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Signed shape tilings of squares

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Abstract

Let T be a tile made up of finitely many rectangles whose corners have rational coordinates and whose sides are parallel to the coordinate axes. This paper gives necessary and sufficient conditions for a square to be tilable by finitely many $\mathbb{Q}$-weighted tiles with the same shape as T, and necessary and sufficient conditions for a square to be tilable by finitely many $\mathbb{Z}$-weighted tiles with the same shape as T. The main tool we use is a variant of F.W. Barnes's algebraic theory of brick packing, which converts tiling problems into problems in commutative algebra.

Keywords: Tile; Shape; Polynomial

1. Introduction

In [3] Dehn proved that an $a \times b$ rectangle R can be tiled by finitely many nonoverlapping squares if and only if a/b is rational. More generally, suppose we allow the squares to have weights from $\mathbb{Z}$. An arrangement of weighted squares is a tiling of R if the sum of the weights of the squares covering a region is 1 inside of R and 0 outside. Dehn's argument applies in this more general setting, and shows that R has a $\mathbb{Z}$-weighted tiling by squares if and only if a/b is rational. In [4] this result is generalized to give necessary and sufficient conditions for a rectangle R to be tilable by $\mathbb{Z}$-weighted rectangles with particular shapes. In this paper we consider a related question: Given a tile T in the plane made up of finitely many weighted rectangles, is there a weighted tiling of a square by tiles with the same shape as T?

We define a *rectangle* in $\mathbb{R} \times \mathbb{R}$ to be a product $[b_1, b_2) \times [c_1, c_2)$ of half-open intervals, with $b_1 < b_2$ and $c_1 < c_2$. Let A be a commutative ring with unity. An *A-weighted tile* is represented by a finite A-linear combination $L = a_1R_1 + \cdots + a_nR_n$ of disjoint rectangles. Associated to each such L there is a function $f_L : \mathbb{R}^2 \to A$ which is supported on $\bigcup R_i$ and whose value on R_i is a_i. We say that L_1 and L_2 represent the same

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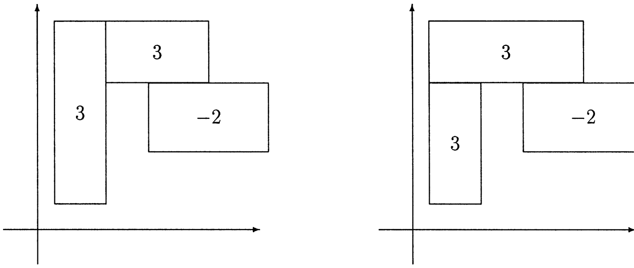

Fig. 1. Two rectangle decompositions of the same $\mathbb{Z}$-weighted tile.

tile if $f_{L_1} = f_{L_2}$. An example of a $\mathbb{Z}$-weighted tile is given in Fig. 1. We may form the sum $T_1 + T_2$ of two weighted tiles T_1, T_2 by superposing them in the natural way. For $a \in A$ the tile aT is formed from T by multiplying all the weights of T by a. The set of all A-weighted tiles forms an A-module under these operations.

Let U be an A-weighted tile and let $\{T_\lambda\colon \lambda \in \Lambda\}$ be a set of A-weighted tiles. We say that the set $\{T_\lambda\colon \lambda \in \Lambda\}$ *A-tiles* U if there are weights $a_1, \ldots, a_n \in A$ and tiles $\tilde{T}_1, \ldots, \tilde{T}_n$, each of which is a translation of some T_{λ_i}, such that $a_1\tilde{T}_1 + \cdots + a_n\tilde{T}_n = U$. Note that we are allowed to use as many translated copies of each prototile T_λ as we need, but we are not allowed to rotate or reflect the prototiles. Given an A-weighted tile T and a real number $\rho > 0$ we define $T(\rho)$ to be the image of T under the rescaling $(x, y) \mapsto (\rho x, \rho y)$. We say that an A-weighted tile T' has the same shape as T if there exists $\rho > 0$ such that T' is a translation of $T(\rho)$. We say that T *A-shapetiles* U if $\{T(\rho)\colon \rho > 0\}$ A-tiles U. If U' has the same shape as U then T A-shapetiles U' if and only if T A-shapetiles U.

In this paper we consider tiles T constructed from rectangles whose corners have rational coordinates. We prove two main results about such tiles. First, we show that if T is a $\mathbb{Q}$-weighted tile whose weighted area is not 0, then T $\mathbb{Q}$-shapetiles a square. Second, if T is a $\mathbb{Z}$-weighted tile we give necessary and sufficient conditions for T to $\mathbb{Z}$-shapetile a square.

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2. Polynomials and tiling

Say that T is a *lattice tile* if T is an A-weighted tile made up of unit squares in $\mathbb{R}^2$ whose corners are in $\mathbb{Z}^2$. We will associate a (generalized) polynomial f_T to each A-weighted lattice tile T. Our approach is similar to that used by Barnes [2], except that the polynomials that we construct differ from Barnes's polynomials by a factor $(X-1)(Y-1)$. Including this extra factor will allow us to generalize the construction to non-lattice tiles at the end of the section.

Our polynomials will be elements of the ring

$$A[X^{\mathbb{Z}}, Y^{\mathbb{Z}}] := A[X, Y, X^{-1}, Y^{-1}]$$

which is naturally isomorphic to the group ring of $\mathbb{Z} \times \mathbb{Z}$ with coefficients in A. To begin we associate the polynomial $X^i Y^j (X-1)(Y-1)$ to the unit square S_{ij} with lower left corner $(i,j) \in \mathbb{Z} \times \mathbb{Z}$. Given an A-weighted lattice tile

$$T = \sum_{i,j} w_{ij} S_{ij},$$

by linearity we associate to T the polynomial

$$f_T(X,Y) = \sum_{i,j} w_{ij} X^i Y^j (X-1)(Y-1).$$

One consequence of this definition is that translating a tile by a vector $(i,j) \in \mathbb{Z} \times \mathbb{Z}$ corresponds to multiplying its polynomial by $X^i Y^j$. The map $T \mapsto f_T$ gives an isomorphism between the A-module of A-weighted lattice tiles in the plane and the principal ideal in $A[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ generated by $(X-1)(Y-1)$.

Example 2.1. Let a, b, c, d be integers such that $a, b \geqslant 1$ and let T be the $a \times b$ rectangle whose lower left corner is at (c,d). Then the polynomial associated to T is

$$f_T(X,Y) = \sum_{i=c}^{c+a-1} \sum_{j=d}^{d+b-1} X^i Y^j (X-1)(Y-1)$$
$$= X^c Y^d (X^a - 1)(Y^b - 1).$$

In Section 4 we will need to work with non-lattice tiles. To represent these more general tiles systematically we introduce a new set of building blocks to play the role that the unit squares S_{ij} play in the theory of lattice tiles. For $\alpha, \beta \in \mathbb{R}^{\times}$ let $R_{\alpha\beta}$ denote the oriented rectangle with vertices $(0,0)$, $(\alpha,0)$, (α,β), $(0,\beta)$. Note that if exactly k of α, β are negative then $R_{\alpha\beta}$ is equal to $(-1)^k$ times a translation of $R_{|\alpha|,|\beta|}$. We can express any rectangle in terms of the rectangles $R_{\alpha\beta}$:

Example 2.2. Let $\alpha, \beta > 0$ and let $R'_{\alpha\beta}$ be the translation of the rectangle $R_{\alpha\beta}$ by the vector $(\sigma, \tau) \in \mathbb{R}^2$. Then $R'_{\alpha\beta} = R_{\alpha+\sigma,\beta+\tau} - R_{\alpha+\sigma,\tau} - R_{\sigma,\beta+\tau} + R_{\sigma\tau}$. In particular, we have $S_{ij} = R_{i+1,j+1} - R_{i+1,j} - R_{i,j+1} + R_{ij}$.

In fact the following holds:

Lemma 2.3. *Every A-weighted tile T can be expressed uniquely as an A-linear combination of rectangles $R_{\alpha\beta}$* with $\alpha, \beta \in \mathbb{R}^{\times}$.

Proof. By Example 2.2 every rectangle is an A-linear combination of the rectangles $R_{\alpha\beta}$. Therefore, every A-weighted tile is an A-linear combination of the $R_{\alpha\beta}$. Suppose

$$c_1 R_{\alpha_1\beta_1} + c_2 R_{\alpha_2\beta_2} + \cdots + c_n R_{\alpha_n\beta_n} = 0$$

is a linear relation such that the pairs (α_i, β_i) are distinct and $c_i \neq 0$ for $1 \leqslant i \leqslant n$. Choose j to maximize the distance from the origin to the far corner (α_j, β_j) of $R_{\alpha_j\beta_j}$. None of the other rectangles in the sum can overlap the region around (α_j, β_j). Since $c_j \neq 0$, this gives a contradiction. Therefore, the set $\{R_{\alpha\beta}\colon \alpha, \beta \in \mathbb{R}^\times\}$ is linearly independent over A, which implies the uniqueness part of the lemma. □

In order to represent arbitrary A-weighted tiles algebraically we introduce a generalization of the polynomials f_T. Let $A[X^{\mathbb{R}}, Y^{\mathbb{R}}]$ denote the set of 'polynomials' with coefficients from A where the exponents of X and Y are allowed to be arbitrary real numbers. The natural operations of addition and multiplication make $A[X^{\mathbb{R}}, Y^{\mathbb{R}}]$ a commutative ring with unity. The ring $A[X^{\mathbb{R}}, Y^{\mathbb{R}}]$ is naturally isomorphic to the group ring of $\mathbb{R} \times \mathbb{R}$ with coefficients in A, and contains $A[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ as a subring.

For $\alpha, \beta \in \mathbb{R}^\times$ define $f_{R_{\alpha\beta}} = (X^\alpha - 1)(Y^\beta - 1) \in A[X^{\mathbb{R}}, Y^{\mathbb{R}}]$. By Lemma 2.3 this definition extends linearly to give a well-defined element $f_T \in A[X^{\mathbb{R}}, Y^{\mathbb{R}}]$ associated to any A-weighted tile T. It follows from Example 2.2 that this definition agrees with that given earlier if $T = S_{ij}$ is a unit lattice square, and hence also if T is any lattice tile. The map $T \mapsto f_T$ gives an isomorphism between the A-module of A-weighted tiles and an A-submodule of $A[X^{\mathbb{R}}, Y^{\mathbb{R}}]$. The next lemma implies that this A-submodule is actually an ideal in $A[X^{\mathbb{R}}, Y^{\mathbb{R}}]$.

Lemma 2.4. *Let T be an A-weighted tile and let T' be the translation of T by the vector $(\sigma, \tau) \in \mathbb{R} \times \mathbb{R}$. Then $f_{T'} = X^\sigma Y^\tau f_T$.*

Proof. Let $R'_{\alpha\beta}$ be the translation of $R_{\alpha\beta}$ by (σ, τ). Using Example 2.2 we get

$$f_{R'_{\alpha\beta}} = X^\sigma Y^\tau (X^\alpha - 1)(Y^\beta - 1) = X^\sigma Y^\tau f_{R_{\alpha\beta}},$$

so the lemma holds for $T = R_{\alpha\beta}$. Therefore, by Lemma 2.3 the lemma holds for all tiles T. □

The next result gives a further relation between ideals and tiling.

Proposition 2.5. *Let U be a tile, let $\{T_\lambda\colon \lambda \in \Lambda\}$ be a collection of tiles, and let $\tilde{I} \subset A[X^{\mathbb{R}}, Y^{\mathbb{R}}]$ be the ideal generated by the set $\{f_{T_\lambda}\colon \lambda \in \Lambda\}$. Then $\{T_\lambda\colon \lambda \in \Lambda\}$ A-tiles U if and only if $f_U \in \tilde{I}$.*

Proof. We have $f_U \in \tilde{I}$ if and only if

$$f_U(X, Y) = \sum_{i=1}^{k} a_i X^{\sigma_i} Y^{\tau_i} f_{T_{\lambda_i}}(X, Y)$$

for some $a_i \in A$, $\sigma_i, \tau_i \in \mathbb{R}$, and $\lambda_i \in \Lambda$. Since $X^{\sigma_i} Y^{\tau_i} f_{T_{\lambda_i}}(X, Y)$ is the polynomial associated to the translation of T_{λ_i} by the vector (σ_i, τ_i), we have $f_U \in \tilde{I}$ if and only if $U = a_1 \tilde{T}_1 + \cdots + a_k \tilde{T}_k$, with $\tilde{T}_i$ a translation of T_{λ_i}. Therefore, $f_U \in \tilde{I}$ if and only if $\{T_\lambda\colon \lambda \in \Lambda\}$ A-tiles U. □

Corollary 2.6. *Let* $\{T_\lambda\colon \lambda \in \Lambda\}$ *be a collection of lattice tiles, let* I *be the ideal in* $A[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ *generated by the set* $\{f_{T_\lambda}\colon \lambda \in \Lambda\}$, *and let* U *be a lattice tile such that* $f_U \in I$. Then $\{T_\lambda\colon \lambda \in \Lambda\}$ *A-tiles* U.

The last result in this section shows what happens to f_T when we replace T by a rescaling.

Lemma 2.7. *Let* T *be an A-weighted tile and let* ρ *be a positive real number. Then* $f_{T(\rho)} = f_T(X^\rho, Y^\rho)$.

Proof. Let $\alpha, \beta \in \mathbb{R}^\times$. Then $R_{\alpha\beta}(\rho) = R_{\rho\alpha,\rho\beta}$ and hence

$$f_{R_{\alpha\beta}(\rho)} = (X^{\rho\alpha} - 1)(Y^{\rho\beta} - 1) = f_{R_{\alpha\beta}}(X^\rho, Y^\rho).$$

Therefore the lemma holds for $T = R_{\alpha\beta}$. It follows from Lemma 2.3 that the lemma holds for all tiles T. $\square$

3. Tiling with rational weights

This section is devoted to proving the following theorem:

Theorem 3.1. *Let* T *be a* $\mathbb{Q}$*-weighted tile made up of rectangles whose corners all have rational coordinates. Then* T $\mathbb{Q}$*-shapetiles a square if and only if the weighted area of* T *is not zero.*

Proof. It is clear that if the weighted area of T is zero then T cannot shapetile a square with nonzero area. Assume conversely that T has nonzero weighted area. By rescaling and translation we may assume that T is a lattice tile in the first quadrant. Let $T(\mathbb{N})$ denote the set $\{T(k)\colon k \in \mathbb{N}\}$ of positive integer rescalings of T. To complete the proof of Theorem 3.1 it suffices to prove that $T(\mathbb{N})$ $\mathbb{Q}$-tiles a square. First, we will prove that $T(\mathbb{N})$ $\mathbb{C}$-tiles a square; from this it will follow easily that $T(\mathbb{N})$ $\mathbb{Q}$-tiles a square.

Since T is a lattice tile in the first quadrant, $f_T \in \mathbb{Q}[X, Y]$ is a polynomial in the ordinary sense. We begin by interpreting the hypothesis that the weighted area of T is nonzero in terms of f_T.

Lemma 3.2. *There is a polynomial* $f_T^* \in \mathbb{Q}[X, Y]$ *such that*

$$f_T(X, Y) = (X - 1)(Y - 1) f_T^*(X, Y).$$

Moreover, the weighted area of T *is equal to* $f_T^*(1, 1)$, *and hence* $f_T^*(1, 1) \neq 0$.

Proof. Since the polynomial associated to the unit square S_{ij} is

$$f_{S_{ij}}(X, Y) = X^i Y^j (X - 1)(Y - 1),$$

the lemma holds for S_{ij}. It follows by linearity that the lemma holds for all lattice tiles in the first quadrant. $\square$

Let I denote the ideal in $\mathbb{C}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ generated by $\{f_{T(k)}\colon k \in \mathbb{N}\}$ and let

$$g_l(X,Y) = (X^l - 1)(Y^l - 1)$$

be the polynomial associated to an $l \times l$ square with lower left corner $(0,0)$. To show that $T(\mathbb{N})$ $\mathbb{C}$-tiles a square it suffices by Corollary 2.6 to show that $g_l \in I$ for some positive integer l. In order to get information about I we consider the set $V(I) \subset \mathbb{C}^\times \times \mathbb{C}^\times$ of common zeros of the elements of I. The set $V(I)$ is essentially the union of the lines $X=1$ and $Y=1$ with the 'shape variety' of $T(\mathbb{N})$ as defined by Barnes [2, Section 3].

We wish to determine which points $(\alpha, \beta) \in \mathbb{C}^\times \times \mathbb{C}^\times$ might be in $V(I)$. Let m be the X-degree of f_T, let n be the Y-degree of f_T, and define $\Upsilon \subset \mathbb{C}^\times$ by

$$\Upsilon = \{\zeta \in \mathbb{C}^\times\colon \zeta^k = 1 \text{ for some } 1 \leqslant k \leqslant 2mn\}.$$

Lemma 3.3. $V(I) \subset (\mathbb{C}^\times \times \Upsilon) \cup (\Upsilon \times \mathbb{C}^\times)$.

Proof. Let $(\alpha, \beta) \in V(I)$, and suppose neither α nor β is in Υ. By Lemmas 2.7 and 3.2 we have

$$0 = f_{T(k)}(\alpha, \beta) = f_T(\alpha^k, \beta^k) = (\alpha^k - 1)(\beta^k - 1) f_T^*(\alpha^k, \beta^k)$$

for all $k \geqslant 1$. Since α and β are not in Υ this implies $f_T^*(\alpha^k, \beta^k) = 0$ for $1 \leqslant k \leqslant 2mn$. Therefore, by Lemma 3.4 below there exist $c, d \in \mathbb{Z}$ such that $f_T^*(X^c, X^d) = 0$. It follows that $f_T^*(1,1) = 0$, contrary to Lemma 3.2. We conclude that if $(\alpha, \beta) \in V(I)$ then at least one of α, β must be in Υ. $\square$

Lemma 3.4. *Let K be a field and let $f^* \in K[X,Y]$ be a nonzero polynomial with X-degree $m-1$ and Y-degree $n-1$. Assume there are $\alpha, \beta \in K^\times$ such that*

1. *α and β are not kth roots of 1 for any $1 \leqslant k \leqslant 2mn$, and*
2. *$f^*(\alpha^k, \beta^k) = 0$ for all $1 \leqslant k \leqslant 2mn$.*

Then there exist relatively prime integers c, d with $1 \leqslant c \leqslant n-1$ and $1 \leqslant |d| \leqslant m-1$ such that $f^*(X^c, X^d) = 0$.

Proof. Define an $mn \times mn$ matrix M whose columns are indexed by pairs (i,j) with $0 \leqslant i \leqslant m-1$ and $0 \leqslant j \leqslant n-1$ by letting the kth entry in the (i,j) column of M be $\alpha^{ik}\beta^{jk}$. Since $f^*(\alpha^k, \beta^k) = 0$ for $1 \leqslant k \leqslant mn$, the coefficients of f^* give a nontrivial element of the nullspace of M. Since M is essentially a Vandermonde matrix this implies

$$0 = \det(M) = \alpha^{nm(m-1)/2} \beta^{mn(n-1)/2} \cdot \prod_{(i,j)<(i',j')} (\alpha^{i'}\beta^{j'} - \alpha^i \beta^j)$$

for an appropriate ordering of the pairs (i,j). It follows that $\alpha^{i'}\beta^{j'}=\alpha^i\beta^j$ for some $(i',j')\neq(i,j)$, so $\alpha^{d_0}=\beta^{c_0}$ for some $(c_0,d_0)\neq(0,0)$ with $|c_0|\leqslant n-1$ and $|d_0|\leqslant m-1$. The first assumption implies that $c_0\neq 0$ and $d_0\neq 0$, so we may assume without loss of generality that $c_0\geqslant 1$.

Let $e=\gcd(c_0,d_0)$ and set $c=c_0/e$ and $d=d_0/e$. Then since $(\alpha^e)^d=(\beta^e)^c$ with $\gcd(c,d)=1$ there is a unique $\gamma\in K$ such that $\gamma^c=\alpha^e$ and $\gamma^d=\beta^e$. Let q be an integer such that $1\leqslant q\leqslant 2mn/e$. Then by the second assumption we have

$$0=f^*(\alpha^{eq},\beta^{eq})=f^*(\gamma^{cq},\gamma^{dq}),$$

and so $f^*(X^c,X^d)\in K[X,X^{-1}]$ has zeros at $X=\gamma^q$ for $1\leqslant q\leqslant 2mn/e$. If these zeros are not distinct then for some $1\leqslant r\leqslant 2mn/e$ we have $\gamma^r=1$ and hence $1=\gamma^{cr}=\alpha^{er}$, which violates the first assumption. Therefore $f^*(X^c,X^d)$ has at least $\lfloor 2mn/e\rfloor$ distinct zeros. On the other hand, the degree of the rational function $f^*(X^c,X^d)$ is at most $(m-1)|c|+(n-1)|d|$, and since $|c|=|c_0/e|\leqslant(n-1)/e$ and $|d|=|d_0/e|\leqslant(m-1)/e$ we have

$$(m-1)|c|+(n-1)|d|\leqslant 2(m-1)(n-1)/e<\lfloor 2mn/e\rfloor.$$

Therefore $f^*(X^c,X^d)=0$. □

Let $l\geqslant 1$ and recall that $g_l(X,Y)=(X^l-1)(Y^l-1)$ is the polynomial associated to an $l\times l$ square with lower left corner $(0,0)$. The set $V(g_l)\subset\mathbb{C}^\times\times\mathbb{C}^\times$ of zeros of g_l is the union of the lines $X=\zeta$ and $Y=\zeta$ as ζ ranges over the lth roots of 1. It follows from Lemma 3.3 that if we choose l appropriately (say $l=(2mn)!$) then $V(g_l)\supset V(I)$. This need not imply that g_l is in I, but by Hilbert's Nullstellensatz [5, VII, Theorem 14] we do have $g_l^k\in I$ for some $k\geqslant 1$.

To show there exists l such that $g_l\in I$ we use the theory of *primary decompositions* (see, e.g., Chapters 4 and 7 of Atiyah and Macdonald [1]). Let A be a commutative ring with 1. We say that the ideal $Q\subset A$ is a *primary ideal* if whenever $xy\in Q$ with $x\notin Q$ there exists $a\geqslant 1$ such that $y^a\in Q$. By the Hilbert basis theorem, $\mathbb{C}[X^{\mathbb{Z}},Y^{\mathbb{Z}}]$ is a Noetherian ring [1, Corollary 7.7]. Therefore, there are primary ideals $Q_1,\dots,Q_r$ in $\mathbb{C}[X^{\mathbb{Z}},Y^{\mathbb{Z}}]$ such that $I=Q_1\cap\cdots\cap Q_r$ [1, Theorem 7.13]. The radical ideal

$$P_i=\sqrt{Q_i}=\{f\in\mathbb{C}[X^{\mathbb{Z}},Y^{\mathbb{Z}}]\colon f^r\in Q_i \text{ for some } r\geqslant 1\}$$

of the primary ideal Q_i is automatically prime, and is called the prime *associated* to Q_i. We may also characterize P_i as the smallest prime ideal containing Q_i.

Since $I=Q_1\cap\cdots\cap Q_r$ we need to show that there exists $l\geqslant 1$ such that $g_l\in Q_i$ for all $1\leqslant i\leqslant r$. Observe that if $l\mid l'$ then $g_l\mid g_{l'}$. Therefore, it is enough to show that for each i there is l_i such that $g_{l_i}\in Q_i$, since in that case we have $g_l\in I$ with $l=1\text{cm}\{l_1,\dots,l_r\}$. To accomplish this we first restrict the possibilities for the prime ideals P_i.

Let $q=(2mn)!$. We observed above that $g_q^k\in I$ for some positive integer k. Since $P_i\supset Q_i\supset I$ this implies that $g_q^k\in P_i$. Therefore, some irreducible factor of

$$g_q(X,Y)^k=\prod_{\zeta^q=1}(X-\zeta)^k(Y-\zeta)^k$$

lies in the prime ideal P_i. It follows that $X - \zeta \in P_i$ or $Y - \zeta \in P_i$ for some $\zeta \in \mathbb{C}^\times$ such that $\zeta^q = 1$.

Assume without loss of generality that $X - \zeta \in P_i$. Then P_i contains the prime ideal $(X - \zeta)$ generated by the irreducible polynomial $X - \zeta$. If $P_i \neq (X - \zeta)$ let h be an element of P_i which is not in $(X - \zeta)$. By dividing $X - \zeta$ into $h(X, Y)$ we see that $h(\zeta, Y) \in P_i$. Since P_i is prime and $\mathbb{C}$ is algebraically closed this implies that some linear factor $Y - \alpha$ of $h(\zeta, Y)$ is in P_i. Therefore, P_i contains the maximal ideal $(X - \zeta, Y - \alpha)$, so in fact $P_i = (X - \zeta, Y - \alpha)$. Moreover, we must have $\alpha \neq 0$ since Y is a unit in $\mathbb{C}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$. It follows that if $X - \zeta \in P_i$ then either $P_i = (X - \zeta)$ or $P_i = (X - \zeta, Y - \alpha)$ for some $\alpha \in \mathbb{C}^\times$.

We will make repeated use of the following elementary fact about primary ideals.

Lemma 3.5. *Let Q be a primary ideal and set $P = \sqrt{Q}$. If $gh \in Q$ with $h \notin P$ then $g \in Q$.*

Proof. Since $h \notin P$ we have $h^a \notin Q$ for all $a \geqslant 1$. Therefore, by the definition of primary ideal we have $g \in Q$. □

Assume now that $P_i = (X - \zeta)$ with $\zeta^q = 1$. Then $X^q - 1$ has a simple zero at $X = \zeta$. Therefore by Lemmas 2.7 and 3.2 we have

$$\begin{aligned} f_{T(q)}(X, Y) &= f_T(X^q, Y^q) \\ &= (X^q - 1)(Y^q - 1) f_T^*(X^q, Y^q) \\ &= (X - \zeta) h(X, Y) \end{aligned}$$

for some $h \in \mathbb{C}[X, Y]$. Moreover we have $h(\zeta, Y) \neq 0$, since otherwise $0 = f_T^*(\zeta^q, Y^q) = f_T^*(1, Y^q)$, which would imply $f_T^*(1, 1) = 0$, contrary to Lemma 3.2. Therefore $h \notin P_i = (X - \zeta)$. It follows by Lemma 3.5 that $X - \zeta \in Q_i$, and hence that $g_q \in Q_i$.

Now assume $P_i = (X - \zeta, Y - \alpha)$. If α is an rth root of 1 for some $r \geqslant 1$ then $X^{qr} - 1$ has a simple zero at $X = \zeta$ and $Y^{qr} - 1$ has a simple zero at $Y = \alpha$. As in the previous case this implies

$$\begin{aligned} f_{T(qr)}(X, Y) &= (X^{qr} - 1)(Y^{qr} - 1) f_T^*(X^{qr}, Y^{qr}) \\ &= (X - \zeta)(Y - \alpha) h(X, Y) \end{aligned}$$

for some $h \in \mathbb{C}[X, Y]$. Since $f_T^*(\zeta^{qr}, \alpha^{qr}) = f_T^*(1, 1) \neq 0$, we have $h(\zeta, \alpha) \neq 0$, and hence $h \notin P_i$. Applying Lemma 3.5 we get $(X - \zeta)(Y - \alpha) \in Q_i$, and hence $g_{qr} \in Q_i$. If α is not a root of 1 we may choose $r \geqslant 1$ so that $f_T^*(\zeta^{qr}, \alpha^{qr}) = f_T^*(1, \alpha^{qr}) \neq 0$, since $f_T^*(1, 1) \neq 0$ implies that $f_T^*(1, Y)$ has only finitely many zeros. Then $X^{qr} - 1$ has a simple zero at $X = \zeta$ and $Y^{qr} - 1$ is nonzero at $Y = \alpha$. By an argument similar to those used above we have $f_{T(qr)}(X, Y) = (X - \zeta) h(X, Y)$ for some $h \in \mathbb{C}[X, Y]$ such that $h(\zeta, \alpha) \neq 0$. This implies $h \notin P_i$, so by Lemma 3.5 we get $X - \zeta \in Q_i$, and hence $g_q \in Q_i$.

We have shown now that for each $1 \leqslant i \leqslant r$ there is $l_i \geqslant 1$ such that $g_{l_i} \in Q_i$. Therefore we have $g_l \in I$ with $l = \text{lcm}\{l_1, \ldots, l_r\}$. It follows from Corollary 2.6 that $T(\mathbb{N})$ $\mathbb{C}$-tiles an $l \times l$ square. To prove that $T(\mathbb{N})$ $\mathbb{Q}$-tiles a square it is sufficient to prove that g_l is in the ideal I_0 in $\mathbb{Q}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ generated by $T(\mathbb{N})$. Equivalently, we need to show that g_l is in the $\mathbb{Q}$-span of the set

$$\mathscr{E} = \{X^i Y^j f_{T(k)}: i, j, k \in \mathbb{Z}, k \geqslant 1\}.$$

We have shown that g_l is in the $\mathbb{C}$-span of $\mathscr{E}$. Since g_l and the elements of $\mathscr{E}$ are all in $\mathbb{Q}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$, and

$$\mathbb{C}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}] \cong \mathbb{Q}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}] \otimes_{\mathbb{Q}} \mathbb{C},$$

it follows immediately that g_l is in the $\mathbb{Q}$-span of $\mathscr{E}$. This completes the proof of Theorem 3.1. □

Corollary 3.6. *Let T be a $\mathbb{Z}$-weighted tile made up of rectangles whose corners all have rational coordinates. Assume that the weighted area of T is not zero. Then there exists a positive integer w such that $T(\mathbb{N})$ $\mathbb{Z}$-tiles a square with weight w.*

Proof. By Theorem 3.1 we know that $T(\mathbb{N})$ $\mathbb{Q}$-tiles a square R, so there are rational numbers $a_1, \ldots, a_n$ and tiles $T_1, \ldots, T_n$, each a translation of some $T(k_i) \in T(\mathbb{N})$, such that $R = a_1 T_1 + \cdots + a_n T_n$. Let $w \geqslant 1$ be a common denominator for $a_1, \ldots, a_n$. Then $wR = wa_1 T_1 + \cdots + wa_n T_n$, and $wa_i \in \mathbb{Z}$ for $1 \leqslant i \leqslant n$. Therefore $T(\mathbb{N})$ $\mathbb{Z}$-tiles wR. □

4. Tiling with integer weights

Let T be a $\mathbb{Z}$-weighted lattice tile, and assume that the weighted area of T is not zero. By Corollary 3.6 we know that T $\mathbb{Z}$-shapetiles a square with weight w for some positive integer w. We wish to find necessary and sufficient conditions for T to $\mathbb{Z}$-shapetile a square with weight 1. To express these conditions we need a definition. Given $\mu \in \mathbb{Q} \cup \{\infty\}$ we say that two lattice squares S_{ij} and $S_{i'j'}$ belong to the same *μ-slope class* if the line joining their centers has slope μ. The tile T can be decomposed into a sum $T = C_1 + \cdots + C_k$ of lattice tiles such that for each i the unit lattice squares which make up C_i all belong to the same μ-slope class.

Proposition 4.1. *Let T be a $\mathbb{Z}$-weighted lattice tile and let n be a positive integer. Let c and d be relatively prime integers and set $\mu = -c/d$. Then the μ-slope classes of T all have weighted area divisible by n if and only if f_T is an element of the ideal $((X^d - Y^c)(X-1)(Y-1), n(X-1)(Y-1))$ in $\mathbb{Z}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$.*

Proof. The μ-slope classes of T all have weighted area divisible by n if and only if we can write $T = T_1 + nT_2$, where T_1 and T_2 are $\mathbb{Z}$-weighted lattice tiles such that the μ-slope classes of T_1 all have weighted area zero. Write the decomposition of T_1 into its μ-slope classes as $T_1 = C_1 + \cdots + C_k$. Since $\mu = -c/d$ with c and d relatively prime, the lattice squares S_{ij} and $S_{i'j'}$ are in the same μ-slope class if and only if $S_{i'j'}$ is the translation of S_{ij} by $(dr, -cr)$ for some $r \in \mathbb{Z}$. Therefore, if C_t is the μ-slope class of T_1 containing S_{ij} we have

$$f_{C_t}(X, Y) = g(X^d Y^{-c}) X^i Y^j (X - 1)(Y - 1)$$

for some $g \in \mathbb{Z}[X^{\mathbb{Z}}]$. Since the weighted area of C_t is zero we see that $0 = f^*_{C_t}(1, 1) = g(1)$, which implies $X - 1 \mid g(X)$. It follows that $(X^d Y^{-c} - 1)(X - 1)(Y - 1)$ divides f_{C_t} for $1 \leqslant t \leqslant k$, and hence also that $(X^d Y^{-c} - 1)(X - 1)(Y - 1)$ divides f_{T_1}. Conversely, if $(X^d Y^{-c} - 1)(X - 1)(Y - 1)$ divides f_{T_1}, it is easy to check that the μ-slope classes of T_1 all have weighted area zero. It follows that the μ-slope classes of T all have area divisible by n if and only if we can write

$$f_T(X, Y) = (X^d Y^{-c} - 1)(X - 1)(Y - 1)h_1(X, Y) + n(X - 1)(Y - 1)h_2(X, Y)$$

for some $h_1, h_2 \in \mathbb{Z}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$. Since Y^c is a unit in $\mathbb{Z}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ this is equivalent to $f_T \in ((X^d - Y^c)(X - 1)(Y - 1), n(X - 1)(Y - 1))$. $\square$

Theorem 4.2. *Let T be a $\mathbb{Z}$-weighted lattice tile. Then T $\mathbb{Z}$-shapetiles a square if and only if the following two conditions hold*:

1. *The weighted area of T is not zero.*
2. *For every $\mu \in \mathbb{Q}^{\times}$ the gcd of the weighted areas of the μ-slope classes of T is* 1.

Proof. Let T be a tile which satisfies conditions 1 and 2. To show that T $\mathbb{Z}$-shapetiles a square it is sufficient by Corollary 3.6 to show that $T(\mathbb{N}) \cup \{wR\}$ $\mathbb{Z}$-tiles a square, where R is an $l \times l$ square and l, w are positive integers. Let $S = S_{00}$ be the unit lattice square with lower left corner $(0, 0)$. If $T(\mathbb{N}) \cup \{wS\}$ $\mathbb{Z}$-tiles an $a \times a$ square then by rescaling we see that $T(\mathbb{N}) \cup \{wR\}$ $\mathbb{Z}$-tiles an $la \times la$ square. Therefore, it is sufficient to show that $T(\mathbb{N}) \cup \{wS\}$ $\mathbb{Z}$-tiles a square. Let J be the ideal in $\mathbb{Z}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ generated by $\{f_{T(k)}\colon k \in \mathbb{N}\} \cup \{w(X - 1)(Y - 1)\}$. By Corollary 2.6 it is sufficient to show that $g_l \in J$ for some $l \geqslant 1$.

By the Hilbert basis theorem $\mathbb{Z}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ is a Noetherian ring. Therefore the ideal J has a primary decomposition $J = Q_1 \cap \cdots \cap Q_t$. We need to show that there exists $l \geqslant 1$ such that $g_l \in Q_i$ for all i. As in the proof of Theorem 3.1 it is enough to show that for each i there is $l_i \geqslant 1$ such that $g_{l_i} \in Q_i$. Let $P_i = \sqrt{Q_i}$ be the prime associated to Q_i, and suppose $w \notin P_i$. Then since $w(X - 1)(Y - 1) \in Q_i$, by Lemma 3.5 we see that $(X - 1)(Y - 1) = g_1$ is in Q_i. If $w \in P_i$ then since P_i is a prime ideal it follows that P_i contains a prime integer p which divides w, and hence that $P_i \cap \mathbb{Z} = p\mathbb{Z}$.

For $f \in \mathbb{Z}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ let $\bar{f} \in \mathbb{F}_p[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ be the reduction of f modulo p, where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is the field with p elements. Let $\bar{P}_i$ be the ideal in $\mathbb{F}_p[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ consisting of the reductions modulo p of the elements of P_i. Since $p \in P_i$ the ideal $\bar{P}_i$ is prime. Let $\bar{J} \subset \mathbb{F}_p[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ be the ideal consisting of the reductions modulo p of the elements of J. Then $\bar{J}$ is generated by $\{\bar{f}_{T(k)}\colon k \geqslant 1\}$. Since $P_i \supset J$, we have $\bar{P}_i \supset \bar{J}$.

Let K be an algebraic closure of $\mathbb{F}_p$ and let $V(\bar{J}) \subset K^{\times} \times K^{\times}$ be the set of common zeros of the elements of $\bar{J}$. Let m be the X-degree of $\bar{f}_T$, let n be the Y-degree of $\bar{f}_T$, and define $\bar{\Upsilon} \subset K^\times \times K^\times$ by

$$\bar{\Upsilon} = \{\zeta \in K^\times : \zeta^k = 1 \text{ for some } 1 \leqslant k \leqslant 2mn\}.$$

Lemma 4.3. $V(\bar{J}) \subset (K^\times \times \bar{\Upsilon}) \cup (\bar{\Upsilon} \times K^\times)$.

Proof. Let $(\alpha, \beta) \in V(\bar{J})$ and suppose neither α nor β is in $\bar{\Upsilon}$. Then for $1 \leqslant k \leqslant 2mn$ we have

$$0 = \bar{f}_{T(k)}(\alpha, \beta) = \bar{f}_T(\alpha^k, \beta^k) = (\alpha^k - 1)(\beta^k - 1)\bar{f}_T^*(\alpha^k, \beta^k).$$

Since α and β are not in $\bar{\Upsilon}$ this implies that $\bar{f}_T^*(\alpha^k, \beta^k) = 0$ for $1 \leqslant k \leqslant 2mn$. Therefore, by Lemma 3.4 there are relatively prime integers c, d with $c \geqslant 1$ and $d \neq 0$ such that $\bar{f}_T^*(X^c, X^d) = 0$. Let $\mathscr{A}$ be the quotient ring $\mathbb{F}_p[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]/(X^d - Y^c)$, and let x, y denote the images of X, Y in $\mathscr{A}$. Then x and y are units in $\mathscr{A}$ satisfying $x^d = y^c$ with $\gcd(c, d) = 1$, so there is $z = x^a y^b$ in $\mathscr{A}^\times$ such that $x = z^c$ and $y = z^d$. Therefore, the image of $\bar{f}_T^*$ in $\mathscr{A}$ is given by $\bar{f}_T^*(x, y) = \bar{f}_T^*(z^c, z^d)$, which equals zero since $\bar{f}_T^*(X^c, X^d) = 0$. It follows that $X^d - Y^c$ divides $\bar{f}_T^*$, and hence that f_T^* is in the ideal $(X^d - Y^c, p)$ in $\mathbb{Z}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$. Therefore, $f_T = (X-1)(Y-1)f_T^*$ is in the ideal

$$((X^d - Y^c)(X-1)(Y-1), p(X-1)(Y-1))$$

in $\mathbb{Z}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$. Proposition 4.1 now implies that every μ-slope class of T has area divisible by p. This violates condition 2 of the theorem, so we have a contradiction. □

Set $q = (2mn)!$ and let $V(\bar{g}_q) \subset K^\times \times K^\times$ be the set of zeros of $\bar{g}_q$. Since $X^q - 1$ has zeros at all elements of $\bar{\Upsilon}$, we have $V(\bar{g}_q) \supset (K^\times \times \bar{\Upsilon}) \cup (\bar{\Upsilon} \times K^\times)$. Therefore Lemma 4.3 implies $V(\bar{g}_q) \supset V(\bar{J})$. Since $\bar{P}_i \supset \bar{J}$ we have $V(\bar{J}) \supset V(\bar{P}_i)$, and hence $V(\bar{g}_q) \supset V(\bar{P}_i)$. As in Section 3 Hilbert's Nullstellensatz implies that $\bar{g}_q^k \in \bar{P}_i$ for some $k \geqslant 1$. Since $\bar{P}_i$ is prime and

$$\bar{g}_q(X, Y)^k = (X^q - 1)^k (Y^q - 1)^k$$

we have either $X^q - 1 \in \bar{P}_i$ or $Y^q - 1 \in \bar{P}_i$. It follows that P_i contains one of the ideals $(X^q - 1, p)$ or $(Y^q - 1, p)$. We may assume without loss of generality that $P_i \supset (X^q - 1, p)$.

By [1, Proposition 7.14] we have $Q_i \supset P_i^u$ for some $u \geqslant 1$. Therefore, it is enough to prove that for every $u \geqslant 1$ there is $l \geqslant 1$ such that $g_l \in P_i^u$. Let t be a positive integer. Expanding $X^{qt} - 1$ in powers of $X^q - 1$ gives

$$\begin{aligned} X^{qt} - 1 &= -1 + ((X^q - 1) + 1)^t \\ &= \sum_{j=1}^{t} \binom{t}{j} (X^q - 1)^j. \end{aligned}$$

If we choose t to be divisible by a large power of p then for small values of $j \geqslant 1$ the binomial coefficient $\binom{t}{j}$ is divisible by a large power of p. Thus, every term in this expansion is divisible either by a large power of p or a large power of $X^q - 1$. It follows that there exists $t \geqslant 1$ such that $X^{qt} - 1 \in (X^q - 1, p)^u$. Since $P_i^u \supset (X^q - 1, p)^u$ we get $g_{qt} \in P_i^u$, as required.

Assume conversely that T $\mathbb{Z}$-shapetiles a square. Then the weighted area of T is clearly not equal to zero, so condition 1 of Theorem 4.2 is satisfied. We need to show that for every $\mu \in \mathbb{Q}^\times$ the gcd of the weighted areas of the μ-slope classes of T is equal to 1. If we knew that the scale factors and the coordinates of the translation vectors used in shapetiling the square were all in $\mathbb{Z}$, or even in $\mathbb{Q}$, we could prove this using polynomials in $\mathbb{Z}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$. Since we have no right to make this assumption, we need to work in the ring $\mathbb{Z}[X^{\mathbb{R}}, Y^{\mathbb{R}}]$.

We may assume that the square which is shapetiled by T is $S = S_{00}$, the unit square with lower left corner $(0,0)$. We have then $S = a_1 T_1 + \cdots + a_k T_k$, where $a_i \in \mathbb{Z}$ and each T_i is a translation of some $T(\rho_i)$. Let p be prime and suppose that for some $\mu \in \mathbb{Q}^\times$ the areas of the μ-slope classes of T are all divisible by p. Let c, d be integers such that $\gcd(c,d) = 1$ and $\mu = -c/d$. Let $\bar{f}_T \in \mathbb{F}_p[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ be the reduction of f_T modulo p, and for $1 \leqslant i \leqslant n$ let $\bar{f}_{T_i} \in \mathbb{F}_p[X^{\mathbb{R}}, Y^{\mathbb{R}}]$ be the reduction of f_{T_i}. Then by Proposition 4.1 we see that $(X^d - Y^c)(X-1)(Y-1)$ divides $\bar{f}_T$ (in $\mathbb{F}_p[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$, and hence also in $\mathbb{F}_p[X^{\mathbb{R}}, Y^{\mathbb{R}}]$). Therefore, by Lemmas 2.7 and 2.4 we see that $\bar{f}_{T_i}$ is divisible by

$$(X^{\rho_i d} - Y^{\rho_i c})(X^{\rho_i} - 1)(Y^{\rho_i} - 1).$$

Define a ring homomorphism $\Psi : \mathbb{F}_p[X^{\mathbb{R}}, Y^{\mathbb{R}}] \to \mathbb{F}_p[X^{\mathbb{R}}]$ by setting $\Psi(f) = f(X^c, X^d)$. Since $\Psi(X^{\rho_i d} - Y^{\rho_i c}) = 0$, the divisibility relation from the preceding paragraph implies that $\Psi(\bar{f}_{T_i}) = 0$ for $1 \leqslant i \leqslant n$. On the other hand, since $\bar{f}_S = \bar{g}_1 = (X-1)(Y-1)$, we have

$$\Psi(\bar{f}_S) = X^{c+d} - X^c - X^d + 1,$$

which is nonzero since c and d are nonzero. Since $S = a_1 T_1 + \cdots + a_k T_k$ we have $\bar{f}_S = \bar{a}_1 \bar{f}_{T_1} + \cdots + \bar{a}_k \bar{f}_{T_k}$ with $\bar{a}_i \in \mathbb{F}_p$, which gives a contradiction. Therefore the areas of the μ-slope classes of T can't all be divisible by p, so condition 2 is satisfied. This completes the proof of Theorem 4.2. □

Example 4.4. Let T be the lattice tile pictured in Fig. 2a. Since T has area $4 \neq 0$, it follows from Theorem 3.1 that T $\mathbb{Q}$-shapetiles a square. But since the nonempty 1-slope classes of T both have area 2, Theorem 4.2 implies that T does not $\mathbb{Z}$-shapetile a square.

Example 4.5. Let a, b, c, d be positive integers with $a > c$ and $b > d$. We construct a lattice tile T by removing a $c \times d$ rectangle from the upper right corner of an $a \times b$ rectangle, as in Fig. 2b. The area of T is $ab - cd > 0$, so the first condition of Theorem 4.2 is satisfied. If $\mu > 0$ there is a μ-slope class of T consisting of just the upper left corner square, while if $\mu < 0$ there is a μ-slope class of T consisting of just the lower left corner square. In either case T has a μ-slope class whose area is 1. Therefore the second condition of Theorem 4.2 is also satisfied, so T $\mathbb{Z}$-shapetiles a square.

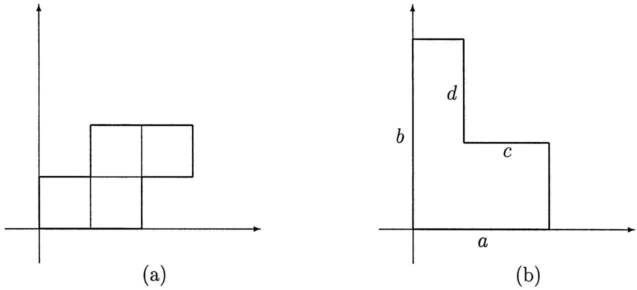

Fig. 2. Two tiles.

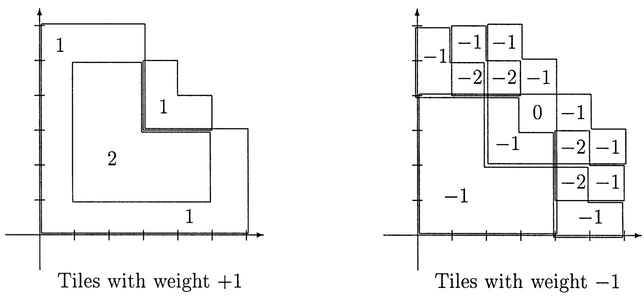

Fig. 3. A $\mathbb{Z}$-shapetiling of a square.

Example 4.6. The simplest case of Example 4.5 occurs when $a=b=2$ and $c=d=1$. In this case we have $f_T(X,Y)=(1+X+Y)(X-1)(Y-1)$. A straightforward calculation shows that

$$XYg_3(X,Y)=(X^3Y^3-X^2Y^2-X^4-X^4Y-X^4Y^2-Y^4-XY^4-X^2Y^4)f_T(X,Y)$$
$$+(XY-1)f_{T(2)}(X,Y)+f_{T(3)}(X,Y).$$

This gives the $\mathbb{Z}$-tiling of a 3×3 square with lower left corner $(1,1)$ depicted in Fig. 3. The left side of Fig. 3 has tiles with weight 1 and the right side has tiles with weight -1. The total weights of the tiles covering each region are indicated.

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