stops, if G is empty or there is no isolated node in G. In the first case $C_{\phi,\psi}$ is a weak (algebraic) Monge matrix, in the latter one C is not a permuted weak (algebraic) Monge matrix.

This approach can also be adapted for the higher-dimensional case. Starting from the algorithm for constructing and detecting d-dimensional (weak) Monge sequences given in [9] we are able to modify this algorithm in the same way as before to recognize permuted weak (algebraic) Monge arrays.

For the ease of illustration let us assume that we only treat $n \times n \times \cdots \times n$ arrays (an extension to $n_1 \times n_2 \times \cdots \times n_d$ arrays is straightforward).

Again we start by building a directed graph G having n^d nodes corresponding to all possible d-tuples of indices. The arcs of G are constructed in the following way. For each d-tuple of indices $(i^1,\dots,i^d)$ we investigate all minimal and simple coverings $\mathcal{F}$ w.r.t. $(i^1,\dots,i^d)$ and look at all *active* inequalities, i.e. we identify each set of d permutations $\phi_1,\dots,\phi_d$ such that

$$c[i^1,i^2,\dots,i^d] + \sum_{l=1}^{q-1} c[i^1_{\phi_1(l)}, i^2_{\phi_2(l)},\dots,i^d_{\phi_d(l)}] < \sum_{(j^1_l,\dots,j^d_l)\in\mathcal{F}} c[j^1_l,\dots,j^d_l], \tag{9}$$

where $i_l^k \in I^k(\mathscr{F})$ for all $1 \leqslant l \leqslant q-1$, $k = 1,\dots,d$. Then define for each active inequality directed arcs from each d-tuple of indices occurring on the left side of inequality (9) to each d-tuple occurring on the right and associate with each of these arcs a common unique label.

After having constructed G, we are able to fix d permutations $\psi_1,\dots,\psi_d$ such that $C_{\psi_1,\dots,\psi_d}$ is a weak Monge array step by step (if they exist at all): In each step j, $j = 1,\dots,n$, we are looking for a node $v \in G$ with indegree equal to zero. If no such node v exists, we can stop, C is not an permuted weak Monge array. Otherwise — assuming that $(i^1,\dots,i^d)$ is the corresponding d-tuple to node v — fix $\psi_k(j) = i^k$ for $k = 1,\dots,d$. The remaining part in this step concerns the update of the graph G. Apart from node $(i^1,\dots,i^d)$ we also delete all nodes corresponding to $(j^1,\dots,j^d)$ with $j^k = i^k$ for at least one k, all incident arcs to these nodes and all arcs having the same labels. Note that this is equivalent to canceling all active inequalities of type (9) in which at least one index i^k is involved, or in other words we delete all $(d-1)$-dimensional subarrays of C which contain the entry $c[i^1,\dots,i^d]$.

Finally, after n steps, all permutations are completely determined.

Theorem 8. *The algorithm above either determines permutations $\psi_1,\dots,\psi_d$ such that $C_{\psi_1,\dots,\psi_d}$ is a weak Monge array or proves that no such permutations exist.*
Its running time is $O(d^2(d-1)!^d n^{d^2})$.

Proof. The correctness of the algorithm is straightforward: Whenever the algorithm stops while G is non-empty, then each node $v \in G$ has an indegree greater than zero. This is equivalent that all d-tuples of indices occur at least once on the right-hand side of an active inequality of type (9) and therefore no d-tuple satisfies condition (6).

If the algorithm stops with d permutations $\psi_1,\dots,\psi_d$, we have to show that $B := C_{\psi_1,\dots,\psi_d}$ is a weak Monge array. Assume the contrary, i.e. that B is not a weak Monge array, then there exists a d-tuple of indices $(i,\dots,i)$ and a simple, upper covering $\mathscr{F}$ w.r.t. $(i,\dots,i)$ such that for all permutations $\phi_1,\dots,\phi_d$

$$b[i,i,\dots,i] + \sum_{l=1}^{q-1} b[i^1_{\phi_1(l)}, i^2_{\phi_2(l)},\dots,i^d_{\phi_d(l)}] > \sum_{(j_l^1,\dots,j_l^d)\in\mathscr{F}} b[j_l^1,\dots,j_l^d], \tag{10}$$

where $i_l^k \in I^k(\mathscr{F})$ for all $1 \leqslant l \leqslant q-1$, $k = 1,\dots,d$. Note that this means that during the ith step the node in G corresponding to $(\psi_1(i),\dots,\psi_d(i))$ has indegree greater than zero, a contradiction to its choice.

Next, we prove the complexity bound of the algorithm. First observe that the directed graph G described above can be constructed in $O(d^2(d-1)!^d n^{d^2})$ time. For each fixed d-tuple $(i_1^1, i_1^2,\dots,i_1^d)$ we have $O(n^{d(d-1)})$ possible simple, minimal coverings. For each such covering there are $O((d-1)!^d)$ possible sets of permutations $\phi_1,\dots,\phi_d$ and therefore, the same maximal number of active inequalities. And since each active inequality generates at most d^2 arcs, G can be constructed in overall $O(d^2(d-1)!^d n^{d^2})$

time. Since during the deletion steps only arcs and nodes constructed in the initialization step are removed, we end up with the same number of basic steps. $\Box$

Since the above algorithm for recognizing permuted weak Monge arrays is mainly based on condition (10), we can simply replace the operation $+$ by $\oplus$ and $>$ by $\succ$ in (10) and finally arrive at an algorithm for recognizing permuted weak algebraic Monge arrays having the same number of basic steps.

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