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Decomposition of 3-connected cubic graphs

J.L. Fouquet
13 bis rue du tertre St Pierre, 72000 Le Mans, France

H. Thuillier
Université d'Orléans (L.I.F.O.), Informatique, B.P.6759, 45067 Orleans Cedex 2, France

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Abstract

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We solve a conjecture of Foulds and Robinson (1979) on decomposable triangulations in the plane, in the more general context of a decomposition theory of cubic 3-connected graphs. The decomposition gives us a natural way to obtain some known results about specific homeomorphic subgraphs and the extremal diameter of 3-connected cubic graphs.

1. Introduction

Foulds and Robinson [4] define a *decomposable* triangulation in the plane as a triangulation having a nonfacial triangle. From Jordan's lemma (see [11]), a decomposable triangulation is obtained by gluing two smaller triangulations (smaller since they have less vertices) along a triangle. The converse operation (cutting along a nonfacial triangle) gives us two smaller triangulations. It is, thus, clear that any triangulation can be repeatedly cut up into smaller ones without such triangles. At this point, Foulds and Robinson [4] conjectured:

Every plane triangulation can be uniquely decomposed.

Since the dual of a plane triangulation is a 3-connected cubic graph, a separating triangle becomes a *cyclic* 3-*edge cocycle* (a cocycle which leaves a cycle in the two

Correspondence to: J.L. Fouquet, L.R.I Bat 490, Universite de Paris-sud, 91405 Orsay Cedex, France.
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distinct components). The above conjecture can, thus, be written in a dual form. We can even drop the planarity condition and conjecture (the definition of a decomposable 3-connected cubic graph is postponed to the next section):

Every 3-connected cubic graph is uniquely decomposable.

Lehel [8] notes the evidence of the first conjecture; we give here a proof of the second one. The decomposition of a 3-connected cubic graph is used later in the study of two distinct problems.

2. Definitions and notations

Let G be a 3-connected cubic graph, G is said to be *cyclically k-edge-connected* if we cannot disconnect G into two parts C_1 and C_2, each of them containing a cycle, without deleting at least k edges. If G has a cyclic 3-edge cocycle L(G is exactly cyclically 3-edge-connected), we shall say that G is *decomposable following L*. The two 3-connected cubic graphs G' and G'' obtained from G by cutting the edges of L, adding a new vertex to each of them with the same name v_L, one is adjacent to the ends of L in C_1, the other in C_2, are the graphs of the *decomposition of G following L* (see Fig. 1).

A cubic 3-connected graph is *indecomposable* if it is cyclically 4-edge connected or K_4 or $K_{3,3}$. We shall say that the family

$$\mathscr{F} = \{F_1, F_2, \ldots, F_k\}$$

of cubic 3-connected graphs is an *$\mathscr{F}$-decomposition* of G if and only if

– F_i is indecomposable for all i, $1 \leqslant i \leqslant k$,

– $\mathscr{F}$ is the family of graphs obtained from G while using recursively the decomposition procedure.

We can associate with this decomposition a binary tree with root G and leaves the graphs of $\mathscr{F}$. An internal node H of a *decomposition tree of G* has two sons H' and H'',

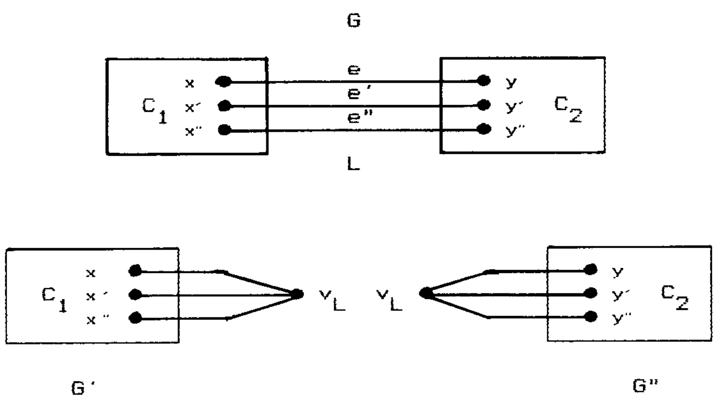

Fig. 1.

which have H as *ancestor*. In a previous paper [5] we have studied the *reducible* (or *removable*) and *nonreducible* (or *nonremovable*) edges (disjoint sets $S(G)$ and $N(G)$ which partition $E(G)$). These notions are derived from the construction of 3-connected cubic graphs which is a specialization of Tutte's [12] construction of 3-connected graphs. When G is a 3-connected cubic graphs on n vertices, we get a new 3-connected cubic graph on $n+2$ vertices by *extension* between two edges e and e' of G (the resulting graph is denoted by $\varepsilon_{e,e'}(G)$); the following figure (Fig. 2) is a straightforward illustration of this operation.

The *reduction* of G relatively to the edge f, denoted as $\varepsilon_f^{-1}(G)$), is the cubic graph obtained from f by deleting f and its end vertices as shown in Fig. 2. This reduction can be done on the edge f if and only if f does not belong to any 3-edge cocycle. Then f is called *reducible* and belongs to $S(G)$ ($N(G)$ will be the set $E(G)\setminus S(G)$). It is clear that the notions of decomposability of a cubic 3-connected graph and reducible edges are two complementary points of view of the same problem. In this paper we develop the notion of decomposability. Decomposition of 3-connected cubic graphs is related to the decomposition theory of Cunningham and Edmonds [2]. However, we do not see actually how to describe our problem in the more general frame of their theory. In particular, using the decomposition of submodular functions [1], we can hope to obtain an efficient algorithm for finding cyclic 3-edge cocycle. This problem is easily solved in $O(n^3)$ by 'brute force' (examine each triple of edges).

A θ-graph is defined to be a graph homeomorphic to θ (the graph on 2 vertices and 3 edges).

3. Decomposition and uniqueness

Let $\mathscr{L}(G)=\{L_1, L_2, \ldots, L_p\}$ be the set of cyclic 3-edge cocycles of G. When G is decomposed following L_i (for some i, $1 \leqslant i \leqslant p$), what can happen for L_j $(i \neq j)$?

Lemma 3.1. *Let G be a 3-connected cubic graph, $L \in \mathscr{L}(G)$. Then there is a θ-graph H in G such that every chain of H contains one (and only one) edge of L.*

Proof. L divides the vertices of G into two 2-connected components C_1 and C_2. Let x in C_1 and y in C_2; from Menger's [9] theorem, we know that there exist three

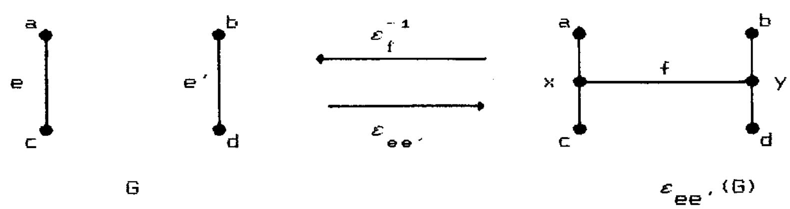

Fig. 2.

internally disjoint chains between x and y. Each of these three chains must intersect L; the result follows. □

Lemma 3.2. *Let G be a 3-connected cubic graph, $L \in \mathscr{L}(G)$, G' and G'' be the sons of G when G is decomposed following L. Every edge in $N(G')$ $(N(G''))$ arises from an edge of $N(G)$ by decomposing G. Furthermore, every cyclic 3-edge cocycle of $G'(G'')$ is also a cyclic 3-edge cocycle of G, or arises from a cyclic 3-edge cocycle of G by decomposing G.*

Proof. Let v_L be the vertex of G' which is created when one decomposes G following L. Let $e \in N(G')$ and $L' \in \mathscr{L}(G')$, which contains e:

– If v_L is not incident with L', L' is clearly in $\mathscr{L}(G)$, and e is in $N(G)$.

– Assume that v_L is incident with some edge e_1 of L', which arises from the decomposition of G following L. If $e_1 = e$ then there is an edge of $N(G)$ which is transformed into e by the decomposition. Otherwise, v_L is not incident with e and it can be easily seen that e is in $N(G)$. □

For brievity, we shall preserve the names of the cyclic 3-edge cocycles in the decomposition. The following lemma is obvious.

Lemma 3.3. *Let G be a 3-connected cubic graph, G' and G'' its two sons in the decomposition following $L \in \mathscr{L}(G)$. Then $ww' \in E(G')$ comes from one of the following two situations:*

(i) *w and $w' \in V(G) \cap V(G')$ and $ww' \in E(G)$,*

(ii) *$w \in V(G)$, w' is the vertex v_L, w is one of the ends of one edge of L.*

One can verify (cf. [5]), on the other hand, that two cyclic 3-edge cocycles are totally disjoint or have exactly on edge in common.

Lemma 3.4. *Let G be a 3-connected cubic graph, L and $L' \in \mathscr{L}(G)$; then one and only one of the following conditions is true:*

(i) $L \cap L' = \emptyset$,

(ii) $|L \cap L'| = 1$,

(iii) $L = L'$.

We are now able to prove the main theorem of this section.

Theorem 3.5. *The decomposition of a 3-connected cubic graph G is unique.*

Proof. Let $\mathscr{F}$ and $\mathscr{H}$ be two decompositions of G:

$$\mathscr{F} = \{F_1, F_2, \ldots, F_1\},$$

$$\mathscr{H} = \{H_1, H_2, \ldots, H_k\}.$$

A vertex in one of the graphs of one decomposition is a vertex of G or a cyclic 3-edge cocycle of G. We, thus, have

$$V(\mathscr{H})=V(\mathscr{F})=\bigcup_{i=1}^{l} V(F_i)=\bigcup_{j=1}^{k} V(H_j)=V(G)\cup\mathscr{L}\{G\}.$$

Let w and w' be two vertices of F_i $(1\leqslant i\leqslant l)$. Suppose, by contradiction, that $w\in H_j$ and $w'\in H_{j'}(1\leqslant j<j'\leqslant k)$. Consider the binary tree representing the $\mathscr{H}$-decomposition of G. Let J be the first common ancestor of H_j and $H_{j'}$ when going from the leaves to the root G. Let J' and J'' the two sons of J in the H-decomposition (J' ancestor of H_j, J'' ancestor of $H_{j'}$). There is a cyclic 3-edge cocycle L in $\mathscr{L}(J)$ which separates J' and J''. From Lemma 3.2, we know that L comes from a cyclic 3-edge cocycle in $\mathscr{L}(G)$ and we have

$$\mathscr{L}(J)=\mathscr{L}(J')\cup\mathscr{L}(J'')\cup\{L\},$$

$$V(H_j)=V(J')\cup\mathscr{L}(J'),$$

$$V(H_{j'})=V(J'')\cup\mathscr{L}(J'').$$

There is exactly one vertex in H_j which has the same name as a vertex in $H_{j'}$; this is the vertex v_L itself. In J every chain linking w to w' (as elements of $V(J)\cup\mathscr{L}(J)$) goes through L and so does every chain linking w and w' (as elements of $V(G)\cup\mathscr{L}(G)$) in G. Every chain from w to w' must use the vertex v_L (w, w' and v_L considered as vertices of the F-decomposition). We, thus, have a contradiction, since v_L is now an articulation point in F_i. Hence, we have $H_j=H_{j'}$ and, thus, $V(H_j)=V(F_i)$.

Let us now show that ww' is an edge of $E(F_i)$ if and only if ww' is an edge of $E(H_j)$. Let ww' be an edge of F_i; Lemma 4.3, applied to every ancestor of F_i (from F_i to G), gives us one of the following three situations for ww':

(i) $ww'\in S(G)$,

(ii) $w\in V(G)$, $w'\in\mathscr{L}(G)$ and w is incident with one edge of w',

(iii) $w\in\mathscr{L}(G)$, $w'\in\mathscr{L}(G)$ and w and w' have exactly one edge in common.

Indeed, if ww' is an edge of G, it is certainly an edge of $S(G)$; otherwise, there exists $L\in\mathscr{L}(G)$ containing ww' and the two vertices w and w' would be separated by the decomposition following L. When ww' is not an edge of G, this means that at least one vertex (w or w') belongs to $\mathscr{L}(G)$. If $w\in\mathscr{L}(G)$ and $w'\in V(G)$ then, by Lemma 4.3, w' is the end of one edge of w. Finally if $w\in\mathscr{L}(G)$ and $w'\in\mathscr{L}(G)$, let $L=\{xy, x'y', x''y''\}$ and $L'=\{uv, u'v', u''v''\}$ such that $w=v_L$ and $w'=v_{L'}$, we have two cases:

Case 1: $L\cap L'=\emptyset$.

When we split a cyclic 3-edge cocycle, we create two new vertices with the same name (one in each son of the decomposition) without deleting any vertex; thus, the decomposition following L creates the vertex w whose neighborhood is contained in $\{x, x', x'', y, y', y''\}$, while the decomposition following L' creates the vertex w' whose neighborhood is contained in $\{u, u', u'', v, v', v''\}$. It is, thus, clear that w and w' cannot be adjacent.

Case 2: $L \cap L' \neq \emptyset$.

In this case (Lemma 3.4) L and L' have exactly one edge in common, thus, insuring the link ww'.

From (i)–(iii) above, it is easy to verify that w and w' are adjacent in any $\mathscr{H}$-decomposition.

F_i and H_j, thus, are the same graphs; hence, we have got the uniqueness of the decomposition. □

One consequence of the previous theorem is that the order of decomposition does not matter; in some problems it will be helpful to begin the decomposition of a cubic 3-connected graph following a convenient cyclic 3-edge cocycle.

The binary tree representing the decomposition has $|\mathscr{L}(G)|+1$ leaves (the number of graphs in $\mathscr{F}$). An easy lower bound for this number is as follows.

Proposition 3.6. *The decomposition of a cubic* 3*-connected graph* G *contains at least as many graphs as the number of connected components of* $S(G)$.

Proof. Any edge e in $S(G)$ appears in one graph of the decomposition, its two ends being vertices of this graph, thus, the whole connected component of $S(G)$ containing e is itself included in this graph by connectivity. The result follows. □

A 3-connected cubic graph is said to be *k-decomposable* if every graph of its decomposition has the same number k of vertices, and *K-decomposable* if they all are isomorphic to a given graph K. We are interested now in the following problem: Is there any property of G induced by the graphs of its decomposition? In this direction we have two results.

Theorem 3.7. *Let* G *and* H *be* k*-decomposable* 3*-connected cubic graphs. If the two decompositions have the same number of graphs then* G *and* H *have the same number of vertices.*

Proof (*By induction on the number of graphs in the decompositions of* G *and* H). Let $\mathscr{G}=\{G_1, G_2, \ldots, G_p\}$ be the decomposition of G, $\mathscr{H}=\{H_1, H_2, \ldots, H_p\}$ the decomposition of H and

$$|V(G_i)| = |V(H_i)| = k \quad \text{for all } 1 \leqslant i \leqslant p.$$

If $k=1$, G is reduced to G_1 and H to H_1; G and H, thus, have the same number of vertices. Let us suppose that the property is true for any pair of k-decomposable 3-connected cubic graphs whose decomposition contains $p-1$ graphs.

In G (H), let $L \in \mathscr{L}(G)$ ($M \in L(H)$) such that the decomposition of G following L (M) gives two sons G' and G'' (H' and H'') such that

$$G' \text{ or } G'' \in \mathscr{G} \qquad (H' \text{ or } H'' \in \mathscr{H}).$$

Since the order of the decomposition is meaningless, it is sufficient to find $L \in \mathscr{L}(G)$ $(M \in \mathscr{L}(H))$ such that a connected component of $G - L$ $(H - M)$ is a connected component of $S(G)$ $(S(H))$. In [5] we have shown that such a cyclic 3-edge cocycle exists (the connected components of $S(G)$ with this property are the so-called *s-ends*). G is, thus, decomposed into G' and G'' (H in H' and H''), with

$$|V(G')| = k \qquad (|V(G'')| = k),$$

$$|V(H')| = k \qquad (|V(H'')| = k).$$

Let us suppose that $|V(G')| = |V(H')| = k$, G'' and H'' are k-decomposable, and their decompositions having $p-1$ graphs; G'' and H'', thus, have the same number of vertices (from the induction hypothesis) and

$$|V(G)| = |V(G')| + |V(G'')| - 2 = |V(H)| = |V(H')| + |V(H'')| - 2. \qquad \square$$

Let us note here that, under the additional hypothesis that all graphs in the decompositions are isomorphic, we cannot insure the isomorphism between G and H, as it can be seen for the two graphs in Fig. 3.

Theorem 3.8. *Let G be a cubic 3-connected graph. If G is 4-decomposable then G is planar.*

Proof. In the decomposition of G, every graph is isomorphic to a complete graph with 4 vertices. Let us show our property by induction on the number of graphs in the

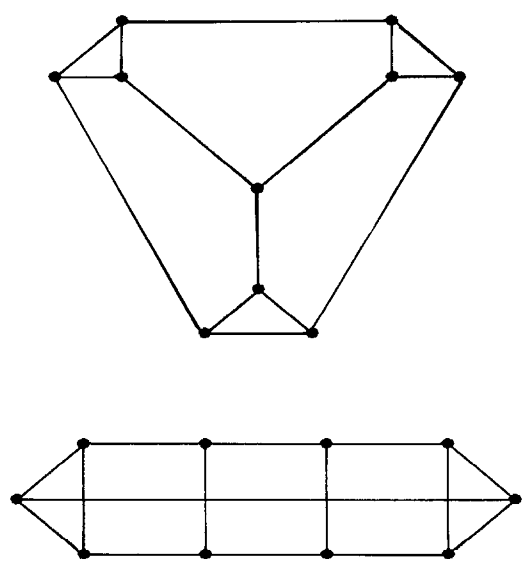

Fig. 3.

decomposition. The planarity of K_4 being well known, let us suppose that every graph 4-decomposable in $p-1$ graphs ($p \geq 2$) is planar. Let $L \in \mathscr{L}(G)$ such that one of the two sons G' and G'' of G, say G', is a K_4. G'' is a cubic 3-connected graph which is 4-decomposable in $p-1$ K_4. G'' is, thus, planar (from the induction hypothesis). One can obtain G from G'' while transforming v_L in a triangle (reconstruction of G from G' and G''); this operation clearly preserves the planarity. □

In fact, one can prove the more general theorem (the proof is left to the reader).

Theorem 3.9. *Let G be a cubic 3-connected graph, the orientable or non orientable genus is less than the sum of the genus of the graphs of the decomposition.*

4. *F*-Homeomorphism

Let H be a subgraph of a cubic 3-connected graph G, homeomorphic to an indecomposable graph. Then what is the behaviour of H under the decomposition? We shall study here this problem.

Let G be a cubic 3-connected graph and H a subgraph of G homeomorphic to a cubic graph K. A vertex of H of degree 3 will be called a *major* vertex (we shall consider a major vertex as a vertex of H or a vertex of G as well). A major vertex is linked in a natural way to three other major vertices (its neighbours a, b, c in K) by three chains P_{va}, P_{vb} and P_{vc} called here *v-major chains.*

Lemma 4.1. *Let G be a cubic 3-connected graph, H a subgraph homeomorphic to an indecomposable graph K, $e \in N(G) \cap E(H)$, $L \in \mathscr{L}(G)$ containing e. For any major vertex v, we have one of the following situations:*

(i) *L does not intersect any v-major chain,*
(ii) *L intersects exactly once every v-major chain,*
(iii) *L intersects twice one v-major chain only; the third edge of L is not in $E(H)$.*

Proof. Let C_1 and C_2 be the two 2-connected components of $G-L$. Let us colour C_1 in blue and C_2 in red. This colouring induces a 'natural' colouring of $V(K)$.

If K has vertices in the two colours, it means that $H-L$ is not connected and, thus, $K-L'$ (where L' is the set of edges of K, whose homeomorphic images in H are chains containing the edges of L) is not connected. L' is, thus, a 3-edge cocycle of K, which is indecomposable. L' is, thus, the elementary cocycle of one vertex of K. Hence, we have one of the cases (i) or (ii) for every vertex of H.

If the vertices of K are in the same class (say red), it means that either H is connected in $G-L$ ($E(H) \cap L = \emptyset$) and we, thus, have situation (i) for every vertex or H is disconnected. In that case, it is easy to see that we are in situation (iii) for two vertices v and v' of H linked by the chain. □

Proposition 4.2. *Let G be a cubic* 3-*connected graph and* $L \in \mathscr{L}(G)$, *and H a subgraph of G homeomorphic to an indecomposable graph K. One (and only one) of the two sons G′ and G″ of G in the decomposition following L contains a subgraph H′ homeomorphic to K whose set of major vertices is the set of major vertices of H except possibly* v_L, *which has been put in place of one major vertex of H.*

Proof. We have two cases:

– $L \cap E(H) = \emptyset$. H is, thus, a subgraph of G'. or (exclusively) G''. The property is obvious.

– $L \cap E(H) \neq \emptyset$. Let H' and H'' be the two graphs (subgraphs of G' and G'', respectively) which we have obtained from the decomposition of G following L. H being homeomorphic to an indecomposable graph, it has at least 4 major-vertices. From Lemmas 3.1 and 4.1, H' or H'' (say H'') is homeomorphic to a θ-graph or a cycle. If H'' is a cycle, H' is homeomorphic to K and has the same set of major vertices as H; while if H'' is a θ-graph, H' is homeomorphic to H and V_L is the only new major vertex possible. □

The unique subgraph H' homeomorphic to H so defined is said to be *$\mathscr{F}$-homeomorphic* to H. We shall say that two graphs H and H' are *$\mathscr{F}$-homeomorph* when it is possible to construct H from H' by a series of $\mathscr{F}$-homeomorphisms (along the path between the two nodes containing H and H' of the binary tree of the decomposition).

From this point, we obtain in an obvious way the main result in this section.

Theorem 4.3. *Let G be a cubic* 3-*connected graph, and H a subgraph homeomorphic to an indecomposable graph. If* $\mathscr{F}$ *is a decomposition of G then there is only one graph F in the decomposition which contains a subgraph H′* $\mathscr{F}$*-homeomorphic to H.*

This proposition allows us to prove the following enhanced version of a result due to Jackson [6].

Theorem 4.4. *If G is a cubic* 3-*connected graph, then G contains a subgraph H homeomorphic to one of the two indecomposable graphs on* 8 *vertices (the 'cube' and the 'twisted cube' of Fig.* 4) *if and only if G is not* $\{K_4, K_{3,3}\}$*-decomposable.*

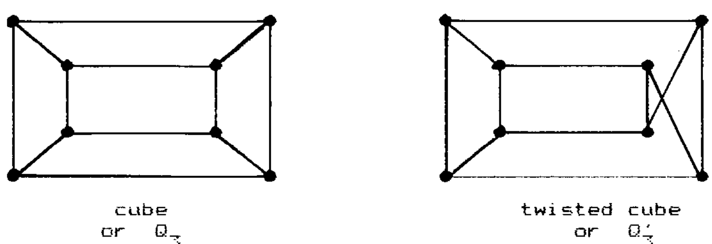

Fig. 4.

Proof. If G is $\{K_4, K_{3,3}\}$-decomposable, we cannot find a subgraph of G homeomorphic to an indecomposable graph of order at least 8. Indeed, from Theorem 4.3, one of the graphs in the decomposition must contain also such a subgraph, but these graphs have 4 or 6 vertices, which is impossible.

If G is not $\{K_4, K_{3,3}\}$-decomposable, let us show by induction on n, the number of vertices, that G contains a homeomorph to the cube or the twisted cube. For $n=8$, the property is easily verified on the four 3-connected cubic graphs of order 8. Let us suppose the property for graphs up to $n'<n$ vertices. Let us examine G with n vertices.

Case 1: *G is decomposable.*

One of the graph in the decomposition has at least 8 vertices (G is not $\{K_4, K_{3,3}\}$-decomposable); from the induction hypothesis, we clearly have the result.

Case 2: *G is indecomposable.*

Let e be an edge, $\varepsilon_e^{-1}(G)=G'$ is a cubic 3-connected graph on $n-2$ vertices. If G' is not $\{K_4, K_{3,3}\}$-decomposable, it contains a graph homeomorphic to the cube or the twisted cube (from the induction hypothesis), it can be easily verified that it is also a subgraph of G. If G' is $\{K_4, K_{3,3}\}$-decomposable, it means that G' is decomposable in exactly two graphs, each of them being a K_4 or a $K_{3,3}$. In each possible case, one can easily verify that G' contains a subgraph, as claimed, which is also a subgraph of G. $\square$

5. Diameter extremal graphs

What is the maximum diameter of a cubic 3-connected graph of order n? A related question (What is the minimum order of a cubic 3-connected graph of a given diameter d, a *(d, 3, 3)-graph* in the literature?) has been answered by various authors (see [3, 7, 10]). We shall give here an answer to this problem via the decomposition theory developed in the previous sections. As a by-product, we shall obtain a complete description of (d, 3, 3)-graphs.

$D_{\max}(n)$ denotes the maximum diameter of a 3-connected cubic graph of order n. We shall say that a graph in this family having a diameter equal to $D_{\max}(n)$ is an *extremal* graph. Two vertices x and y at maximum distance in an extremal graph constitute an *extremal pair* of vertices, a vertex is *extremal* when it belongs to an extremal pair. Let us recall that in a 3-connected graph any two vertices are joined by three internal disjoint paths (Menger's theorem), a *3-rail* in the sequel.

5.1. *Arithmetical properties of extremal graphs*

Lemma 5.1.

$$D_{\max}(n) \leqslant \left\lfloor \frac{n+1}{3} \right\rfloor$$

Proof. Let x and y be a pair of extremal vertices in an extremal graph G of order n, P_i $(1 \leqslant i \leqslant 3)$ the 3 paths of a 3-rail joining this two vertices. Since the diameter of G is $D_{\max}(n)$, we have

$$|V(P_i)| \geqslant D_{\max}(n) \quad \text{for all } 1 \leqslant i \leqslant 3.$$

From

$$\left| \bigcup_{i=1}^{3} V(P_i) \right| = 2 + \sum_{i=1}^{3} (|V(P_i)| - 2),$$

we have

$$n = |V(G)| \geqslant 2 + \sum_{i=1}^{3} (|V(P_i)| - 2).$$

Thus,

$$n \geqslant 2 + 3(D_{\max}(n) - 1)$$

and the result follows. □

We shall construct now, for each $n \geqslant 1$, a 3-connected cubic graph whose diameter is exactly $\lfloor (n+1)/3 \rfloor$. Inspection of the 3-connected cubic graphs on $n = 4, 6, 8$ vertices shows that this bound is attained for these 3 distinct values of n. In Fig. 5, we have drawn the extremal graphs of order at most 8 with a pair of extremal vertices (in white).

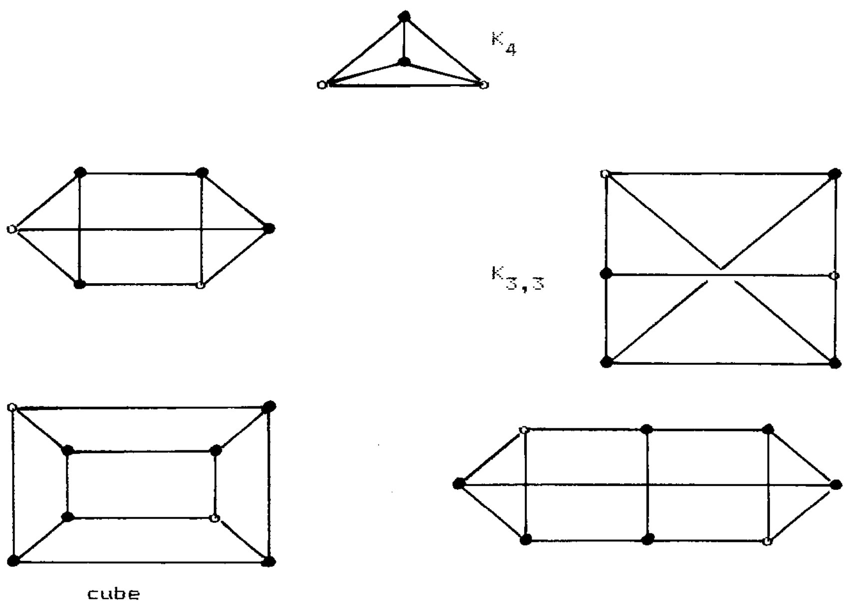

Fig. 5.

Let G be an extremal graph on n vertices, x and y an extremal pair joined by P_1, P_2 and P_3 a 3-rail such that the length of each P_i is $D_{\max}(n)$ (this is possible for $n=3p+2$).

Let e_i $(1 \leqslant i \leqslant 3)$ be the edge of P_i with end x. It is an easy matter to verify that the following 3 graphs, G_1, G_2 and G_3, on, respectively, $n+2$, $n+4$ and $n+6$ vertices are extremal graphs:

$G_1 = \varepsilon_{e_2 e_3}(G)$ (x and y is a pair of extremal vertices, e_2 and e_3 are transformed into e_2 and e'_3).

$G_2 = \varepsilon_{e_1 e'_3}(G)$ (x and y is a pair of extremal vertices, e_1 and e'_3 are transformed into e'_1 and e''_3).

$G_3 = \varepsilon_{e'_1 e'_2}(G)$ (x and y is a pair of extremal vertices, e'_1 and e'_2 are transformed into e''_1 and e''_2).

The sequence of extensions transforms the original 3-rail into a 3-rail for which the length of each path is, respectively,

$$D_{\max}(n), \quad D_{\max}(n)+1, \quad D_{\max}(n)+1 \quad \text{for } n \equiv 4 \text{ modulo } 6,$$

$$D_{\max}(n), \quad D_{\max}(n), \quad D_{\max}(n)+1 \quad \text{for } n \equiv 0 \text{ modulo } 6,$$

$$D_{\max}(n), \quad D_{\max}(n), \quad D_{\max}(n) \quad \text{for } n \equiv 2 \text{ modulo } 6.$$

This last relation ensures that we can repeat the construction (with G_3 instead of G). It can be pointed out that these extremal graphs are such that there is no vertices outside the 3-rails. In the general case we have, in fact, the following result.

Lemma 5.2. *Let G be an extremal graph on n vertices, and P_i $(1 \leqslant i \leqslant 3)$ the three paths of a 3-rail between x and y, the two extremal vertices. If k is the number of vertices which are not on one of the P_i $(1 \leqslant i \leqslant 3)$ then:*

(i) *$k=0$ if $n \equiv 2$ modulo 6,*

(ii) *$k=0$, 1 or 2 if $n \equiv 4$ modulo 6,*

(iii) *$k=0$ or 1 if $n=0$ modulo 6,*

Proof. We clearly have

$$n = k + |V(P_1)| + |V(P_2)| + |V(P_3)| - 4;$$

since $|V(P_i)|$ $(1 \leqslant i \leqslant 3)$ is at least $D_{\max}(n)+1$, we get

$$k \leqslant n - 3D_{\max}(n) + 1,$$

but $D_{\max}(n)$ is exactly $\lfloor(n+1)/3\rfloor$ (Lemma 5.1 and the above construction). Hence, we have

$$n=6p, \qquad D_{\max}(n)=2p \quad \text{and} \quad k\leqslant 1,$$

$$n=6p+2, \qquad D_{\max}(n)=2p+1 \quad \text{and} \quad k=0,$$

$$n=6p+4, \qquad D_{\max}(n)=2p+1 \quad \text{and} \quad k\leqslant 2. \qquad \square$$

The following lemma gives the possible lengths of the paths in a 3-rail between two extremal vertices of an extremal graph.

Lemma 5.3. *Let G be an extremal graph on n vertices, P_i $(1\leqslant i\leqslant 3)$ the three paths of a 3-rail between x and y, the two extremal vertices, and k the number of vertices which are not on the 3-rail. Then we have one of the following possibilities:*

(i) $n=6p$, $k=0$ *and* $D_{\max}(n)=2p$, $l(P_1)=l(P_2)=D_{\max}(n)$, $l(P_3)=D_{\max}(n)+1$,

(ii) $n=6p$, $k=1$ *and* $D_{\max}(n),=2p$, $l(P_1)=l(P_2)=l(P_3)=D_{\max}(n)$,

(iii) $n=6p+2$, $k=0$ *and* $D_{\max}(n)=2p+1$, $l(P_1)=l(P_2)=l(P_3)=D_{\max}(n)$,

(iv) $n=6p+4$, $k=0$ *and* $D_{\max}(n)=2p+1$, $l(P_1)=l(P_2)=D_{\max}(n)$, $l(P_3)=D_{\max}(n)+2$,

(v) $n=6p+4$, $k=0$ *and* $D_{\max}(n)=2p+1$, $l(P_1)=D_{\max}(n)$, $l(P_2)=l(P_3)=D_{\max}(n)+1$,

(vi) $n=6p+4$, $k=1$ *and* $D_{\max}(n)=2p+1$, $l(P_1)=l(P_2)=D_{\max}(n)$, $l(P_3)=D_{\max}(n)+1$,

(vii) $n=6p+4$, $k=2$ *and* $D_{\max}(n)=2p+1$, $l(P_1)=l(P_2)=l(P_3)=D_{\max}(n)$.

The proof is an easy arithmetical manipulation (use Lemma 5.2) and we have it to the reader.

5.2. *Cutting and gluing extremal graphs*

We are concerned now with the following problem: Under what condition(s) does the reconstruction (in the sense of Section 2) from two extremal graphs give an extremal graph?

Theorem 5.4. *Let G_1 and G_2 be two extremal graphs on n_1 and n_2 vertices, (x_i, v_i) $(i=1,2)$ an extremal pair in each of them. Let H be one of the six graphs obtained by deleting v_1* in *G_1 and v_2 in G_2 and adding a matching between the two neighbourhoods. Then H is extremal and (x_1, x_2) is an extremal pair in each of the following* cases:

(i) $n_1=6p_1$, $n_2=6p_2$,

(ii) $n_1=6p_1$, $n_2=6p_2+2$,

(iii) $n_1=6p_1+2$, $n_2=6p_2+2$,

(iv) $n_1=6p_1+2$, $n_2=6p_2+4$.

Proof. Let P be a path of minimum length between x_1 and x_2 in H, and let P_1 (P_2) be the trace of P in G_1 (G_2) between x_1 and v_1 (x_2 and v_2). We have

$$l(P) \geqslant l(P_1) + l(P_2) - 1.$$

Since G_i ($i = 1, 2$) is extremal, we certainly have (Lemma 5.1)

$$L(P_i) \geqslant \lfloor (n_i + 1)/3 \rfloor,$$

which gives the following:

Case (i): $n_1 = 6p_1$, $n_2 = 6p_2$.

In this case we have (cf. proof of Lemma 5.2)

$$D_{\max}(n_i) = 2p_i$$

and, thus,

$$l(P) \geqslant 2(p_1 + p_2) - 1$$

but

$$n = 6(p_1 + p_2) - 2 = 6(p_1 + p_2 - 1) + 4$$

and (cf. proof of Lemma 5.2)

$$D_{\max}(n) = 2(p_1 + p_2 - 1) + 1 = 2(p_1 + p_2) - 1.$$

The two vertices x_1 and x_2 are, thus, extremal in H.

Case (ii): $n_1 = 6p_1$, $n_2 = 6p_2 + 2$.

In this case we have

$$D_{\max}(n_1) = 2p_1 \quad \text{and} \quad D_{\max}(n_2) = 2p_2 + 1,$$

$$l(P) \geqslant 2(p_1 + p_2),$$

$$n = 6(p_1 + p_2),$$

$$D_{\max}(n) = 2(p_1 + p_2).$$

Case (iii): $n_1 = 6p_1 + 2$, $n_2 = 6p_2 + 2$.

In this case we have

$$D_{\max}(n_1) = 2p_1 + 1 \quad \text{and} \quad D_{\max}(n_2) = 2p_2 + 1,$$

$$l(P) \geqslant 2(p_1 + p_2) + 1,$$

$$n = 6(p_1 + p_2) + 2$$

$$D_{\max}(n) = 2(p_1 + p_2) + 1,$$

Case (iv): $n_1 = 6p_1 + 2$, $n_2 = 6p_2 + 4$.

In this case we have

$$D_{\max}(n_1) = 2p_1 + 1 \quad \text{and} \quad D_{\max}(n_2) = 2p_2 + 1,$$

$$l(P) \geqslant 2(p_1 + p_2) + 1,$$

$$n = 6(p_1 + p_2) + 4$$

$$D_{\max}(n) = 2(p_1 + p_2) + 1. \qquad \square$$

It can be pointed out that in the two other cases ($n_1 = 6p_1$ and $n_2 = 6p_2 + 4$; $n_1 = 6p_1 + 4$ and $n_2 = 6p_2 + 4$), we can obtain a graph H which is not extremal (see Fig. 6), but in these cases analogous calculus gives $D_{\max}(n) - 1$ as diameter for H.

Conversely, we cannot ensure that the decomposition of an extremal graph gives two extremal graphs. However, in some cases the decomposition leads to the expected result; we shall describe now one of them. We need for this purpose some definitions.

Let G be an extremal graph, x an extremal vertex, and V_i^x the set of vertices which are at distance i from x (or V_i when there is no possible confusion). From the 3-connectivity of G, each V_i^x has at least 3 vertices. When V_i^x has exactly 3 vertices, this set is a vertex separator of G. Since the vertex connectivity and the edge connectivity are equal in cubic graphs, it is not difficult to see that there is certainly a 3-edge cocycle which is incident to the 3 vertices of V_i^x. Let $L \in \mathscr{L}(G)$, we shall say that L is *natural* when L is a cyclic 3-edge cocycle whose ends are in V_i^x and V_{i+1}^x (for some i, $1 \leqslant i \leqslant D_{\max}(n) - 2$).

Theorem 5.5. *Let G be an extremal graph on n vertices, (x, y) an extremal pair and $L \in \mathscr{L}(G)$ such that x and y are in distinct component of $G - L$. If L is natural then the two sons of G, G' and G'', in the decomposition following L are extremal. Moreover, (v_L, x) and (v_L, y) are extremal pairs in these graphs.*

We omit the straightforward but tedious proof.

5.3. *Construction of extremal graphs*

We are concerned now with the construction of extremal graphs. We shall give a symbolic construction of the so-called *fundamental primitive graphs* (defined later),

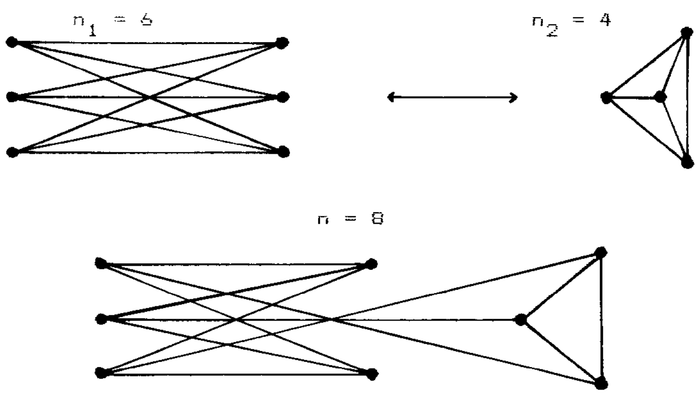

Fig. 6.

which can be seen as the generating family of extremal graphs. All the needed information is clearly contained in this description and it is an easy matter to construct an example of fundamental primitive graphs in each case. On the other hand, an extremal graph on $6p$ or $6p+2$ is merely a $(2p,3,3)$ minimum ($(2p+1,3,3)$ *minimum*) graph since, for $n<6p$, the diameter is at most $2p-1$ (at most $2p$ for $n<6p+1$). These classes of graphs having been depicted in the literature (see [3, 7, 10]) from another point of view, we refer the reader to these works.

An *n-primitive graph* in the sequel denotes an extremal graph on n vertices without natural 3-edge cocycle. In view of Theorems 5.4 and 5.5, it is clear that the construction of extremal graphs on n vertices will be achieved with the knowledge of the whole set of n'-primitive graphs, with $n' \leqslant n$. In fact, it is not necessary to know all the primitive graphs, as can be seen now.

For this purpose, we need to define a transformation of a cubic graph in another one. Let ab and cd be two independent edges in a cubic graph G; we get a new cubic graph G' from G by deleting the edges ab and cd and adding the new edges ac and bd. Let G be an extremal graph with (x, y) as extremal pair, let $V_i^x = \{a_1, a_2, a_3\}$ and $V_{i+1}^x = \{b_1, b_2, b_3\}$ and let us suppose that we have a_1b_1, a_2b_2, a_3b_3 the 3 edges of the 3-rail between x and y, a_1b_2 and a_2b_1 or a_1b_2 and a_2b_3 as only edges between V_i^x and V_{i+1}^x (Fig. 7). It is easy to see that the graph G' obtained by the above transformation on a_1b_2 and a_2b_1 in the former, on a_1b_2 and a_2b_3 in the latter one, remains extremal, since the two sets V_i^x and V_{i+1}^x are not changed. Moreover, we can see that V_i^x and V_{i+1}^x are joined by a natural 3-edge cocycle L in the transformed graph.

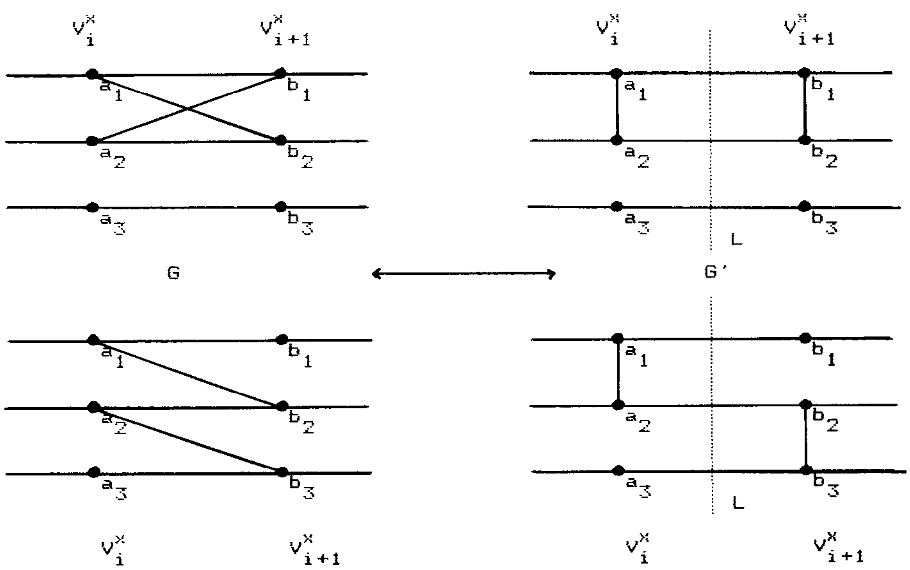

Fig. 7.

Proposition 5.6. *Let G be an n-primitive graph with (x, y) as extremal pair. Let us suppose that, for some i, $(2 \leqslant i \leqslant D_{\max}(n)-2)$. We have $|V_i^x| = |V_{i+1}^x| = |V_{i+2}^x| = 3$; then V_{i+1}^x is an independent set and exactly one of the following is true:*

(i) *V_i^x and V_{i+1}^x are joined by 4 edges and V_{i+1}^x and V_{i+2}^x are joined by 5 edges,*

(ii) *V_i^x and V_{i+1}^x are joined by 5 edges and V_{i+1}^x and V_{i+2}^x are joined by 4 edges.*

Proof. If V_{i+1}^x is not independent, then it contains exactly one edge (the 3-rail going through V_{i+1}^x uses the 3 edges between V_i^x and V_{i+1}^x, and the 3 edges between V_{i+1}^x and V_{i+2}^x). There is, thus, one edge more between V_i^x and V_{i+1}^x or between V_{i+1}^x and V_{i+2}^x. We, thus, get a natural 3-edge cocycle between V_{i+1}^x and V_{i+2}^x or V_i^x and V_{i+1}^x, respectively, a contradiction. □

In view of the previous transformation in the class of extremal graphs, this proposition means that the construction of n-primitive graphs is reduced to the construction of n-primitive graphs without 3 consecutive sets V_i^x, V_{i+1}^x and V_{i+2}^x having 3 vertices each (*fundamental primitive* graphs).

We shall give now a quick description of these fundamental primitive graphs. For this purpose (except for $n \leqslant 8$, for which all the fundamental graphs are depicted in Fig. 5), we give only an abstract partition of the vertex sets. For example, $x\, V_1\, V_2\, V_3^{\bullet}\, V_4\, \{y, z\}$, is the symbolic representation of a fundamental 16-primitive graph with (x, y) or (x, z) as extremal pairs $(d(x, y) = d(x, z) = 5)$; each V_i has 3 vertices except for V_3, which has 4 vertices. The other interpretations are straightforward.

Fundamental $6p+2$-primitive graphs

$x\, V_1\, V_2\, y, \quad n = 8.$

Fundamental $6p$-primitive graphs

$x\, V_1\, \{y, z\}, \quad n = 6,$

$x\, V_1\, V_2^{\bullet}\, V_3\, y, \quad n = 12,$

$x\, V_1\, V_2\, V_3^{\bullet}\, V_4\, V_5\, y, \quad n = 18.$

Fundamental $6p+4$-primitive graphs

$x\, \{y, z, t\}, \quad n = 4,$

$x\, V_1\, V_2\, \{y, z, t\}, \quad n = 10,$

$x\, V_1\, V_2^{\bullet}\, \{y, z\}, \quad n = 10,$

$x\, V_1\, V_2^{\bullet\bullet}\, y, \quad n = 10,$

$x\, V_1\, V_2^{\bullet}\, V_3\, V_4\, \{y, z\}, \quad n = 16,$

$$xV_1\,V_2\,V_3^{\bullet}\,V_4\{y,z\}, \qquad n=16,$$

$$xV_1\,V_2\,V_3^{\bullet\bullet}\,V_4\,y, \qquad n=16,$$

$$xV_1\,V_2^{\bullet\bullet}\,V_3\,V_4\,y, \qquad n=16,$$

$$xV_1\,V_2^{\bullet}\,V_3\,V_4\,V_5^{\bullet}\,V_6\,y, \qquad n=22,$$

$$xV_1\,V_2^{\bullet}\,V_3\,V_4^{\bullet}\,V_5\,V_6\,y, \qquad n=22,$$

$$xV_1\,V_2\,V_3^{\bullet}\,V_4^{\bullet}\,V_5\,V_6\,Y, \qquad n=22,$$

$$xV_1\,V_2\,V_3^{\bullet}\,V_4\,V_5\,V_6^{\bullet}\,V_7\,V_8\,y, \qquad n=28,$$

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