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Linear Algebra and its Applications 378 (2004) 159–166
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

A lower bound for the minimum eigenvalue of the Hadamard product of matrices ☆

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Received 5 December 2002; accepted 18 September 2003

Submitted by R.A. Horn

Abstract

Suppose both A and B are $n \times n$ nonsingular M-matrices. An estimate from below for the smallest eigenvalue $\tau(A \circ B^{-1})$ (in modulus) of the Hadamard product $A \circ B^{-1}$ of A and B^{-1} is derived. As a special case, we obtain the inequality $\tau(A \circ A^{-1}) \geqslant \frac{2}{n}$ $(n \geqslant 2)$.

AMS classification: 15A15; 15A48

Keywords: M-matrix; Hadamard product; Minimum eigenvalue; Perron eigenvectors

1. Introduction

For a positive integer n, N denotes the set $\{1, 2, \ldots, n\}$ throughout.

For two real matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same size, the Hadamard product of A and B is defined as the matrix $A \circ B = (a_{ij}b_{ij})$. We write $A \geqslant B$ if $a_{ij} \geqslant b_{ij}$ for all $i, j \in N$.

We denote by Z_n the class of all $n \times n$ real matrices all of whose off-diagonal entries are nonpositive. An $n \times n$ matrix A is called an M-matrix if there exists an $n \times n$ nonnegative matrix B and some nonnegative real number λ such that $A = \lambda I - B$ and $\lambda \geqslant \rho(B)$, where $\rho(B)$ is the spectral radius of B, I is an identity matrix; if $\lambda > \rho(B)$, we call A a nonsingular M-matrix, and denote it by $A \in M_n$; if $\lambda = \rho(B)$, we call A a singular M-matrix.

☆ Supported by Foundation to the Educational Committee of Fujian, China (grant no. JB02084) and the Science and Technical Development Foundation of Fuzhou University (2003-XQ-22).

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0024-3795/$ - see front matter
doi:10.1016/j.laa.2003.09.011

Let $A \in Z_n$ and denote $\tau(A) = \min\{R_e(\lambda): \lambda \in \sigma(A)\}$, where $\sigma(A)$ is the set of all eigenvalues of A. Basic for our purpose is the following simple facts (see Problem 16, 19 and 28 in Section (2.5) of [1]):

(i) $\tau(A) \in \sigma(A)$; $\tau(A)$ is called the minimum eigenvalue of A.
(ii) If $A \in M_n$, $B \in M_n$ and $A \geqslant B$, then $\tau(A) \geqslant \tau(B)$.
(iii) If $A \in M_n$, then $\rho(A^{-1})$ is the Perron eigenvalue of the nonnegative matrix A^{-1}, and $\tau(A) = \frac{1}{\rho(A^{-1})}$ is a positive real eigenvalue of A.

Let A be an irreducible nonsingular M-matrix. It is well known that there exist positive vectors u and v such that $Au = \tau(A)u$, and $v^{\mathrm{T}}A = \tau(A)v^{\mathrm{T}}$, u and v are called right and left Perron eigenvectors of A respectively.

For $A \in M_n$, $n \geqslant 2$, Fiedler and Markham [2] proved that $\tau(A \circ A^{-1}) \geqslant \frac{1}{n}$, and proposed the following conjecture:

$$\tau(A \circ A^{-1}) \geqslant \frac{2}{n}.$$

Yong [3] and Song [4] have independently proved this conjecture affirmatively.

For two independent nonsingular M-matrices $A, B \in M_n$, we exhibit lower bounds for $\tau(A \circ B^{-1})$. These bounds are strong enough to yield, upon specialization, the conjectured lower bound of $\frac{2}{n}$ for $\tau(A \circ A^{-1})$.

2. Main results

In this section, we state and prove our main results.

Lemma 2.1 [5]. *If P is irreducible, and $P \in M_n$, $Pz \geqslant kz$ for a nonnegative non-zero vector z, then $k \leqslant \tau(P)$.*

Lemma 2.2 [5]

(a) *If $A = (a_{ij})$ is an $n \times n$ strictly diagonally dominant matrix by row, that is,*

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \quad \forall i \in N$$

then $A^{-1} = (b_{ij})$ exists, and

$$|b_{ji}| \leqslant \frac{\sum_{k \neq j} |a_{jk}|}{|a_{jj}|} |b_{ii}| \quad \text{for all } i \neq j.$$

(b) *If $A = (a_{ij})$ is an $n \times n$ strictly diagonally dominant matrix by column, that is,*

$$|a_{ii}| > \sum_{j \neq i} |a_{ji}| \quad \forall i \in N.$$

then $A^{-1}=(b_{ij})$ *exists, and*

$$|b_{ij}| \leqslant \frac{\sum_{k\neq j}|a_{kj}|}{|a_{jj}|}|b_{ii}| \quad \text{for all } i \neq j.$$

Theorem 2.3. *If* $A=(a_{ij})\in M_n,\ B=(b_{ij})\in M_n,\ B^{-1}=(\beta_{ij})$, *then*

$$\tau(A\circ B^{-1}) \geqslant \tau(A)\tau(B)\min_{1\leqslant i\leqslant n}\left\{\left(\frac{a_{ii}}{\tau(A)}+\frac{b_{ii}}{\tau(B)}-1\right)\frac{\beta_{ii}}{b_{ii}}\right\}. \tag{1}$$

Proof. It is quite evident that (1) holds with equality for $n=1$.

Below we assume that $n\geqslant 2$, let us distinguish two cases:

Case 1. Both A and B are irreducible. Since $B-\tau(B)I$ is a singular irreducible M-matrix, Theorem 6.4.16 of [6] yields that

$$b_{ii}-\tau(B)>0 \quad \forall\, i\in N.$$

Let $u=(u_i)$, $v=(v_i)$ and $y=(y_i)$ be the right Perron eigenvectors of B, B^{T} and A respectively.

Define $C=DB$, where $D=\operatorname{diag}(v_1,v_2,\ldots,v_n)$, then $C^{-1}=B^{-1}D^{-1}$.

Since the matrix C is strictly diagonally dominant by column, by Lemma 2.2, for all $i\neq j$, we have

$$\frac{\beta_{ij}}{v_j} \leqslant \frac{\sum_{k\neq j}|v_k b_{kj}|}{v_j b_{jj}}\cdot\frac{\beta_{ii}}{v_i} = \frac{(b_{jj}-\tau(B))v_j}{v_j b_{jj}}\cdot\frac{\beta_{ii}}{v_i}$$

hence

$$\beta_{ij} \leqslant \frac{(b_{jj}-\tau(B))v_j\beta_{ii}}{b_{jj}v_i}.$$

Now let z be the vector (z_i), where

$$z_i=\frac{y_i b_{ii}}{v_i(b_{ii}-\tau(B))}>0 \quad \forall\, i\in N.$$

We define $P=A\circ B^{-1}$. Since B^{-1} is positive by Theorem 6.2.7 of [6], then P is irreducible as well, and for any $i\in N$,

$$\begin{aligned}
(Pz)_i &= a_{ii}\beta_{ii}z_i-\sum_{j\neq i}|a_{ij}|\beta_{ij}z_j\\
&\geqslant a_{ii}\beta_{ii}z_i-\sum_{j\neq i}|a_{ij}|\cdot\frac{(b_{jj}-\tau(B))v_j\beta_{ii}}{b_{jj}v_i}\cdot\frac{y_j b_{jj}}{v_j(b_{jj}-\tau(B))}\\
&= a_{ii}\beta_{ii}z_i-\frac{\beta_{ii}}{v_i}\sum_{j\neq i}|a_{ij}y_j|\\
&= a_{ii}\beta_{ii}z_i-\frac{\beta_{ii}}{v_i}\cdot(a_{ii}-\tau(A))y_i
\end{aligned}$$

$$
\begin{aligned}
&= \beta_{ii} z_i \left[a_{ii} - \frac{1}{b_{ii}} (a_{ii} - \tau(A))(b_{ii} - \tau(B)) \right] \\
&= \frac{\beta_{ii}}{b_{ii}} \cdot \tau(A)\tau(B) \left[\frac{a_{ii}}{\tau(A)} + \frac{b_{ii}}{\tau(B)} - 1 \right] z_i \\
&\geqslant \tau(A)\tau(B) \min_{1 \leqslant k \leqslant n} \left\{ \left(\frac{a_{kk}}{\tau(A)} + \frac{b_{kk}}{\tau(B)} - 1 \right) \frac{\beta_{kk}}{b_{kk}} \right\} z_i
\end{aligned}
$$

By Lemma 2.1, this shows that Theorem 2.3 is valid.

Case 2. One of A and B is reducible. It is well known that a matrix in Z_n is a nonsingular M-matrix if and only if all its leading principal minors are positive (see condition (E_{17}) of Theorem 6.2.3 of [6]). If we denote by T the $n \times n$ permutation matrix (t_{ij}) with $t_{12} = t_{23} = \cdots = t_{n-1,n} = t_{n1} = 1$, the remaining t_{ij} zero, then both $A - \varepsilon T$ and $B - \varepsilon T$ are irreducible nonsingular M-matrices for any chosen positive real number ε, sufficiently small such that all the leading principal minors of both $A - \varepsilon T$ and $B - \varepsilon T$ are positive. Now we substitute $A - \varepsilon T$ and $B - \varepsilon T$ for A and B respectively in the previous case, and then letting $\varepsilon \to 0$, the result follows by continuity. □

Remark 2.4. Under the hypotheses of Theorem 2.3, in view of that

$$\operatorname{diag}(a_{11}, a_{22}, \ldots, a_{nn}) \geqslant A$$

we have $\min_{1 \leqslant i \leqslant n} a_{ii} \geqslant \tau(A)$. Thus

$$\left(\frac{a_{ii}}{\tau(A)} + \frac{b_{ii}}{\tau(B)} - 1 \right) \frac{\beta_{ii}}{b_{ii}} \geqslant \frac{\beta_{ii}}{\tau(B)}.$$

Therefore

$$\tau(A \circ B^{-1}) \geqslant \tau(A)\tau(B) \min_{1 \leqslant i \leqslant n} \left\{ \left(\frac{a_{ii}}{\tau(A)} + \frac{b_{ii}}{\tau(B)} - 1 \right) \frac{\beta_{ii}}{b_{ii}} \right\} \geqslant \tau(A) \min_{1 \leqslant i \leqslant n} \beta_{ii}.$$

This shows that Theorem 2.3 is better than Theorem 5.7.31 of [1].

Theorem 2.5. *Let $A = (a_{ij}) \in M_n$, $B = (b_{ij}) \in M_n$. Suppose B is irreducible, $u = (u_i)$ and $v = (v_i)$ are right and left Perron eigenvectors of B respectively, such that* $\min_{1 \leqslant i \leqslant n} \{u_i v_i\} = 1$. *Then*

$$\text{(a)} \quad \tau(A \circ B^{-1}) \geqslant \frac{\tau(A)}{\tau(B)} \min_{1 \leqslant i \leqslant n} \left\{ \frac{\frac{a_{ii}}{\tau(A)} + \frac{b_{ii}}{\tau(B)} - 1}{1 + \left(\frac{b_{ii}}{\tau(B)} - 1 \right) \sum_{k=1}^{n} u_k v_k} \right\}, \qquad (2)$$

$$\text{(b)} \quad \tau(B \circ B^{-1}) \geqslant \frac{2}{\sum_{k=1}^{n} u_k v_k}, \quad n \geqslant 2. \qquad (3)$$

Proof. It is not difficult to verify that (2) holds with equality for $n = 1$. Now we assume that $B^{-1} = (\beta_{ij})$ and $n \geqslant 2$. The strategy is to estimate $\frac{\beta_{ii}}{b_{ii}}$, and thus $\min_{1\leqslant i\leqslant n} \frac{\beta_{ii}}{b_{ii}}$, and then apply Theorem 2.3.

Partition B as $B = \begin{pmatrix} b_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, where B_{22} is a matrix of order $n-1$. Since $Bu = \tau(B)u$, and $v^{\mathrm{T}}B = \tau(B)v^{\mathrm{T}}$, we have

$$b_{11}u_1 + B_{12}(u_2, \ldots, u_n)^{\mathrm{T}} = u_1\tau(B), \tag{4}$$

$$B_{21}u_1 + B_{22}(u_2, \ldots, u_n)^{\mathrm{T}} = \tau(B)(u_2, \ldots, u_n)^{\mathrm{T}}, \tag{5}$$

$$b_{11}v_1 + (v_2, \ldots, v_n)B_{21} = \tau(B)v_1, \tag{6}$$

$$B_{12}v_1 + (v_2, \ldots, v_n)B_{22} = \tau(B)(v_2, \ldots, v_n). \tag{7}$$

From (5), we have

$$B_{22}^{-1}B_{21}u_1 + (u_2, \ldots, u_n)^{\mathrm{T}} = \tau(B)B_{22}^{-1}(u_2, \ldots, u_n)^{\mathrm{T}},$$
$$B_{12}B_{22}^{-1}B_{21}u_1 + B_{12}(u_2, \ldots, u_n)^{\mathrm{T}} = \tau(B)B_{12}B_{22}^{-1}(u_2, \ldots, u_n)^{\mathrm{T}}.$$

Using (4), we obtain

$$B_{12}B_{22}^{-1}B_{21}u_1 + (\tau(B) - b_{11})u_1 = \tau(B)B_{12}B_{22}^{-1}(u_2, \ldots, u_n)^{\mathrm{T}},$$
$$(b_{11} - B_{12}B_{22}^{-1}B_{21})u_1v_1 = \tau(B)\big[u_1v_1 - v_1B_{12}B_{22}^{-1}(u_2, \ldots, u_n)^{\mathrm{T}}\big]. \tag{8}$$

On the other hand, (7) implies that

$$v_1B_{12}B_{22}^{-1} + (v_2, \ldots, v_n) = \tau(B)(v_2, \ldots, v_n)B_{22}^{-1},$$
$$v_1B_{12}B_{22}^{-1}(u_2, \ldots, u_n)^{\mathrm{T}} + \sum_{k=2}^{n} u_kv_k = \tau(B)(v_2, \ldots, v_n)B_{22}^{-1}(u_2, \ldots, u_n)^{\mathrm{T}}.$$

By (8), we deduce that

$$(b_{11} - B_{12}B_{22}^{-1}B_{21})u_1v_1 = \tau(B)\left[\sum_{k=1}^{n} u_kv_k - \tau(B)(v_2, \ldots, v_n)B_{22}^{-1}(u_2, \ldots, u_n)^{\mathrm{T}}\right]. \tag{9}$$

Let

$$B_{22}^{-1}(u_2, \ldots, u_n)^{\mathrm{T}} = (u_2y_2, \ldots, u_ny_n)^{\mathrm{T}}, \quad y_i = \min_{2\leqslant k\leqslant n} y_k.$$

Then

$$B_{22}(u_2y_2, \ldots, u_ny_n)^{\mathrm{T}} = (u_2, \ldots, u_n)^{\mathrm{T}},$$

$$
\begin{aligned}
u_i &= \sum_{j=2}^{n} b_{ij} u_j y_j \leqslant y_i \sum_{j=2}^{n} b_{ij} u_j = (\tau(B)u_i - b_{i1}u_1) y_i, \\
u_i v_i &\leqslant [\tau(B)u_i v_i - v_i b_{i1} u_1] y_i \\
&\leqslant \left[\tau(B)u_i v_i - u_1 \sum_{k=2}^{n} b_{k1} v_k\right] y_i \\
&= \left[\tau(B)u_i v_i + (b_{11} - \tau(B))u_1 v_1\right] y_i.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
y_i &\geqslant \frac{u_i v_i}{\tau(B)u_i v_i + (b_{11} - \tau(B))u_1 v_1}, \\
(v_2, \ldots, v_n) B_{22}^{-1} (u_2, \ldots, u_n)^{\mathrm{T}} &= \sum_{k=2}^{n} u_k v_k y_k \geqslant y_i \sum_{k=2}^{n} u_k v_k \\
&\geqslant \frac{u_i v_i \sum_{k=2}^{n} u_k v_k}{\tau(B)u_i v_i + u_1 v_1 (b_{11} - \tau(B))}.
\end{aligned}
$$

According to (9), we infer that

$$
\begin{aligned}
&(b_{11} - B_{12} B_{22}^{-1} B_{21}) u_1 v_1 \\
&\quad \leqslant \tau(B) \left[\sum_{k=1}^{n} u_k v_k - \frac{\tau(B) u_i v_i \sum_{k=2}^{n} u_k v_k}{\tau(B)u_i v_i + u_1 v_1 (b_{11} - \tau(B))}\right] \\
&\quad = \frac{u_1 v_1 \tau(B) \left[\tau(B)u_i v_i + (b_{11} - \tau(B)) \sum_{k=1}^{n} u_k v_k\right]}{\tau(B)u_i v_i + u_1 v_1 (b_{11} - \tau(B))}, \\
&b_{11} - B_{12} B_{22}^{-1} B_{21} \leqslant \frac{\tau(B) \left[\tau(B)u_i v_i + (b_{11} - \tau(B)) \sum_{k=1}^{n} u_k v_k\right]}{\tau(B)u_i v_i + u_1 v_1 (b_{11} - \tau(B))}, \\
&\beta_{11} = \frac{\det B_{22}}{\det B} = \frac{1}{b_{11} - B_{12} B_{22}^{-1} B_{21}} \\
&\quad \geqslant \frac{\tau(B)u_i v_i + u_1 v_1 (b_{11} - \tau(B))}{\tau(B) \left[\tau(B)u_i v_i + (b_{11} - \tau(B)) \sum_{k=1}^{n} u_k v_k\right]}.
\end{aligned}
$$

Taking into account that $u_k v_k \geqslant 1$ $(\forall k \in N)$, we have

$$
\begin{aligned}
&(u_i v_i - 1) \sum_{k=1}^{n} u_k v_k + u_1 v_1 - u_i v_i \\
&\quad = (u_i v_i - 1) \sum_{k=1}^{n} u_k v_k - (u_i v_i - 1) + (u_1 v_1 - 1) \\
&\quad = (u_i v_i - 1) \left(\sum_{k=1}^{n} u_k v_k - 1\right) + (u_1 v_1 - 1) \geqslant 0.
\end{aligned}
$$

which yields that

$$\left[\tau(B)u_iv_i + u_1v_1(b_{11} - \tau(B))\right]\left[\tau(B) + (b_{11} - \tau(B))\sum_{k=1}^{n} u_kv_k\right]$$
$$- [\tau(B) + (b_{11} - \tau(B))]\left[\tau(B)u_iv_i + (b_{11} - \tau(B))\sum_{k=1}^{n} u_kv_k\right]$$
$$= \tau(B)(b_{11} - \tau(B))\left[(u_iv_i - 1)\sum_{k=1}^{n} u_kv_k + u_1v_1 - u_iv_i\right]$$
$$+ (b_{11} - \tau(B))^2(u_1v_1 - 1)\sum_{k=1}^{n} u_kv_k \geqslant 0.$$

This means that

$$\frac{\tau(B)u_iv_i + u_1v_1(b_{11} - \tau(B))}{\tau(B)u_iv_i + (b_{11} - \tau(B))\sum_{k=1}^{n} u_kv_k} \geqslant \frac{b_{11}}{\tau(B) + (b_{11} - \tau(B))\sum_{k=1}^{n} u_kv_k},$$
$$\beta_{11} \geqslant \frac{b_{11}}{\tau(B)\left[\tau(B) + (b_{11} - \tau(B))\sum_{k=1}^{n} u_kv_k\right]}.$$

We can similarly prove

$$\beta_{ii} \geqslant \frac{b_{ii}}{\tau(B)\left[\tau(B) + (b_{ii} - \tau(B))\sum_{k=1}^{n} u_kv_k\right]} \quad \forall\, i \in N,$$
$$\frac{\beta_{ii}}{b_{ii}} \geqslant \frac{1}{\tau(B)\left[\tau(B) + (b_{ii} - \tau(B))\sum_{k=1}^{n} u_kv_k\right]}.$$

Substitution into the inequality (1) of Theorem 2.3 yields the asserted inequality (2).

When $n \geqslant 2$, $\sum_{k=1}^{n} u_kv_k \geqslant 2$. For any $i \in N$, observe that

$$\frac{\frac{2b_{ii}}{\tau(B)} - 1}{1 + \left(\frac{b_{ii}}{\tau(B)} - 1\right)\sum_{k=1}^{n} u_kv_k} = \frac{2\left(\frac{b_{ii}}{\tau(B)} - 1\right) + 1}{\left(\frac{b_{ii}}{\tau(B)} - 1\right)\sum_{k=1}^{n} u_kv_k + 1} \geqslant \frac{2}{\sum_{k=1}^{n} u_kv_k}.$$

This means that (3) holds by (2). □

Corollary 2.6. *Let $B \in M_n$, $n \geqslant 2$. Then*

$$\tau(B \circ B^{-1}) \geqslant \frac{2}{n}.$$

Proof. By examining the known proof of Theorem 3 of [2] carefully, we may assume that B is irreducible, and B^{-1} is a doubly stochastic matrix, in this case, Corollary 2.6 follows immediately from Theorem 2.5(b), since both Perron eigenvectors $u = (u_i)$ and $v = (v_i)$ can be chosen as e, the vector of all ones. □

Acknowledgements

The author would like to thank Professor Roger A. Horn and the anonymous referee for giving valuable suggestions and pointing out some errors in the original version of the paper.

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