JOURNAL OF FUNCTIONAL ANALYSIS **135**, 50–75 (1996)
Article No. 0003

Non-symmetric Dirichlet Forms on Semifinite von Neumann Algebras

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Received July 29, 1993; revised November 23, 1994

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Markovian semigroups and sub-Markovian resolvents within this context. Some results on the allowed functional calculus for closed derivations on Hilbert algebras are obtained. Examples of non symmetric Dirichlet forms given by derivations on Hilbert algebras are studied. © 1996 Academic Press, Inc.

INTRODUCTION

The theory of non commutative Dirichlet forms, which originated from the pioneering examples of L. Gross [G] and the general analysis of S. Albeverio and R. Høegh-Krohn [AH] (see also [AHO]), has nowadays drawn a renewed interest between researchers ([DL1], [DL2], [DR], [D3], [Sa], [GL] and [Ci]). There are different reasons which, in our opinion, explain (and justify) the recent activity in this area. On the one side the presence of a feed-back effect due to the increasing ability showed by the commutative theory in handling successfully analytic and probabilistic problems during the last fifteen years ([AR], [AMR], [MR], [D2], and ref. therein). On the other side the great recent development of other new branches of mathematics such as A. Connes' non commutative geometry ([Co] and ref. therein) and quantum probability ([Pa], [AW] and ref. therein) within which the theory of non commutative Dirichlet forms naturally fits. Let us remark that up to now all works on non commutative Dirichlet forms treated the generalization to a non abelian setting, possibly in the non tracial case (see [Ci], [GL]), of the symmetric classical theory (see [F]).

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0022-1236/96 $12.00

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In this paper, we develop the general theory of non symmetric Dirichlet forms on a semifinite von Neumann algebra $\mathscr{A}$. This means that we study sesquilinear forms on the Hilbert space $L^2(\mathscr{A}, \tau)$, requiring only the so called "weak sector condition", which, at the form level, roughly means the antisymmetric part of the form must be controlled by the symmetric one. In this sense our work can be seen as a non commutative extension of the theory of Dirichlet forms as it has been recently presented in [MR], where this condition is assumed from the very beginning. It is worthwhile to notice that these authors are able to produce a large amount of examples of Dirichlet forms (see [MR] Chap. II). Among their examples let us quote the following simple one: consider the form

$$\mathscr{E}(u, v) := \sum_{i,j=1}^{n} \int b_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx, \tag{0.1}$$

where u and v are $\mathscr{C}^\infty$ functions with compact support in an open set U in $\mathbf{R}^n$. If the functions $b_{ij}(x)$ are locally summable on U, the symmetric part of the matrix-valued function $[b_{ij}(x)]$ is uniformly bounded from below by a positive constant and the entries of the antisymmetric part are L^∞ functions, it can be proven that the form $\{\mathscr{E}, \mathscr{D}(\mathscr{E})\}$ is closable and its closure is a Dirichlet form. As it will be explained in Section 5, a natural generalization of the preceding example is given by the form

$$\mathscr{E}(x, y) := \sum_{i,j=1}^{n} (d_j x, a_{ij} \, d_i y), \tag{0.2}$$

where $a_{ij} := \delta_{ij} + c_{ij}$ and $[c_{ij}]$ is an antisymmetric matrix whose entries are in the center of $\mathscr{A}$. As a consequence of the theory developed in this paper, we are able to prove that, if d_i are closable *-derivations and the intersection of their domains is dense, then the form in (0.2) gives rise to a Dirichlet form.

To better illustrate our results, we recall that throughout this paper $\mathscr{A}$ is a von Neumann algebra with a faithful, normal, semifinite trace τ. Forms, semigroups, resolvents etc. are defined on the complex Hilbert space $L^2(\mathscr{A}, \tau)$, even though many of their properties and relations require the real Hilbert space $L^2(\mathscr{A}, \tau)_h$ and its underlying order structure in an essential way.

In Section 1 we collect some preliminary material taken from [MR] on the relationships between coercive closed forms on a Hilbert space and strongly continuous contraction resolvents (resp. semigroups and their generators) satisfying a sector condition. The last paragraph of the section recall the essentials of I. E. Segal's theory of non commutative L^p spaces on $\mathscr{A}$ (see [N], [Se], [St]).

In Section 2 we establish the correspondence between Dirichlet forms, sub-Markovian semigroups and sub-Markovian resolvents, thus generalizing the results of S. Albeverio and R. Høegh-Krohn ([AH], see also [DL1]) to the non symmetric case. This is made adapting the non-symmetric abelian definitions and results in [MR] to the non commutative (semifinite) case.

Section 3 is devoted to the extension of some properties of sub-Markovian semigroups, already studied in [DL1], to the non symmetric context. In particular we prove that sub-Markovian semigroups may be extended to L^p spaces and study a class of sub-Markovian semigroups on $L^\infty(\mathscr{A}, \tau)$, showing a correspondence between such semigroups and those on $L^2(\mathscr{A}, \tau)$.

In Section 4 we study derivations on Hilbert algebras and, based on previous results in [Sa] and [DL1], we prove that, for a closed derivation δ on a Hilbert algebra, the self-adjoint part of its domain is closed under Lipschitz functional calculus and the whole domain is closed under the modulus operation. We also show that the corresponding norm inequalities (see (4.4) and (4.6)) hold, in particular a *-derivation is a Dirichlet derivation in the sense of E. B. Davies and J. M. Lindsay [DL1]. Moreover, a non-abelian chain rule holds for the $\mathscr{C}^1$ functional calculus of a self-adjoint operator. We notice that δ need not be a *-derivation for the previous results to hold. Finally we show how derivations which are not *-invariant give rise naturally to (non symmetric) Dirichlet forms.

In Section 5 we prove a theorem which gives rise to new examples of non commutative Dirichlet forms (and related semigroups). These examples are of the previously mentioned type. They were already studied in [DL1] in the symmetric case: this simply corresponds to requiring the antisymmetric part $[c_{ij}]$ in (0.2) to vanish.

Lastly let us mention that these results may be useful in the context of open quantum systems and quantum statistical mechanics which, as is known, represent a natural physical arena where these mathematical theories have found interesting applications (see e.g. references in [AH], [DL1] and [DL2]).

1. Preliminaries

In this section we first collect definitions and facts about strongly continuous semigroups and related objects, referring to [MR] for proofs and further results, and then definitions and facts about L^p spaces on $\{\mathscr{A}, \tau\}$, a von Neumann algebra with a faithful semifinite normal trace, referring to classic works of [Se], [K], [St] and [N] for more detailed analysis and proofs, and to [KR] and [T] for the general theory of von Neumann algebras.

It is well known that there is a bijective correspondence between strongly continuous contraction resolvents $\{G_\alpha\}_{\alpha>0}$ on a Banach space X, strongly continuous contraction semigroups $\{T_t\}_{t>0}$ on X, and closed, densely defined linear operators $\{L, \mathscr{D}(L)\}$ on X, with the properties that $(0, \infty) \subset \rho(L)$, and $\|\alpha(\alpha-L)^{-1}\| \leqslant 1$, $\forall \alpha>0$. These objects are related by

$$\begin{aligned}
&G_\alpha = (\alpha - L)^{-1}, \qquad \alpha>0,\\
&G_\alpha x = \int_0^\infty e^{-\alpha t} T_t x\, dt, \qquad x \in X,\\
&Lx = \lim_{t\downarrow 0} \frac{T_t x - x}{t}, \qquad x \in \mathscr{D}(L) := \left\{ x \in X : \lim_{t\downarrow 0} \frac{T_t x - x}{t} \text{ exists} \right\},\\
&T_t x = \lim_{\alpha\to\infty} e^{-\alpha t} \sum_{n=0}^{\infty} \frac{(t\alpha)^n}{n!} (\alpha G_\alpha)^n x, \qquad x \in X.
\end{aligned} \tag{1.1}$$

Recall now the theory of coercive closed forms.

Let $\{\mathscr{H}, (\cdot, \cdot)\}$ be a complex Hilbert space, $\mathscr{K} \subset \mathscr{H}$ a real vector subspace s.t. $\mathscr{K} + i\mathscr{K} = \mathscr{H}$ and $(x, y) \in \mathbf{R}$, $\forall x, y \in \mathscr{K}$ (this implies $\mathscr{K} \cap i\mathscr{K} = \{0\}$), and denote with $M_h := M \cap \mathscr{K}$, the real part of $M \subset \mathscr{H}$, and with $x^* := y - iz$, the adjoint of $x = y + iz$, $y, z \in \mathscr{K}$. Let $\mathscr{E}$: $\mathscr{D}(\mathscr{E}) \times \mathscr{D}(\mathscr{E}) \to \mathbf{C}$, where $\mathscr{D}(\mathscr{E})$ is a *-invariant subspace of $\mathscr{H}$, be a real-positive, sesquilinear form on $\mathscr{H}$, that is, $\forall x, y, z \in \mathscr{D}(\mathscr{E})$, $\alpha, \beta \in \mathbf{C}$,

$$\begin{aligned}
\mathscr{E}(x, \alpha y + \beta z) &= \alpha \mathscr{E}(x, y) + \beta \mathscr{E}(x, z)\\
\mathscr{E}(\alpha y + \beta z, x) &= \bar{\alpha} \mathscr{E}(y, x) + \bar{\beta} \mathscr{E}(z, x)\\
\mathscr{E}(x^*, y^*) &= \overline{\mathscr{E}(x, y)},
\end{aligned}$$

and

$$\mathscr{E}(x, x) \geqslant 0, \qquad x \in \mathscr{D}(\mathscr{E})_h.$$

Notice that since $\mathscr{D}(\mathscr{E})$ is *-invariant, $\mathscr{D}(\mathscr{E}) = \mathscr{D}(\mathscr{E})_h + i\mathscr{D}(\mathscr{E})_h$. Therefore the sesquilinear form $\mathscr{E}$ may be seen as the complex-linear extension of a bilinear positive real-valued form on $\mathscr{D}(\mathscr{E})_h \times \mathscr{D}(\mathscr{E})_h$.

We denote by ${}^s\mathscr{E}$ the symmetric part of $\mathscr{E}$,

$${}^s\mathscr{E}(x, y) := \tfrac{1}{2}[\mathscr{E}(x, y) + \overline{\mathscr{E}(y, x)}], \qquad x, y \in \mathscr{D}(\mathscr{E}),$$

and by ${}^a\mathscr{E}$ the antisymmetric part of $\mathscr{E}$,

$${}^a\mathscr{E}(x, y) := \tfrac{1}{2}[\mathscr{E}(x, y) - \overline{\mathscr{E}(y, x)}], \qquad x, y \in \mathscr{D}(\mathscr{E}).$$

Finally, denote by $\mathscr{E}_\alpha$, $\alpha \geqslant 0$, the form $\mathscr{E}_\alpha(x, y) := \mathscr{E}(x, y) + \alpha(x, y)$, $\forall x, y \in \mathscr{D}(\mathscr{E})$.

1.1. Definition. $\{\mathscr{E}, \mathscr{D}(\mathscr{E})\}$ is said to satisfy the weak sector condition if $\exists K>0$ s.t. $|\mathscr{E}_1(x, y)| \leqslant K\mathscr{E}_1(x, x)^{1/2}\, \mathscr{E}_1(y, y)^{1/2}$, $x, y \in \mathscr{D}(\mathscr{E})_h$.

Notice that the above definition is equivalent to $\exists K'>0$ s.t. $|{}^a\mathscr{E}_1(x, y)| \leqslant K'\mathscr{E}_1(x, x)^{1/2}\, \mathscr{E}_1(y, y)^{1/2}$, $x, y \in \mathscr{D}(\mathscr{E})_h$.

1.2. Definition. $\{\mathscr{E}, \mathscr{D}(\mathscr{E})\}$ is said a coercive closed form on $\mathscr{H}$ if

(i) $\mathscr{D}(\mathscr{E})$ is dense in $\mathscr{H}$

(ii) $\{{}^s\mathscr{E}, \mathscr{D}(\mathscr{E})\}$ is closed [i.e. $\{\mathscr{D}(\mathscr{E}), {}^s\mathscr{E}_1\}$ is a Hilbert space]

(iii) $\{\mathscr{E}, \mathscr{D}(\mathscr{E})\}$ is real-positive and satisfies the weak sector condition.

1.3. Definition. A positive linear operator $\{A, \mathscr{D}(A)\}$ on $\mathscr{H}$ is said to satisfy the sector condition if $\exists K>0$ s.t. $|(x, Ay)| \leqslant K(x, Ax)^{1/2}\,(y, Ay)^{1/2}$, $x, y \in \mathscr{D}(A)_h$.

1.4. Theorem ([MR, Theorem 2.15]). *There is a bijective correspondence between coercive closed forms $\{\mathscr{E}, \mathscr{D}(\mathscr{E})\}$ and strongly continuous contraction resolvents $\{G_\alpha\}_{\alpha>0}$ s.t. G_α satisfies the sector condition for some (hence for all) $\alpha>0$. These objects are related by*

$$\mathscr{E}_\alpha(x, G_\alpha y) = (x, y), \qquad x \in \mathscr{D}(\mathscr{E}), \quad y \in \mathscr{H},$$

and, if L is the generator of $\{G_\alpha\}_{\alpha>0}$,

$$\mathscr{E}(x, y) = (x, -Ly), \qquad x \in \mathscr{D}(\mathscr{E}), \quad y \in \mathscr{D}(L),$$

where $\mathscr{D}(\mathscr{E})$ is the completion of $\mathscr{D}(L)$ w.r.t. ${}^s\mathscr{E}_1^{1/2}$.

1.5. Proposition ([MR, Lemma 2.11, Theorem 2.13]). *Let $\{\mathscr{E}, \mathscr{D}(\mathscr{E})\}$ be a coercive closed form on $\mathscr{H}$, and $\{G_\alpha\}_{\alpha>0}$, the associated resolvent. Then, setting $\mathscr{E}^{(\beta)}(x, y) := \beta(x, y-\beta G_\beta y)$, $x, y \in \mathscr{H}$, we get*

(i) $|\mathscr{E}_1^{(\beta)}(x, y)| \leqslant (K+1)\, \mathscr{E}_1(x, x)^{1/2}\, \mathscr{E}_1^{(\beta)}(y, y)^{1/2}$, $x \in \mathscr{D}(\mathscr{E})$, $y \in \mathscr{H}$

(ii) Let $x \in \mathscr{H}$. Then $x \in \mathscr{D}(\mathscr{E}) \Leftrightarrow \sup_{\beta>0} \mathscr{E}^{(\beta)}(x, x) < \infty$

(iii) $\lim_{\beta\to\infty} \mathscr{E}^{(\beta)}(x, y) = \mathscr{E}(x, y)$, $x, y \in \mathscr{D}(\mathscr{E})$.

Let $(\mathscr{A}, \tau)$ be a von Neumann algebra with a faithful normal semifinite trace, identified with its GNS representation $L^\infty(\mathscr{A}, \tau)$. Then $\tilde{\mathscr{A}}$ denotes the set of all closed densely defined operators x affiliated to $\mathscr{A}$ such that $\tau(e_{|x|}(\lambda, \infty)) < \infty$ for some $\lambda>0$, where $e_{|x|}$ is the spectral measure of $|x|$. $\tilde{\mathscr{A}}$ is a topological *-algebra, with algebraic operations understood in a strong sense [Se] and topology given by convergence in measure [St].

On the positive part of $\tilde{\mathscr{A}}$, the trace extends by $\tau(x)=\int \lambda \, d\nu_x(\lambda)$ where $\nu_x := \tau \circ e_x$. One can naturally define L^p spaces, which are Banach spaces and enjoy most of the properties of classical L^p spaces, including duality and Riesz–Thorin interpolation ([N], [K]). The properties of these non-commutative L^p spaces which are required here are conveniently collected in Section 1 of [DL1] where further references may also be found. We quote just some elementary properties which will be used later. If $p \in [1, \infty]$, p' the conjugate exponent, we have

$$x \in L^p(\mathscr{A}, \tau)_+, \qquad y \in L^{p'}(\mathscr{A}, \tau)_+ \Rightarrow \tau(xy) \geqslant 0, \tag{1.2}$$

$$x \in L^p \Rightarrow x e_{|x|}\left(\left(\frac{1}{n}, n\right)\right) \to x \qquad \text{in } L^p, \tag{1.3}$$

$$\begin{aligned} &x \in \tilde{\mathscr{A}}, \quad ax \in L^1 \quad \text{and} \\ &\tau(ax) = 0 \ \forall a \in L^1(\mathscr{A}, \tau) \cap L^\infty(\mathscr{A}, \tau) \Rightarrow x = 0, \end{aligned} \tag{1.4}$$

$$x \in \tilde{\mathscr{A}}_h \Rightarrow \|\varphi(x)\|_p = \|\varphi\|_{L^p(\mathbf{R}, \nu_x)} \tag{1.5}$$

where $\varphi \colon \mathbf{R} \to \mathbf{C}$ is a Borel measurable function. If M is a linear positive operator from $(L^1 \cap L^\infty, \|\cdot\|_\infty)$ in L^∞, then

$$\|M\| = \sup_{0 < x < 1} \|Mx\|. \tag{1.6}$$

2. Markov Semigroups and Dirichlet Forms

In this section we give the basic definitions and prove the main theorems which constitute the basis of the theory of non symmetric Dirichlet forms in a non commutative setting. In our exposition we generalize to the non abelian case results and techniques of Chap. I, Sec. 4 in [MR]. In particular, the classical space of square integrable functions on a measure space is replaced by the space of the operators affiliated to a von Neumann algebra $\mathscr{A}$ which are square integrable w.r.t. a normal, semifinite, faithful trace τ.

2.1. Definition. (i) A bounded linear operator G on $L^2(\mathscr{A}, \tau)$ is called sub-Markovian if

$$0 \leqslant x \leqslant 1 \Rightarrow 0 \leqslant Gx \leqslant 1, \qquad \forall x \in L^2(\mathscr{A}, \tau).$$

A strongly continuous contraction resolvent $\{G_\alpha\}_{\alpha > 0}$, resp. semigroup $\{T_t\}_{t > 0}$, is called sub-Markovian if all αG_α, $\alpha > 0$, resp. T_t, $t > 0$, are sub-Markovian.

(ii) A closed densely defined operator $\{L, \mathscr{D}(L)\}$ on $L^2(\mathscr{A}, \tau)$ is called a Dirichlet operator if $(Lx, (x-1)^+) \leqslant 0$, for each $x \in \mathscr{D}(L)_h$.

(iii) A coercive closed form on $L^2(\mathscr{A}, \tau)$ is called a Dirichlet form if, for all $x \in \mathscr{D}(\mathscr{E})_h$, $x^+ \wedge 1 \in \mathscr{D}(\mathscr{E})$ and

$$\begin{aligned} &\mathscr{E}(x - x^+ \wedge 1, x + x^+ \wedge 1) \geqslant 0 \\ &\mathscr{E}(x + x^+ \wedge 1, x - x^+ \wedge 1) \geqslant 0. \end{aligned} \tag{2.1}$$

If only the first inequality in (2.1) holds, the form is called 1/2-Dirichlet.

As in the classical case, if the form $\mathscr{E}$ is symmetric each of the two inequalities in (2.1) is equivalent to the usual definition of Dirichlet form (see e.g. [AH]).

The following two theorems state the equivalence among the objects described in Definition 2.1.

2.2. THEOREM. *Let $\{\mathscr{E}, \mathscr{D}(\mathscr{E})\}$ be a coercive closed form on $L^2(\mathscr{A}, \tau)$ with corresponding semigroup $\{T_t\}_{t>0}$, resolvent $\{G_\alpha\}_{\alpha>0}$ and generator $\{L, \mathscr{D}(L)\}$. Then the following are equivalent*:

(a) *The form $\mathscr{E}$ is* 1/2-*Dirichlet.*

(b) *The semigroup $\{T_t\}_{t>0}$ is sub-Markovian.*

(c) *The resolvent $\{G_\alpha\}_{\alpha>0}$ is sub-Markovian.*

(d) *The generator $\{L, \mathscr{D}(L)\}$ is a Dirichlet operator.*

2.3. THEOREM. *Under the same hypotheses of the preceding theorem, the following are equivalent*:

(a) *The form $\mathscr{E}$ is Dirichlet.*

(b) *The semigroups $\{T_t\}_{t>0}$ and $\{T_t^*\}_{t>0}$ are sub-Markovian.*

(c) *The resolvents $\{G_\alpha\}_{\alpha>0}$ and $\{G_\alpha^*\}_{\alpha>0}$ are sub-Markovian.*

(d) *The generators L and L^* are Dirichlet operators.*

The proof of the preceding theorems follows directly from Propositions 2.5 and 2.6.

2.4. LEMMA. (i) *A bounded linear operator G on $L^2(\mathscr{A}, \tau)$ is sub-Markovian iff*

$$\begin{cases} x \geqslant 0 \Rightarrow Gx \geqslant 0, \\ x \leqslant 1 \Rightarrow Gx \leqslant 1, \end{cases} \qquad \forall x \in L^2(\mathscr{A}, \tau).$$

(ii) *Let $\{x_n\}$ be a sequence converging to x in $L^p(\mathscr{A}, \tau)$, $p \in [1, \infty]$, for which $0 \leqslant x_n \leqslant 1$, $\forall n \in \mathbf{N}$. Then $0 \leqslant x \leqslant 1$.*

2.5. PROPOSITION. *Let* $\{G_\alpha\}_{\alpha>0}$ *be a strongly continuous contraction resolvent on* $L^2(\mathscr{A},\tau)$ *with corresponding generator* L *and semigroup* $\{T_t\}_{t>0}$. *Then the following are equivalent*:

(i) $\{G_\alpha\}_{\alpha>0}$ *is sub-Markovian.*

(ii) $\{T_t\}_{t>0}$ *is sub-Markovian.*

(iii) L *is a Dirichlet operator.*

Proof. (i) $\Rightarrow$ (ii). Let $x \in L^2(\mathscr{A},\tau)$ and $0 \leqslant x \leqslant 1$. Then, for all $\beta>0$, $x_\beta := \beta G_\beta x$ is in $\mathscr{D}(L)$ and $0 \leqslant x_\beta \leqslant 1$ since the resolvent G_β is sub-Markovian, therefore, by formula (1.1) and Lemma 2.4(ii), $0 \leqslant T_t x_\beta \leqslant 1$. Moreover $x_\beta \to x$ in $L^2(\mathscr{A},\tau)$ when $\beta \to \infty$, therefore, again by Lemma 2.4(ii), $0 \leqslant T_t x \leqslant 1$, i.e. T_t is sub-Markovian.

(ii) $\Rightarrow$ (iii). Let $x \in L^2(\mathscr{A},\tau)_h$. Then

$$((x-1)^+, T_t(x-1)^+) \leqslant ((x-1)^+, (x-1)^+) = ((x-1)^+, (x-1))$$

by the Schwartz inequality and the fact that $(x-1)^+$ and $(x-1)^-$ are orthogonal in $L^2(\mathscr{A},\tau)$. Moreover $T_t(x\wedge 1) \leqslant 1$ by Lemma 2.4(i). Therefore, since $x = (x-1)^+ + x\wedge 1$, we have

$$\begin{aligned}((x-1)^+, T_t x) &= ((x-1)^+, T_t(x-1)^+) + ((x-1)^+, T_t(x\wedge 1))\\ &\leqslant ((x-1)^+, (x-1)) + \tau((x-1)^+)\\ &= ((x-1)^+, x).\end{aligned}$$

Therefore we get

$$((x-1)^+, Lx) = \lim_{t\downarrow 0} \frac{1}{t}((x-1)^+, T_t x - x) \leqslant 0, \qquad \forall x \in \mathscr{D}(L).$$

(iii) $\Rightarrow$ (i). Let $x \in L^2(\mathscr{A},\tau)_h$, $\alpha>0$ and $y := \alpha G_\alpha x$. We want to prove that if $0 \leqslant x \leqslant 1$ then $0 \leqslant y \leqslant 1$. Indeed, for $x \leqslant 1$, we have

$$\begin{aligned}\alpha((y-1)^+, y) &= ((y-1)^+, \alpha y - Ly) + ((y-1)^+, Ly)\\ &\leqslant \alpha((y-1)^+, x) \leqslant \alpha\tau((y-1)^+).\end{aligned}$$

As a consequence,

$$\|(y-1)^+\|_2 = ((y-1)^+, y) - \tau((y-1)^+) \leqslant 0,$$

hence $y \leqslant 1$. On the other hand, if $x \geqslant 0$, then $-nx \leqslant 1$ $\forall n \in \mathbf{N}$, therefore, by the previous result, $-ny \leqslant 1$, $\forall n \in \mathbf{N}$, i.e. $y \geqslant 0$. ∎

2.6. PROPOSITION. *Let $\{\mathscr{E}, \mathscr{D}(\mathscr{E})\}$ be a coercive closed form on $L^2(\mathscr{A}, \tau)$ with resolvent $\{G_\alpha\}_{\alpha>0}$. Then the following are equivalent*:

(i) *For all $x \in \mathscr{D}(\mathscr{E})_h$ and $\alpha \geqslant 0$, $x \wedge \alpha \in \mathscr{D}(\mathscr{E})$ and $\mathscr{E}(x - x \wedge \alpha, x \wedge \alpha) \geqslant 0$.*

(ii) *For all $x \in \mathscr{D}(\mathscr{E})_h$, $x^+ \wedge 1 \in \mathscr{D}(\mathscr{E})$ and $\mathscr{E}(x - x^+ \wedge 1, x^+ \wedge 1) \geqslant 0$.*

(iii) *$\mathscr{E}$ is a 1/2-Dirichlet form.*

(iv) *$\{G_\alpha\}_{\alpha>0}$ is sub-Markovian.*

The analogous equivalences hold when $\{G_\alpha\}_{\alpha>0}$ is replaced by its adjoint and $\mathscr{E}$ by the form $\mathscr{E}^\dagger(x, y) := \overline{\mathscr{E}(y, x)}$.

Proof. (i) $\Rightarrow$ (ii). Let $x \in \mathscr{D}(\mathscr{E})_h$, then, by (i), we get x^-, x^+, $x^+ \wedge 1 \in \mathscr{D}(\mathscr{E})$. As a consequence

$$\begin{aligned}\mathscr{E}(x - x^+ \wedge 1, x^+ \wedge 1) &= \mathscr{E}(x^+ - x^+ \wedge 1, x^+ \wedge 1) - \mathscr{E}(x^-, x^+ \wedge 1)\\ &\geqslant -\mathscr{E}((x \wedge 1)^-, (x \wedge 1)^+).\end{aligned}$$

Now for any $y \in \mathscr{D}(\mathscr{E})_h$ we have, again by (i),

$$\mathscr{E}(y^-, y^+) = \mathscr{E}(y^+ - y, y^+) = -\mathscr{E}((-y) - (-y) \wedge 0, (-y) \wedge 0) \leqslant 0;$$

therefore $\mathscr{E}(x - x^+ \wedge 1, x^+ \wedge 1) \geqslant 0$.

(ii) $\Rightarrow$ (iii). Since $\mathscr{E}$ is a real-positive sesquilinear form and (ii) holds, we get, for all $x \in \mathscr{D}(\mathscr{E})_h$,

$$\begin{aligned}\mathscr{E}(x - x^+ \wedge 1, x + x^+ \wedge 1) &= \mathscr{E}(x - x^+ \wedge 1, x - x^+ \wedge 1)\\ &\quad + 2\mathscr{E}(x - x^+ \wedge 1, x^+ \wedge 1) \geqslant 0.\end{aligned}$$

(iii) $\Rightarrow$ (iv). Let $y \in L^2(\mathscr{A}, \tau)$, $0 \leqslant y \leqslant 1$. We have to show that $x := \alpha G_\alpha y$ satisfies $0 \leqslant x \leqslant 1$. Indeed

$$\begin{aligned}&\|x - x^+ \wedge 1\|^2 + (x - x^+ \wedge 1, x^+ \wedge 1 - y)\\ &\quad = (x - x^+ \wedge 1, x - y)\\ &\quad = -\frac{1}{\alpha}\,\mathscr{E}(x - x^+ \wedge 1, x)\\ &\quad = -\frac{1}{2\alpha}\,(\mathscr{E}(x - x^+ \wedge 1, x + x^+ \wedge 1) + \mathscr{E}(x - x^+ \wedge 1, x - x^+ \wedge 1))\\ &\quad \leqslant 0, \qquad (2.2)\end{aligned}$$

where the equality in the second line follows from Theorem 1.4. Let us introduce the functions $f, g, h\colon \mathbf{R} \to \mathbf{R}$,

$$f(t) = t\chi_{(-\infty, 0]}(t),$$
$$g(t) = (t \wedge 1)\, \chi_{[0, \infty)}(t),$$
$$h(t) = (t - 1)\, \chi_{[1, \infty)}(t).$$

Then $fg \equiv 0$, $gh \equiv h$, $g(x) = x^+ \wedge 1$ and $x - g(x) = f(x) + h(x)$. Therefore

$$\begin{aligned}(x - x^+ \wedge 1, x^+ \wedge 1 - y) &= \tau((x - x^+ \wedge 1)(x^+ \wedge 1 - y)) \\ &= \tau(f(x)(g(x) - y)) + \tau(h(x)(g(x) - y)) \\ &= \tau((-f(x))\, y) + \tau(h(x)(1 - y)) \geqslant 0 \qquad (2.3)\end{aligned}$$

where we used property (1.2). Finally equations (2.2) and (2.3) imply $\|x - x^+ \wedge 1\| = 0$, i.e. $0 \leqslant x \leqslant 1$.

(iv) $\Rightarrow$ (i). Let $x \in \mathcal{D}(\mathcal{E})_h$, $\alpha \geqslant 0$. Now we prove that $x \wedge \alpha \in \mathcal{D}(\mathcal{E})$: since $x = (x - \alpha)^+ + x \wedge \alpha$, it suffices to prove $(x - \alpha)^+ \in \mathcal{D}(\mathcal{E})$. Recalling that, as defined in Proposition 1.5, $\mathcal{E}^{(\beta)}(y, z) = \beta\tau(y^*(z - \beta G_\beta z))$, for $y, z \in L^2(\mathcal{A}, \tau)$, we have

$$\begin{aligned}\mathcal{E}^{(\beta)}((x - \alpha)^+, x \wedge \alpha) &= \beta\tau((x - \alpha)^+\, (x \wedge \alpha)) - \beta\tau((x - \alpha)^+\, \beta G_\beta(x \wedge \alpha)) \\ &\geqslant \alpha\beta\tau((x - \alpha)^+) - \alpha\beta\tau((x - \alpha)^+) = 0, \qquad (2.4)\end{aligned}$$

where, since $x \wedge \alpha \leqslant \alpha$, the inequality in (2.4) follows from Lemma 2.4(i), property (1.2) and the fact that βG_β is sub-Markovian.

Therefore,

$$\begin{aligned}&\mathcal{E}_1^{(\beta)}((x - \alpha)^+, (x - \alpha)^+) \\ &\quad = \mathcal{E}_1^{(\beta)}((x - \alpha)^+, x - x \wedge \alpha) \\ &\quad = \mathcal{E}_1^{(\beta)}((x - \alpha)^+, x) - \mathcal{E}_1^{(\beta)}((x - \alpha)^+, x \wedge \alpha) \\ &\quad = \mathcal{E}_1^{(\beta)}((x - \alpha)^+, x) - \mathcal{E}^{(\beta)}((x - \alpha)^+, x \wedge \alpha) - ((x - \alpha)^+, x \wedge \alpha) \\ &\quad \leqslant \mathcal{E}_1^{(\beta)}((x - \alpha)^+, x) \\ &\quad \leqslant (K + 1)\, \mathcal{E}_1(x, x)^{1/2}\, \mathcal{E}_1^{(\beta)}((x - \alpha)^+, (x - \alpha)^+)^{1/2},\end{aligned}$$

where the last inequality follows from Proposition 1.5(i). As a consequence,

$$\begin{aligned}\mathcal{E}^{(\beta)}((x - \alpha)^+, (x - \alpha)^+) &\leqslant \mathcal{E}_1^{(\beta)}((x - \alpha)^+, (x - \alpha)^+) \\ &\leqslant (K + 1)^2\, \mathcal{E}_1(x, x). \qquad (2.5)\end{aligned}$$

Now Proposition 1.5(ii) and (2.5) imply $(x - \alpha)^+ \in \mathcal{D}(\mathcal{E})$.

Finally we prove that $\mathscr{E}(x-x\wedge\alpha, x\wedge\alpha)\geqslant 0$: we have

$$\mathscr{E}^{(\beta)}(x-x\wedge\alpha, x\wedge\alpha)=\mathscr{E}^{(\beta)}((x-\alpha)^+, x\wedge\alpha)\geqslant 0$$

by (2.4); hence the result follows by Proposition 1.5(iii). ∎

We conclude this section with a theorem in which it is shown that a smooth version of the definition of a Dirichlet form can be given. More precisely, the *normal contraction* $x^+\wedge 1$ in (2.1) may be substituted by a family of $\mathscr{C}^\infty$ contractions.

2.7. THEOREM. *Let $\{\mathscr{E}, \mathscr{D}(\mathscr{E})\}$ be a coercive closed form on $L^2(\mathscr{A}, \tau)$. Then, the following are equivalent*:

(i) *$\mathscr{E}$ is a Dirichlet form.*

(ii) *For each $x\in\mathscr{D}(\mathscr{E})_h$ there exists a family of functions φ_ε: $\mathbf{R}\to[-\varepsilon, 1+\varepsilon]$, $\varepsilon\geqslant 0$, such that*

(a) *$\varphi_\varepsilon(t)=t$ for all $t\in[0,1]$.*

(b) *φ_ε is Lipschitz continuous with Lipschitz constant* 1.

(c) *$\varphi_\varepsilon(x)\in\mathscr{D}(\mathscr{E})$*

(d) *$\liminf_{\varepsilon\to 0}\mathscr{E}(x\mp\varphi_\varepsilon(x), x\pm\varphi_\varepsilon(x))\geqslant 0$.*

Proof. The implication (i) ⇒ (ii) is trivial.

(ii) ⇒ (i). Let $\varphi_0(t):=(t\vee 0)\wedge 1$ then, by (1.5) and Lebesgue's dominated convergence theorem, for $x\in L^2(\mathscr{A}, \tau)_h$,

$$\lim_{\varepsilon\to 0}\|\varphi_\varepsilon(x)-x^+\wedge 1\|_2=\lim_{\varepsilon\to 0}\|\varphi_\varepsilon-\varphi_0\|_{L^2(\mathbf{R}, \nu_x)}=0.$$

Summing the two inequalities in (d) it follows

$$\limsup_{\varepsilon\to 0}\mathscr{E}(\varphi_\varepsilon(x), \varphi_\varepsilon(x))\leqslant\mathscr{E}(x, x);$$

therefore, applying [MR, Lemma 2.12] to $\varphi_\varepsilon(x)$ and the form $\mathscr{E}$, we get a sequence $\varepsilon_n\to 0$ such that $\varphi_{\varepsilon_n}(x)$ converges weakly in $\{\mathscr{D}(\mathscr{E}), {}^s\mathscr{E}_1\}$ to $x^+\wedge 1$, so that $x^+\wedge 1\in\mathscr{D}(\mathscr{E})$, and

$$\mathscr{E}(x^+\wedge 1, x^+\wedge 1)\leqslant\liminf_{n\to\infty}\mathscr{E}(\varphi_{\varepsilon_n}(x), \varphi_{\varepsilon_n}(x)).$$

Moreover, by the weak sector condition, the functional $\mathscr{E}_1(\cdot, x)$ is continuous in $\{\mathscr{D}(\mathscr{E}), {}^s\mathscr{E}_1\}$; therefore

$$\lim_{n\to\infty}\mathscr{E}(x, \varphi_{\varepsilon_n}(x))=\mathscr{E}(x, x^+\wedge 1).$$

Finally, we have

$$
\begin{aligned}
&\mathscr{E}(x \pm (x^+ \wedge 1), x \mp (x^+ \wedge 1)) \\
&\quad \geqslant \mathscr{E}(x, x) \mp \lim_{n \to \infty} \mathscr{E}(x, \varphi_{\varepsilon_n}(x)) \pm \lim_{n \to \infty} \mathscr{E}(\varphi_{\varepsilon_n}(x), x) \\
&\qquad - \liminf_{n \to \infty} \mathscr{E}(\varphi_{\varepsilon_n}(x), \varphi_{\varepsilon_n}(x)) \\
&\quad = \limsup_{n \to \infty} \mathscr{E}(x \pm \varphi_{\varepsilon_n}(x), x \mp \varphi_{\varepsilon_n}(x)) \geqslant 0,
\end{aligned}
$$

where the last inequality follows by hypothesis (d). ∎

3. L^p Extensions of Sub-Markovian Semigroups and Complete Positivity

This section is devoted to the extension of some properties of sub-Markovian semigroups, already studied in [DL1], to the non symmetric context. In particular we prove that sub-Markovian semigroups may be extended to L^p spaces and study a class of sub-Markovian semigroups on $L^\infty(\mathscr{A}, \tau)$, showing a correspondence between such semigroups and those on $L^2(\mathscr{A}, \tau)$.

In this section $\{p, p'\}$ will always be a pair of conjugate exponents.

3.1. Definition. (i) Let $M \in \mathscr{B}(L^p(\mathscr{A}, \tau))$, $p \in [1, \infty)$ then we define $M^* \in \mathscr{B}(L^{p'}(\mathscr{A}, \tau))$ as the unique linear operator satisfying $\tau((M^*x)^* y) = \tau(x^*(My))$, $\forall x \in L^p$, $y \in L^{p'}$. We notice that this notation is consistent with that for the adjoint in L^2.

(ii) $M \in \mathscr{B}(L^p(\mathscr{A}, \tau))$, $p \in [1, \infty)$, is said a sub-Markovian operator on L^p if $x \in L^p(\mathscr{A}, \tau)$, $0 \leqslant x \leqslant 1 \Rightarrow 0 \leqslant Mx \leqslant 1$.

(iii) $M \in \mathscr{B}(L^\infty(\mathscr{A}, \tau))$, is said a sub-Markovian operator on L^∞ if it is weak* continuous and $0 \leqslant x \leqslant 1 \Rightarrow 0 \leqslant Mx \leqslant 1$.

[We remark that the sub-Markovian operators on L^∞ are precisely the positive normal contractions, see Proposition 3.2(i).]

(iv) $\{T_t\}_{t>0} \subset \mathscr{B}(L^p(\mathscr{A}, \tau))$, $p \in [1, \infty)$, is said a sub-Markovian semigroup on L^p, if T_t is a sub-Markovian operator on L^p, for all $t > 0$, and $T_t \to I$, $t \to 0$, strongly on $L^p(\mathscr{A}, \tau)$.

(v) $\{T_t\}_{t>0} \subset \mathscr{B}(L^\infty(\mathscr{A}, \tau))$ is said a sub-Markovian semigroup on L^∞, if T_t is a sub-Markovian operator on L^∞, for all $t > 0$, and $T_t \to I$, $t \to 0$, weak* on $L^\infty(\mathscr{A}, \tau)$.

The following proposition lists the main properties of sub-Markovian operators on L^p spaces. In order to prove it we need a transposition argument, i.e. we cannot prove the properties for a single sub-Markovian operator M but we need sub-Markovianity for the adjoint too.

3.2. PROPOSITION. *Let M be a sub-Markovian operator on $L^{p_0}(\mathscr{A}, \tau)$, $p_0 \in [1, \infty)$ whose adjoint is sub-Markovian. Then*:

(i) *$M|_{L^1 \cap L^\infty}$ and $M^*|_{L^1 \cap L^\infty}$ extend uniquely to contractions $M^{(p)}$ and $M^{*(p)}$ on L^p for $p \in [1, \infty)$ satisfying $(M^{(p)})^* = M^{*(p')}$, $(M^{*(p)})^* = M^{(p')}$, $p \in (1, \infty)$.*

(ii) *$M|_{L^1 \cap L^\infty}$ and $M^*|_{L^1 \cap L^\infty}$ have a unique weak*-continuous extension to $L^\infty(\mathscr{A}, \tau)$ given by $M^{(\infty)} := (M^{*(1)})^*$, resp. $M^{*(\infty)} := (M^{(1)})^*$.*

(iii) *$M^{(p)}x = M^{(q)}x$, and $M^{*(p)}x = M^{*(q)}x$, for $x \in L^p \cap L^q$, $p, q \in [1, \infty]$.*

(iv) *$M^{(p)}$ and $M^{*(p)}$ are sub-Markovian on $L^p(\mathscr{A}, \tau)$, $p \in [1, \infty]$.*

Proof. By (1.6) and the sub-Markovian property, M: $L^\infty \cap L^1 \subset L^\infty \to L^\infty$ is a contraction, and the same holds for M^*. By a duality argument, the contractivity property for M^* proved above implies the contractivity property for M: $L^\infty \cap L^1 \subset L^1 \to L^1$, and analogously for M^*: $L^\infty \cap L^1 \subset L^1 \to L^1$. By Riesz–Thorin–Kunze interpolation, we get the contractivity property for M: $L^\infty \cap L^1 \subset L^p \to L^p$, $p \in [1, \infty]$. The rest of point (i) and points (ii) and (iii) follow as in [DL1] Proposition 2.2 making use of the transposition argument.

As regards point (iv), since $x \in L^1$ and $0 \leqslant x \leqslant 1$ imply $x \in L^{p_0}$, $M^{(1)}$ and $M^{*(1)}$ are sub-Markovian by (iii). Therefore $M^{*(\infty)}$ and $M^{(\infty)}$, being the dual of a positive contraction, are sub-Markovian. Hence $M^{(p)}$ and $M^{*(p)}$ are sub-Markovian again by (iii). ∎

By obvious modifications of the proofs of [DL1] Proposition 2.14 and Proposition 3.1 we have the following results.

3.3. THEOREM. *Let $\{T_t\}_{t>0}$, $\{T_t^*\}_{t>0}$ be sub-Markovian semigroups on L^2. Then their extensions to L^p are sub-Markovian semigroups on L^p, for $p \in [1, \infty]$.*

3.4. THEOREM. *Let $\{T_t\}_{t>0}$, $\{\hat{T}_t\}_{t>0}$ be sub-Markovian semigroups on $\{\mathscr{A}, \tau\}$ s.t.*

$$\tau(x(T_t y)) = \tau((\hat{T}_t x)\, y), \qquad x, y \in L^1 \cap L^\infty.$$

Then $\{T_t\}_{t>0}$ and $\{\hat{T}_t\}_{t>0}$ are the unique weak-continuous extensions of sub-Markovian semigroups on L^2 which are adjoint to each other.*

3.5. DEFINITION. Let $\{\mathscr{E}, \mathscr{D}(\mathscr{E})\}$ be a sesquilinear form on $L^2(\mathscr{A}, \tau)$. Then

$$\mathscr{E}^{[n]}([a_{ij}], [b_{ij}]) := \sum_{i,j=1}^{n} \mathscr{E}(a_{ij}, b_{ij}), \qquad a_{ij}, b_{ij} \in \mathscr{D}(\mathscr{E}),$$

is a sesquilinear form on $L^2(\mathscr{A} \otimes M_n, \tau \otimes tr) \cong L^2(\mathscr{A}, \tau) \otimes L^2(M_n, tr)$, where tr is the usual trace on n by n matrices.

We say that $\mathscr{E}$ is n-Dirichlet if $\mathscr{E}^{[n]}$ is a Dirichlet form.

If $P \in \mathscr{B}(L^2(\mathscr{A}, \tau))$, we denote by $P^{[n]}$ the operator $P \otimes I$ on $L^2(\mathscr{A}, \tau) \otimes L^2(M_n, tr)$.

3.6. LEMMA. *Let $\{\mathscr{E}, \mathscr{D}(\mathscr{E})\}$ be a coercive closed form on $L^2(\mathscr{A}, \tau)$, and let $\{G_\alpha\}_{\alpha>0}$ be the associated resolvent.*

Then $\mathscr{E}^{[n]}$ is a coercive closed form, $\{G_\alpha^{[n]}\}_{\alpha>0}$ is the associated resolvent and $\{T_t^{[n]}\}_{t>0}$ the associated semigroup.

Proof. Let us observe that

$$\begin{aligned}\mathscr{E}_\alpha^{[n]}([a_{ij}], G_\alpha^{[n]}[b_{ij}]) = \mathscr{E}_\alpha^{[n]}([a_{ij}], [G_\alpha b_{ij}]) &= \sum_{i,j=1}^{n} \mathscr{E}_\alpha(a_{ij}, G_\alpha b_{ij}) \\ &= \sum_{i,j=1}^{n} (a_{ij}, b_{ij}) = ([a_{ij}], [b_{ij}])\end{aligned}$$

so that $\{G_\alpha^{[n]}\}_{\alpha>0}$ is the resolvent associated to $\mathscr{E}^{[n]}$. Let us now prove that $\{G_\alpha^{[n]}\}_{\alpha>0}$ is contractive

$$\|\alpha G_\alpha^{[n]}[a_{ij}]\|_2^2 = \|[\alpha G_\alpha a_{ij}]\|_2^2 = \sum_{i,j=1}^{n} \|\alpha G_\alpha a_{ij}\|_2^2 \leqslant \sum_{i,j=1}^{n} \|a_{ij}\|_2^2 = \|[a_{ij}]\|_2^2.$$

Finally let us prove $\{G_\alpha^{[n]}\}_{\alpha>0}$ satisfies the sector condition [see 1.3]. Let $[a_{ij}], [b_{ij}] \in L^2(\mathscr{A} \otimes M_n, \tau \otimes tr)_h$; then

$$\begin{aligned}|([a_{ij}], G_1^{[n]}[b_{ij}])| &= |([a_{ij}], [G_1 b_{ij}])| \\ &= \left| \sum_{i,j=1}^{n} (a_{ij}, G_1 b_{ij}) \right| \leqslant \sum_{i,j=1}^{n} |(a_{ij}, G_1 b_{ij})| \\ &\leqslant K \sum_{i,j=1}^{n} (a_{ij}, G_1 a_{ij})^{1/2} (b_{ij}, G_1 b_{ij})^{1/2} \\ &\leqslant K \left(\sum_{i,j=1}^{n} (a_{ij}, G_1 a_{ij}) \right)^{1/2} \left(\sum_{i,j=1}^{n} (b_{ij}, G_1 b_{ij}) \right)^{1/2} \\ &= K([a_{ij}], G_1^{[n]}[a_{ij}])^{1/2} ([b_{ij}], G_1^{[n]}[b_{ij}])^{1/2},\end{aligned}$$

that is, $\mathscr{E}^{[n]}$ is a coercive closed form. ∎

3.7. THEOREM. *Let $\{\mathscr{E}, \mathscr{D}(\mathscr{E})\}$ be a Dirichlet form and $\{T_t\}_{t>0}$, $\{T_t^*\}_{t>0}$ the associated semigroups.*
Then $\mathscr{E}$ is n-Dirichlet $\Leftrightarrow \{T_t\}$ and $\{T_t^*\}$ are n-positive.

Proof. Indeed $\mathscr{E}$ is n-Dirichlet $\Leftrightarrow \mathscr{E}^{[n]}$ is Dirichlet $\Leftrightarrow \{T_t^{[n]}\}$ and $\{T_t^{*(n)}\}$ are sub-Markovian, which, by Theorems 3.3 and 3.4, is equivalent to $\{T_t^{(\infty)[n]}\}$ and $\{T_t^{*(\infty)[n]}\}$ are sub-Markovian. By Definition 3.1(iii), this is the same as saying $\{T_t^{(\infty)}\}$ and $\{T_t^{*(\infty)}\}$ are n-positive normal contractions, which implies the thesis again by Theorems 3.3, 3.4. ∎

4. DERIVATIONS ON SQUARE INTEGRABLE OPERATORS

In this section we consider derivations on the space $L^2(\mathscr{A}, \tau)$. By this we mean a linear operator

$$\delta: \mathscr{D} \subseteq L^2(\mathscr{A}, \tau) \to L^2(\mathscr{A}, \tau),$$

where $\mathscr{D}$ is a subalgebra of $L^2(\mathscr{A}, \tau) \cap L^\infty(\mathscr{A}, \tau)$, and δ verifies

$$\delta(ab) = a \cdot \delta b + \delta a \cdot b, \qquad a, b \in \mathscr{D}.$$

We say that a derivation δ is closed under the $\mathscr{C}^1$, resp. Lipschitz functional calculus if, whenever $a \in \mathscr{D}_h$, $f(a) \in \mathscr{D}$ for each $\mathscr{C}^1$, resp. Lipschitz function f such that $f(0) = 0$.

The domain $\mathscr{D}$ of a derivation is said to be *self-adjoint* if it is closed under the * operation. A *-subalgebra of $L^2(\mathscr{A}, \tau) \cap L^\infty(\mathscr{A}, \tau)$ which is dense in $L^2(\mathscr{A}, \tau)$ is called a *Hilbert algebra*. A derivation δ is **-derivation* if $\mathscr{D}$ is self-adjoint and $\delta(a^*) = (\delta a)^*$.

Now we follow an argument in [Sa] which gives rise to a *non-abelian chain rule* [formula (4.3)] for the derivation of the functional calculus of a self-adjoint element. Let us fix a self-adjoint element $a \in \mathscr{A}$ and consider the representation π_a of the $\mathscr{C}^*$-algebra $\mathscr{C}_0(\mathbf{R}) \otimes \mathscr{C}_0(\mathbf{R}) \equiv \mathscr{C}_0(\mathbf{R}^2)$ (by $\otimes$ we mean any C^*-tensor product, see e.g. [T]) on $L^2(\mathscr{A}, \tau)$ given by

$$\pi_a(f \otimes g)\, b = f(a)\, b g(a), \qquad b \in L^2(\mathscr{A}, \tau),$$

and observe that

$$\mathrm{Range}(\pi_a) \subset \mathscr{A} \vee \mathscr{A}'. \tag{4.1}$$

For each real-valued Lipschitz function f, we set

$$\tilde{f}(s, t) = \begin{cases} \dfrac{f(s) - f(t)}{s - t} & s \neq t \\ f'(t) & s = t. \end{cases} \tag{4.2}$$

We observe that if $f \in \mathrm{Lip}_0(\mathbf{R}) := \{f : \mathbf{R} \to \mathbf{R},\ f \text{ Lipschitz},\ f(0) = 0\}$ then $\tilde{f} \in L^\infty(\mathbf{R}^2)$, and

$$\|f\|_{\mathrm{Lip}_0(\mathbf{R})} := \|\tilde{f}\|_\infty \equiv \|f'\|_\infty$$

is a Banach norm on $\mathrm{Lip}_0(\mathbf{R})$. Now we may state the main theorem of this section:

4.1. THEOREM. *Let δ be a closed derivation on $L^2(\mathscr{A}, \tau)$, $a \in \mathscr{D}_h$. Then the following properties hold*:

(i) *δ is closed under the Lipschitz functional calculus.*

(ii) *For each $f \in \mathscr{C}_0^1(\mathbf{R})$, $f(0) = 0$, one has*

$$\delta f(a) = \pi_a(\tilde{f})\, \delta a. \tag{4.3}$$

(iii) *For each $f \in \mathrm{Lip}_0(\mathbf{R})$ one has*

$$\|\delta f(a)\|_2 \leqslant \|f\|_{\mathrm{Lip}_0(\mathbf{R})}\, \|\delta a\|_2. \tag{4.4}$$

Remark. The best constant for the inequality (4.4) is given by $\inf\{\|g\|_{\mathrm{Lip}_0(\mathbf{R})} : g \in \mathrm{Lip}_0(\mathbf{R}),\ g|_{\sigma(a)} \equiv f|_{\sigma(a)}\}$.

4.2. LEMMA. *Let $f \in \mathrm{Lip}_0(\mathbf{R})$, φ a positive $\mathscr{C}^\infty$ function with support in $[-1, 1]$ s.t. $\int \varphi = 1$. Then, the sequence of functions $\{f_n\}$, $f_n(t) := f * \varphi_n(t) - f * \varphi_n(0)$, where $\varphi_n(t) := n\varphi(nt)$, verifies the following properties*:

(a) *$f_n \in \mathscr{C}_0^\infty(\mathbf{R})$ and $f_n(0) = 0$*

(b) $\|f - f_n\|_\infty \leqslant (2/n)\, \|f\|_{\mathrm{Lip}_0(\mathbf{R})}$

(c) $\|f_n\|_{\mathrm{Lip}_0(\mathbf{R})} \leqslant \|f\|_{\mathrm{Lip}_0(\mathbf{R})}$

(d) *$\tilde{f}_n \to \tilde{f}$ weak* in $L^\infty(\mathbf{R})$*

(e) *If g_n is in the convex hull of $\{f_k : k \geqslant n\}$, the sequence $\{g_n\}$ enjoys properties* (a)–(d).

The proof is trivial and is omitted.

4.3. LEMMA. *Let $\{A, \mathscr{D}(A)\}$ be a closed linear operator on $\mathscr{H}$, $\{x_n\} \subset \mathscr{D}(A)$ such that $\|x_n - x\| \to 0$ and exists $k > 0$ s.t. $\|Ax_n\| \leqslant k$. Then there exists w_n in the convex hull of $\{x_k : k \geqslant n\}$ s.t. $w_n \to x$ in the graph-norm of A. As a consequence $x \in \mathscr{D}(A)$ and $\|Ax\| \leqslant k$.*

Proof. See [MR, Lemma 2.12]. ∎

Proof of Theorem 4.1. First we observe that, since $a \in \mathscr{A}$, we may replace $\mathscr{C}_0^1(\mathbf{R})$ by $\mathscr{C}^1(I)$, $I := [-\|a\|, \|a\|]$. Then, equation (4.3) makes sense also for polynomials, and we check it for $f(t) = t^n$.

$$\begin{aligned}\delta(a^n) &= \sum_{j=0}^{n-1} a^j(\delta a)\, a^{n-j-1} \\ &= \sum_{j=0}^{n-1} \pi_a(s^j t^{n-j-1})\, \delta a \\ &= \pi_a\left(\frac{s^n - t^n}{s-t}\right) \delta a = \pi_a(\tilde{f})\, \delta a.\end{aligned}$$

By linearity, (4.3) holds for all polynomials p such that $p(0) = 0$. Finally we observe that for all such polynomials,

$$\begin{aligned}\|\pi_a(\tilde{p})\, \delta a\|_2 &\leqslant \|\pi_a(\tilde{p})\|\ \|\delta a\|_2 \\ &= \|p\|_{\mathrm{Lip}_0(I)}\ \|\delta a\|_2 \\ &= \|p\|_{\mathscr{C}^1(I)}\ \|\delta a\|_2.\end{aligned}$$

Therefore, if p_n is a sequence of polynomials converging to a $\mathscr{C}^1$ function f in the $\mathscr{C}^1(I)$ norm, then $\delta(p_n(a))$ is a Cauchy sequence w.r.t. the graph norm of δ, and (ii) follows by continuity.

In particular, we proved that δ is closed under the $\mathscr{C}^1$ functional calculus. Therefore, if f, φ_n are as in Lemma 4.2, formula (4.3) applies to f_n, and we get, using Lemma 4.2c,

$$\|\delta f_n(a)\|_2 = \|\pi_a(\tilde{f}_n)\, \delta a\|_2 \leqslant \|f\|_{\mathrm{Lip}_0(\mathbf{R})}\ \|\delta a\|_2.$$

Now we prove that

$$\|f(a) - f_n(a)\|_2 \to 0. \tag{4.5}$$

Indeed, choosing $\|f - f_n\|_\infty \leqslant \varepsilon^3$, we have

$$\begin{aligned}\|f(a) - f_n(a)\|_2^2 &= \tau\left(\int |f - f_n|^2\,(\lambda)\, de(\lambda)\right) \\ &\leqslant \int_{-\varepsilon}^{\varepsilon} \left(\left|\frac{f(\lambda)}{\lambda}\right| + \left|\frac{f_n(\lambda)}{\lambda}\right|\right)^2 d\mu(\lambda) \\ &\quad + \frac{1}{\varepsilon^2} \int_{|\lambda| \geqslant \varepsilon} |f - f_n|^2\, d\mu \\ &\leqslant 4\, \|f\|_{\mathrm{Lip}_0(\mathbf{R})}^2\, \mu([-\varepsilon, \varepsilon]) + \|a\|_2^2\, \varepsilon,\end{aligned}$$

where the measure μ is defined by $\mu(\Omega) := \tau(\int_\Omega \lambda^2 \, de(\lambda))$. Since $\mu(\{0\}) = 0$ and $\mu(\mathbf{R}) = \|a\|_2^2$ by definition, we get $\mu([-\varepsilon, \varepsilon]) \to 0$, which proves (4.5).

Finally we apply Lemma 4.3 to the sequence $f_n(a)$ in the domain of δ, and (i) and (iii) are proven. ∎

The following corollary is a consequence of Theorem 4.1.

4.4. COROLLARY. *Let δ_n, $n = 1, ..., N$, be closed derivations. If $\delta := \sum_n \delta_n$, $\mathscr{D}(\delta) = \bigcap_n \mathscr{D}(\delta_n)$, then δ is closed under the Lipschitz functional calculus and, for each $f \in \mathrm{Lip}_0(\mathbf{R})$,*

$$\|\delta f(a)\|_2 \leqslant \|f\|_{\mathrm{Lip}_0(\mathbf{R})} \|\delta a\|_2, \qquad \forall a \in \mathscr{D}(\delta)_h.$$

Proof. By Theorem 4.1 δ is closed under Lipschitz functional calculus and, by linearity, (4.3) holds for $\mathscr{C}^1$ functions. Let f and f_n be as in Lemma 4.2. Then, as in the proof of Theorem 4.1, we get

$$\|\delta f_n(a)\|_2 = \|\pi_a(\tilde{f}_n) \, \delta a\|_2 \leqslant \|f\|_{\mathrm{Lip}_0(\mathbf{R})} \|\delta a\|_2$$

and $\|f_n(a) - f(a)\|_2 \to 0$. Then, by Lemma 4.3, we get a sequence $\{h_n\}$ in the convex hull of $\{f_n\}$ such that $h_n(a) \to f(a)$ in the graph-norm of δ_1. Since $\{h_n(a)\}$ is bounded in the graph-norm of δ_2, applying again Lemma 4.3, we find a sequence in the convex hull of $\{f_n\}$ converging to $f(a)$ in the graph-norms of δ_1 and δ_2. Iterating this procedure N times, we find a sequence $\{g_n\}$ in the convex hull of $\{f_n\}$ s.t. $g_n(a)$ converges to $f(a)$ in the graph-norms of δ_i, for all i. Therefore $g_n(a) \to f(a)$ in the graph-norm of δ. Finally

$$\|\delta f(a)\|_2 = \lim \|\delta g_n(a)\|_2 \leqslant \|f\|_{\mathrm{Lip}_0(\mathbf{R})} \|\delta a\|_2. \quad ∎$$

4.5. *Remark.* We would like to compare Theorem 4.1 with an analogous result of Powers ([Po], cf. Theorem 1.6.2 in [B]) for the derivations on a C^*-algebra. While the difference in the formulation of the non-abelian chain-rule (Equation 4.3) is just a matter of taste, the difference on the allowed functional calculus depends on the different norms. Indeed, let $\mathscr{M}$ be the C^*-algebra generated by a self-adjoint element in $\mathscr{A}$. The representation of the tensor product $\mathscr{M} \otimes \mathscr{M}$ given by the left and right actions $\mathscr{M}$ on $L^2(\mathscr{A}, \tau)$ extends to a representation of the C^*-tensor product. This guarantees that Equation (4.3) holds for the closure of the polynomials in the appropriate norm, i.e. for $\mathscr{C}^1$ functions. On the contrary, if the abelian C^*-algebra $\mathscr{M}$ acts on $\mathscr{A}$, the tensor product is embedded in the Banach algebra $\mathscr{B}(\mathscr{A})$, and this embedding is not necessarily continuous in the C^* norm. Indeed McIntosh disproves one of Power's claims by producing a counterexample to (4.3), i.e. a *-automorphism group of a C^*-algebra, with generator δ, a self-adjoint element $a \in D(\delta)$, and a $\mathscr{C}^1$ function f such that $f(a) \notin D(\delta)$ [Mc].

Theorem 4.1 gives a general answer to the problem of the Lipschitz functional calculus of a self-adjoint operator in the domain of a derivation. On the other hand there is a trivial form of “Lipschitz” functional calculus that makes sense also for non self-adjoint operators, i.e. the modulus of an operator. Now we are going to study this question.

4.6. LEMMA. *Let δ be a derivation on $\mathscr{D}$. Then the operator $\delta^\dagger : \mathscr{D}^* \to L^2(\mathscr{A}, \tau)$ defined by*

$$\delta^\dagger a := (\delta a^*)^*, \qquad a \in \mathscr{D}^*,$$

is a derivation. If δ is closed (closable), also $\delta^\dagger$ is.

Proof. The Leibnitz rule for $\delta^\dagger$ follows by a straightforward calculation. The equivalence of the closability properties follows by the equality

$$\|a\|_{\delta^\dagger} = \|a^*\|_\delta, \qquad \forall a \in \mathscr{D}^*. \quad \blacksquare$$

4.7. THEOREM. *Let δ be a closed derivation on a self-adjoint domain $\mathscr{D}$. Then, if $a \in \mathscr{D}$, $|a| \in \mathscr{D}$ and*

$$\|\delta \mid a|\|_2^2 \leqslant \|\delta a\|_2^2 + \|\delta(a^*)\|_2^2. \tag{4.6}$$

Proof. Consider the functions

$$\varphi_n(t) = \sqrt{t + \frac{1}{n^2}} - \frac{1}{n} \qquad t \geqslant 0 \qquad \text{and} \qquad \psi_n(t) = \sqrt{t^2 + \frac{1}{n^2}} - \frac{1}{n} \qquad t \in \mathbf{R}.$$

Since $a \in \mathscr{D}$, then a^*, and therefore a^*a, are in $\mathscr{D}$ and

$$\psi_n(|a|) = \varphi_n(a^*a) \in \mathscr{D}$$

by the theorem on the Lipschitz functional calculus.

Now consider the chain of inequalities

$$\begin{aligned}
\|\delta\psi_n(|a|)\|_2^2 &\leqslant \|\delta\psi_n(|a|)\|_2^2 + \|\delta\psi_n(|a^*|)\|_2^2 \\
&= \left\| \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} \psi_n(|a|) & 0 \\ 0 & \psi_n(|a^*|) \end{pmatrix} \right\|_2^2 \\
&= \left\| \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} \psi_n\left(\begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix} \right) \right\|_2^2 \\
&\leqslant \left\| \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix} \right\|_2^2 \\
&= \|\delta a\|_2^2 + \|\delta(a^*)\|_2^2,
\end{aligned}$$

where the main inequality follows by the theorem on Lipschitz functional calculus applied to $\left(\begin{smallmatrix} \delta & 0 \\ 0 & \delta \end{smallmatrix}\right)$, which is obviously a closed derivation on $\mathscr{D} \otimes M_2$.

Now it is easy to see that $\psi_n(|a|) \to |a|$ in the L^2 norm, and therefore the result follows from Lemma 4.3. ∎

We remark that, according to the terminology in [DL1], Theorems 4.1 and 4.7 imply that each closed *-derivation on a Hilbert algebra is a Dirichlet derivation.

A natural question related to Theorem 4.1 is the following: when the non-abelian chain rule given in (4.3) extends to the Lipschitz functional calculus? The first problem is that $\tilde{f}$ is not necessarily in the domain of π_a, when f is a Lipschitz function. Indeed π_a may easily be extended to the C^* tensor product of $L^\infty(\mathbf{R})$ with itself, but this space is smaller than $L^\infty(\mathbf{R}^2)$ and therefore does not contain $\tilde{f}$ in general. Even though we do not try to give a general answer to the previous question, in the following proposition we mention two extremal cases in which the addressed question has a positive answer, the abelian case and the type I factor case.

4.8. Proposition. *Let $\mathscr{A}$ be either an abelian algebra or a type I factor, and δ a closed derivation on $L^2(\mathscr{A}, \tau)$. Then, for each self-adjoint a in the domain of δ, the map π_a extends to $L^\infty(\mathbf{R}^2)$. Therefore, the non abelian chain-rule given in* (4.3) *extends to Lipschitz functional calculus.*

Proof. It is well known that, if $\mathscr{A}$ is a type I factor and π_L, resp. π_R, denote the representation of $\mathscr{A}$ in $\mathscr{B}(L^2(\mathscr{A}, \tau))$ as left, resp. right multiplication on $L^2(\mathscr{A}, \tau)$, the von Neumann algebra generated by $\pi_L(\mathscr{A})$ and $\pi_R(\mathscr{A})$ is isomorphic to $\mathscr{A} \otimes \mathscr{A}$ (the von Neumann tensor product) [T]. Therefore in this case the map π_a extends to a normal representation of $L^\infty(\sigma(a) \times \sigma(a))$, for each self-adjoint a in $\mathscr{A}$. Then, let f and f_n be as in Lemma 4.2. By Lemma 4.2(d), $\pi_a(\tilde{f}_n)\, \delta a \to \pi_a(\tilde{f})\, \delta a$ weakly in $L^2(\mathscr{A}, \tau)$, and also $\delta g_n(a) \to \delta f(a)$ in $L^2(\mathscr{A}, \tau)$ for g_n suitably chosen in the convex hull of $\{f_k : k \geqslant n\}$, as in Lemma 4.3. Then the thesis holds by Lemma 4.2(e). If $\mathscr{A}$ is abelian, formula 4.3 becomes the usual chain rule, and by normality of the map $f \to f(a)$, $a \in \mathscr{A}_h$, we get the thesis. ∎

4.9. *Remark.* We remark that the key property we used in the proof of the factor case, i.e. the fact that the von Neumann algebra generated by the right and left action of any abelian subalgebra of $\mathscr{A}$ on $L^2(\mathscr{A}, \tau)$ is isomorphic to the tensor product of $\mathscr{A}$ with itself, is a characterization of type I factors (see Corollary 2.9 in [BDL]), therefore the property we are studying is probably confined to type I algebras.

We conclude this section with an example of a simple Dirichlet form associated with a general derivation.

4.10. PROPOSITION. *Let δ be a closed derivation on a Hilbert algebra $\mathscr{D}$. Then the form*

$$\mathscr{E}(x, y) = Re(\delta x, \delta y) + Im(\delta x, \delta y), \qquad x, y \in \mathscr{D}_h, \tag{4.7}$$

extends by sesquilinearity to a Dirichlet form.

Proof. The form $\mathscr{E}$ is closed iff its symmetric part $Re(\delta x, \delta y)$ is, that is iff δ is closed. The weak sector condition follows by

$$|\mathscr{E}(x, y)|^2 \leqslant 2\, |(\delta x, \delta y)|^2 \leqslant 2\mathscr{E}(x, x)\, \mathscr{E}(y, y).$$

Now let us define the operators

$$\left.\begin{aligned} d_1 a &= \frac{\delta a + \delta^\dagger a}{2}, \\ d_2 a &= \frac{\delta a - \delta^\dagger a}{2i}, \end{aligned}\right. \qquad a \in \mathscr{D}.$$

It is clear that d_1, d_2 are closed *-derivations and

$$\mathscr{E}(x, y) = \sum_{i, j \in \{1, 2\}} a_{ij}(d_i x, d_j y),$$

where $(a_{ij}) = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$. Now Dirichlet property follows from the general theorem in the following section. ∎

As the proof of the preceding proposition shows, the study of the Dirichlet form in (4.7) may be reduced to the case of *-derivations. Therefore in the following section only *-derivations will be considered.

5. EXPLICIT CONSTRUCTIONS OF DIRICHLET FORMS

The aim of this section is to describe a class of Dirichlet forms which can be considered as the non commutative generalization of a class of (generally non symmetric) commutative Dirichlet forms studied in [MR]. At the same time these non commutative examples also extend previous ones constructed in [DL1].

We start considering the following leading example taken from the commutative context. Let $B=[b_{ij}]$ be an element of $L^1_{\mathrm{loc}}(U)\otimes M_n$, $U\subset\mathbf{R}^n$ open. Then we define the bilinear form on $\mathscr{C}_0^\infty(U)$

$$\mathscr{E}(u,v):=\sum_{i,j=1}^{n}\int b_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx. \tag{5.1}$$

Under the following simple assumptions: there exists $0<\nu<\infty$ such that

$$\begin{aligned}&\sum_{i,j=1}^{n}{}^{s}b_{ij}(x)\,\xi_i\xi_j\geqslant\nu\,\|\xi\|^2, &&\forall\xi=(\xi_1,...,\xi_n),\\ &{}^{a}b_{ij}\in L^\infty(U,dx), &&1\leqslant i,j\leqslant n.\end{aligned} \tag{5.2}$$

where ${}^{s}B$ (resp. ${}^{a}B$) is the symmetric (resp. antisymmetric) part of B, it can be proven that the form $\{\mathscr{E},\mathscr{D}(\mathscr{E})\}$ is closable and its closure is a Dirichlet form (cf. [MR]).

Now we discuss some generalizations of the preceding example to the non commutative context. The general setting is the following: we have a sesquilinear form of the type

$$\mathscr{E}(x,y):=\sum_{i=1}^{n}(d_ix,b_{ij}\,d_jy), \tag{5.1$'$}$$

where d_i, $i=1\cdots n$, is a family of *-derivations on $L^2(\mathscr{A},\tau)$ and the b_{ij}'s belong to the center of $\mathscr{A}$. In this case condition (5.2) is replaced by its proper non-commutative analogue:

$$\begin{aligned}&{}^{s}B\geqslant\nu I,\\ &{}^{a}b_{ij}\in L^\infty(\mathscr{A},\tau), \qquad 1\leqslant i,j\leqslant n.\end{aligned} \tag{5.2$'$}$$

Our first result is Theorem 5.1. This theorem deals with *general* *-derivations with a dense common domain. In this case, because of such generality, we need stronger requirements in order to get closedness for the form (5.1′). This is obtained by asking the symmetric part of the matrix B to be the identity matrix, which makes the first condition in (5.2′) automatically fulfilled.

In Theorem 5.2 the derivations d_i are given by commutators $[z_i,\cdot]$, where the z_i's are skew-symmetric elements in L^2+L^∞, which provides the closedness of the form $\mathscr{E}$. In this case conditions (5.2′) suffice to get the result. Such a theorem is a partial generalization of Theorem 6.10 in [DL1].

5.1. THEOREM. *Assume we are given a family*

$$d_i\colon \mathcal{D}_i \subset L^2(\mathcal{A}, \tau) \to L^2(\mathcal{A}, \tau), \qquad i = 1, ..., n,$$

*of *-derivations over Hilbert algebras* $\mathcal{D}_i$ *such that*

(a) *each* d_i *is closable,*

(b) $\mathcal{D} := \bigcap_{i=1}^n \mathcal{D}_i$ *is dense,*

and consider the form $\mathcal{E}$ *given by*

$$\begin{aligned} &\mathcal{D}(\mathcal{E}) := \mathcal{D} \\ &\mathcal{E}(x, y) := \sum_{i=1}^n (d_i x, d_i y) + \sum_{i,j=1}^n (d_i x, c_{ij}\, d_j y), \end{aligned} \tag{5.3}$$

where the c_{ij}*'s are self-adjoint elements in the center of* $\mathcal{A}$ *such that* $c_{ij} = -c_{ji}$. *Then the form is closable and its closure is a Dirichlet form.*

Proof. Sesquilinearity of $\mathcal{E}$ being evident, we prove real positivity. We have

$$\mathcal{E}(x, x) = \sum_{i=1}^n \|d_i x\|^2 + \tau \otimes tr(CA(x, x)) \tag{5.4}$$

with $C = [c_{ij}] \in L^\infty(\mathcal{A}, \tau) \otimes M_n$ and $A(x, y) \in L^1(\mathcal{A}, \tau) \otimes M_n$ is given by $A(x, y) = [d_i y\, d_j x]$. Since C is a real antisymmetric matrix and $A(x, x)$ is a real symmetric matrix when $x \in \mathcal{D}_h$, the last term in the right hand side of (5.4) vanishes, and the real positivity of $\mathcal{E}$ follows. Because of hypothesis (a) the form ${}^s\mathcal{E}$ is closable (see e.g. [DL1]) and therefore, by definition, $\mathcal{E}$ is closable. We now prove that the weak sector condition holds for $\mathcal{E}$, i.e. there exists $0 < K < \infty$ such that

$$|{}^a\mathcal{E}(x, y)| \leqslant K\, {}^s\mathcal{E}_1(x, x)^{1/2}\, {}^s\mathcal{E}_1(y, y)^{1/2}$$

for all $x, y \in \mathcal{D}_h$, where ${}^s\mathcal{E}_1(x, y) := {}^s\mathcal{E}(x, y) + (x, y)$.

Setting $M := \|C\|_\infty$ and applying Hölder and Schwartz inequalities we get

$$\begin{aligned} |{}^a\mathcal{E}(x, y)| = |\tau \otimes tr(CA(x, y))| &\leqslant M\, \|A(x, y)\|_1 \\ &\leqslant M \sum_{i,j=1}^n \|A(x, y)_{ij}\|_1 \leqslant M \sum_{i,j=1}^n \|d_j x\|_2\, \|d_i y\|_2 \\ &\leqslant nM \left(\sum_{j=1}^n \|d_j x\|_2^2 \right)^{1/2} \left(\sum_{i=1}^n \|d_i y\|_2^2 \right)^{1/2} \\ &\leqslant nM\, {}^s\mathcal{E}_1(x, x)^{1/2}\, {}^s\mathcal{E}_1(y, y)^{1/2}. \end{aligned}$$

It remains to prove that the closure of the form (5.3) is a Dirichlet form. Let $\overline{\mathscr{E}}$ denote the closure of the form $\mathscr{E}$, which is obtained replacing the d_i's with their closures $\bar{d}_i$ (cf. [DL1]). We notice that if $x \in \mathscr{D}(\overline{\mathscr{E}})$ then $\varphi_0(x) := x^+ \wedge 1 \in \mathscr{D}(\overline{\mathscr{E}})$ because this holds for each $\mathscr{D}_i$. Hence we claim that

$$\overline{\mathscr{E}}(x \mp \varphi_0(x), x \pm \varphi_0(x)) \geqslant 0, \qquad \forall x \in \mathscr{D}(\overline{\mathscr{E}})_h. \tag{5.5}$$

First we observe that the matrix $[(\bar{d}_i x, c_{ij}\, \bar{d}_j \varphi_0(x))]$ is antisymmetric. It is enough to show this replacing φ_0 with a smooth approximation φ. Then, by Equations (4.1) and (4.3) and by the anti-symmetry of C, we get

$$\begin{aligned}(\bar{d}_i x, c_{ij}\, \bar{d}_j \varphi(x)) &= (\bar{d}_i x, c_{ij} \pi(\tilde{\varphi})\, \bar{d}_j x)\\ &= (\pi(\tilde{\varphi})\, \bar{d}_i x, c_{ij}\, \bar{d}_j x)\\ &= (\bar{d}_i \varphi(x), c_{ij}\, \bar{d}_j x)\\ &= -(\bar{d}_j x, c_{ji}\, \bar{d}_i \varphi(x)).\end{aligned}$$

Therefore, using again the antisymmetry of C, we have

$$\begin{aligned}\overline{\mathscr{E}}(x \mp \varphi_0(x), x \pm \varphi_0(x)) &= \sum_{i=1}^{n} (\bar{d}_i(x \mp \varphi_0(x)), \bar{d}_i(x \pm \varphi_0(x)))\\ &= \overline{{}^s\mathscr{E}}(x \mp \varphi_0(x), x \pm \varphi_0(x)).\end{aligned}$$

Since $\overline{{}^s\mathscr{E}}$ is Dirichlet, by Theorem 4.1 and [DL1], Theorem 6.10, $\overline{\mathscr{E}}$ is Dirichlet too. ∎

5.2. THEOREM. *Let $z_1, ..., z_n$ be skew-adjoint elements in $L^2 + L^\infty$, define*

$$d_i(x) := z_i x - x z_i, \qquad \forall x \in L^2 \cap L^\infty,$$

and let $B = [b_{ij}]$ be a matrix of self-adjoint elements in the center of $\mathscr{A}$ such that condition (5.2′) holds. Then, the form

$$\mathscr{D}(\mathscr{E}) := L^2 \cap L^\infty$$

$$\mathscr{E}(x, y) := \sum_{i,j=1}^{n} (d_i x, b_{ij}\, d_j y)$$

is closable and its closure is a Dirichlet form.

Proof. Let us denote by A the square root of the symmetric part of B as an element in $L^\infty(\mathscr{A}, \tau) \otimes M_n$. We also set $\delta_i := \sum a_{ij}\, d_j$ and

$C := A^{-1}({}^aB)A^{-1}$. We notice that, since ${}^sB \geqslant vI$, A^{-1} is bounded and C is bounded, real and skew-symmetric. Then,

$$\mathscr{E}(x, y) = \sum_{i=1}^{n} (\delta_i x, \delta_j y) + (\delta_i x, c_{ij}\, \delta_j y).$$

Since $z_i \in L^2 + L^\infty$, $i = 1, ..., n$ and A is bounded, $w_i := \sum a_{ij} z_j \in L^2 + L^\infty$. Then δ_i, being implemented by w_i, is a closable derivation on $L^2 \cap L^\infty$ (see e.g. [DL1]) and the thesis follows by Theorem 5.1. ∎

Acknowledgments

The authors thank S. Albeverio for suggesting this line of research and R. Longo for discussions. They also thank the organizers of the Nottingham Conference on Quantum Probability (March 1993), where the results of this paper were announced. The authors are grateful to the referee for pointing out a gap in the original proof of Theorem 4.7 and for a very careful reading of the manuscript.

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