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Potential-Like Operators and Extensions of Hunt's Theorem for σ-Compact Spaces*

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Let Ω be a locally compact (Hausdorff) space. $C(\Omega, F) = \{$all continuous F-valued functions on $\Omega\}$, where $F =$ either the reals, $\mathbb{R}$, or the complexes, $\mathbb{C}$. $C_0(\Omega, F) = \{f \in C(\Omega, F)$: f vanishes at infinity$\}$. $C_{00}(\Omega, F) = \{f \in C(\Omega, F)$: f has compact support$\}$.

A celebrated theorem of Hunt [2, 4] asserts: Suppose A is a linear operator whose domain $D(A)$ contains $C_{00}(\Omega, \mathbb{R})$ and is contained in $C_0(\Omega, \mathbb{R})$, whose range is dense in $C_0(\Omega, \mathbb{R})$, and which satisfies the (Cartan–Deny) complete maximum principle, then there exists a Feller semigroup $\{P_t\}$ such that $Af = \int_0^\infty P_t f\, dt$, $\forall f \in C_{00}(\Omega, \mathbb{R})$, ($\Omega$ is assumed to be σ-compact); more generally, Lion [7], assuming a weak form of maximum principle and without supposing density of the range, shows the existence of a sub-markovian resolvent family $\{R_\lambda\}_{\lambda>0}$ such that $Af = \lim_{\lambda\to 0^+} R_\lambda f$, $\forall f \in C_{00}(\Omega, \mathbb{R})$. This was further extended, quite recently, in a useful and interesting way by Hirsh, [3], replacing maximum principles by codissipativeness (see definition below), and working in $C_0(\Omega, \mathbb{C})$ without "positivity" and without assuming σ-compactness.

Here we place ourselves in the σ-compact case (by far the most important for applications) and show that for any codissipative operator A in $C_0(\Omega, \mathbb{C})$, defined on $C_{00}(\Omega, \mathbb{C})$, $\exists \psi > 0$ in $C_0(\Omega, \mathbb{R})$ such that A extends codissipatively to all $f \in C_0(\Omega, \mathbb{C})$ which in modulus are $O(\psi)$ near infinity (all such f belong to $D(\bar{A})$, $\bar{A} =$ closure of the closeable operator A). This fact of some independent interest, and not available from [3], permits us to give a rather transparent

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"from-scratch" proof of the above mentioned result of Hirsch (and an extension to a somewhat wider class of operators), for Ω σ-compact.

For historical background, a more ample view of the subject and its ramifications, the reader may consult [2, 3, 6] and references given there.

A will henceforth, unless indicated otherwise, denote a linear operator with domain $D(A)$, and range $R(A)$, $\subset C_0(\Omega, \mathbb{C})$ considered as a Banach space under sup-norm. We recall, the following definition (see [3]).

DEFINITION 1. A is called codissipative if $\forall f \in D(A)$,

$$\| f + \lambda A f \| \geqslant \| \lambda A f \| \quad \forall \lambda > 0. \tag{1}$$

It is well known, and also follows from Lemma 9 that a codissipative densely defined operator A is closeable. Below, for $f \in C(\Omega, F)$, we write "supp f" to mean "support of f."

LEMMA 2. *Suppose A is closeable and $C_{00}(\Omega, \mathbb{C}) \subset D(A)$. Then for each open $V \subset \Omega$ with compact closure, $\exists$ a constant $C_V \geqslant 0$ such that $\| Af \| \leqslant C_V \| f \|$ whenever* supp $f \subset V$.

Proof. Let X_1 denote $C_0(V, \mathbb{C})$ as a Banach space under sup-norm. Let $T\colon X_1 \mapsto X = C_0(\Omega, \mathbb{C})$ be defined by $Tf = f$ on V, $= 0$ on $\complement V$. T is isometric on X_1. Set $A_1 = A \circ T$. A_1 is closed, X_1 and X are complete, $D(A_1) = X_1$; hence, A_1 is bounded and $\| A_1 \|$ provides a C_V as required. Q.E.D.

THEOREM 3. *Suppose A is closeable and $C_{00}(\Omega, \mathbb{C}) \subset D(A)$. Then $\exists \psi$ in $C_0(\Omega, \mathbb{R})$, > 0 on all of Ω, such that every $f \in C_0(\Omega, \mathbb{C})$ satisfying*

$$| f | = O(\psi), \text{ near infinity} \tag{2}$$

is in $D(\bar{A})$. ((2) means, of course, for the given f, $\exists K$ compact $\subset \Omega$ and a constant $\alpha \geqslant 0$ such that $| f | \leqslant \alpha \psi$ off K).

Proof. For $n = 1, 2, \ldots$, we define intervals $I_n \subset \mathbb{R}$ by

$$I_n = \left(\left(1 - \frac{1}{10}\right)/2^n, \left(1 + \frac{1}{10}\right)/2^{n-1}\right).$$

Ω being σ-compact $\exists \varphi \in C_0(\Omega, \mathbb{R})$, >0, $\| \varphi \| = 1$. Set $J_n = \varphi^{-1}(I_n)$, and $\Omega_n = \bigcup_{k=1}^{k=n} J_k$. $\bar{\Omega}_n$ is compact; let $C_n = C_{\Omega_n}$ be a corresponding constant as provided by lemma 2. We may assume that $1 \leqslant C_1 \leqslant$

$C_2 \leqslant \cdots$. It is easy to construct $\rho \in C((0, 1], \mathbb{R})$, >0, such that $\rho \leqslant 1/2^{n+1}C_{n+2}$ on I_n.

We shall show that $\psi = \rho \circ \varphi$ has the properties claimed in the statement of this theorem. Let $\varphi_n \in C_{00}(\Omega, \mathbb{R})$, $= 1$ on J_n, $= 0$ on J_k for $|k - n| > 1$, with $0 \leqslant \varphi_n \leqslant 1$ everywhere. Then $\psi_n = \varphi_n(\sum_{k=1}^{\infty} \varphi_k)^{-1}$ is well defined, in $C_{00}(\Omega, \mathbb{R})$, and $\sum_{n=1}^{\infty} \psi_n = 1$. Suppose $f \in C_0(\Omega, \mathbb{C})$ satisfies (2). Let $f_n = \sum_{k=1}^{k=n} f\psi_k$, then $f_n \in D(A)$, $\| f - f_n \| \to 0$. Since $\operatorname{supp}(f\psi_k) \subset \Omega_{k+1}$, we have for k large $\| Af\psi_k \| \leqslant C_{k+1} \| f\psi_k \| \leqslant C_{k+1}\alpha(1/2^k C_{k+1}) = \alpha/2^k$. Thus, the $Af_n = \sum_{k=1}^{k=n} Af\psi_k$ form a Cauchy sequence, and hence, $f \in D(\bar{A})$, $\bar{A}f = \sum_{k=1}^{\infty} Af\psi_k$.

Q.E.D.

REMARKS 4. There is a great deal of freedom in constructing positive functions ψ that have the properties required in the previous theorem. In particular, given K compact in Ω one can easily, by modifying ρ in the proof above, obtain $\psi = 1$ on K, and $0 < \psi \leqslant 1$ everywhere.

Of course, the classical situation which Theorem 3 generalizes to the $C_0(\Omega, \mathbb{C})$ setup, is that of the potential (codissipative) operator A defined on $C_{00}(\mathbb{R}^3, \mathbb{C})$ by

$$(Af)(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x - \xi|} f(\xi)\, d\xi.$$

In this case one may take $\psi(x) = 1/(1 + |x|^3)$, (where of course x stands for $(x_1, x_2, x_3) \in \mathbb{R}^3$), or some other appropriate function which is $O(1/|x|^\alpha)$, $\alpha > 2$, near infinity.

The following characterization of codissipativeness, which we shall need, is proved in [3], but the proof we give below is so simple that it seems worth the slight duplication.

THEOREM 5. *A is codissipative if and only if* $\forall f \in D(A)\ \exists x \in \Omega$ *with* $|(Af)(x)| = \| Af \|$ and

$$\operatorname{Re} f(x)\overline{(Af)(x)} \geqslant 0, \tag{3}$$

"—" *denoting here as usual "complex conjugate of," "*Re*" means "real part of."*

Proof. Suppose A is codissipative. $\forall t > 0$, $\exists x_t \in \Omega$ with $\| tf + Af \| = |(tf + Af)(x_t)| \geqslant \| Af \|$. We may assume $\| Af \| = 1$ and (as $t \to 0$)

$x_t \to x \in \Omega$. Then $1 \leqslant |(Af)(x_t)|^2 + 2t\,\mathrm{Re} f(x_t)\overline{(Af)(x_t)} + O(t^2)$ leads to $\mathrm{Re} f(x)\overline{(Af)(x)} \geqslant 0$, while $|(Af)(x)| = 1$; conversely, if $x \in \Omega$ is as in our statement,

$$\| tf + Af \| \, \| Af \| \geqslant |(tf + Af)(x)\overline{(Af)(x)}| = | tf(x)\overline{(Af)(x)} + \| Af \|^2 | \geqslant \| Af \|^2$$

since (3) holds. Hence, $\| tf + Af \| \geqslant \| Af \|, \forall t > 0$, so A is codissipative Q.E.D.

For what follows, we recall explicitely some general notions and results valid for an arbitrary Banach space X. They can all be found in [3]. $B(X)$ will denote the algebra of all bounded everywhere defined linear operators on X. An L_0-resolvent family (on X), $\{R_\lambda\}_{\lambda>0}$, is a family of operators $R_\lambda \in B(X)$ depending on $\lambda > 0$ in such a way that

$$R_\lambda - R_\mu = (\mu - \lambda)\, R_\lambda R_\mu \qquad \forall \lambda, \mu > 0 \tag{4}$$

and $\forall f \in X$, $s\text{-}\lim_{\lambda\to 0^+} \lambda R_\lambda f = 0$. The family $\{R_\lambda\}_{\lambda>0}$ is called contractive if $\| \lambda R_\lambda \| \leqslant 1, \forall \lambda > 0$. $A = s\text{-}\lim_{\lambda \to 0^+} R_\lambda$ is called the cogenerator of $\{R_\lambda\}_{\lambda>0}$. One calls A a precogenerator if it is closeable and its closure is a cogenerator. One has the following general fact:

(i) Let A be a densely defined linear operator with domain and range in X. A is the precogenerator of a contractive L_0-resolvent family if and only if it is codissipative and for some $\lambda_0 > 0$(and then for all $\lambda_0 > 0$) $R(1 + \lambda_0 A)$ is dense (1 = identity operator).

We now return to our $C_0(\Omega, \mathbb{C})$ setting and prove the following.

THEOREM 6 (Hirsch). *Suppose $D(A) \supset C_{00}(\Omega, \mathbb{C})$. Then A is the precogenerator of a contractive L_0-resolvent family if and only if it is codissipative.*

Proof. Suppose A is codissipative. By (i) it suffices to show that $R(1 + \bar{A})$ is dense. Let ν be a totally finite measure "orthogonal" to $R(1 + \bar{A})$. For ψ as in Theorem 3 define A_ψ by $A_\psi f = \bar{A}(\psi f)$. $D(A_\psi) = C_0(\Omega, \mathbb{C}) = X$, and since $\psi > 0$, it follows from Theorem 5 that A_ψ is codissipative, hence, closed and, thus, bounded. So $(1 + \lambda A_\psi)^{-1} \in B(X)$ for λ small and, thus, by (i) for all $\lambda > 0$. If $f \in X$, $f = (1 + A_\psi)g$, with $\| g \| \leqslant 2 \| f \|$,

$$\int f d\nu = \int (1 + \bar{A})\, \psi g \, d\nu + \int (1 - \psi) g \, d\nu = \int (1 - \psi) g \, d\nu.$$

Hence,

$$\left| \int f\,d\nu \right| \leqslant 2 \|f\| \int | 1 - \psi | \, d|\nu|. \tag{5}$$

Considering Remarks 4, we see that $\int f d\nu = 0$. Thus, $\nu = 0$. Q.E.D.

We shall extend somewhat the above result along closely related lines making further use of Theorem 3. For this purpose we first examine codissipativeness and the previous theorem in terms of "numerical ranges." $X = C_0(\Omega, \mathbb{C})$; let $\{f, g\}$ be a generic element of $X \times X$. Consider a map $\sigma: X \times X \mapsto \Omega$ such that

$$\{f, g\} \to x^{\sigma}_{f,g} \in \{x \in \Omega: |g(x)| = \|g\|\}.$$

There are many such maps; let us consider one, σ, and associate to every ordered pair f, $g \in X$ the number (which may be called the pseudo-inner-product of f and g):

$$[f \mid g]_\sigma = f(x^{\sigma}_{f,g}) \overline{g(x^{\sigma}_{f,g})}. \tag{6}$$

(When there is no need of naming explicitely the map σ we have chosen, we write $[f \mid g]$ in lieu of $[f \mid g]_\sigma$). $[\ \mid\ ]$ has the following properties

$$[g \mid g] = \|g\|^2 \qquad \forall g \in X, \tag{7}$$
$$|\alpha[f \mid g] + \beta[g \mid g]| \leqslant \|\alpha f + \beta g\| \, \|g\| \qquad \forall f, g \in X \quad \text{and} \quad \alpha, \beta \in \mathbb{C}.$$

Given an operator A, we define the numerical corange of A (relative to σ), $N(A)$, (we write $N_\sigma(A)$ if there is need to explicitly name σ), by

$$N(A) = \{[f \mid Af] : f \in D(A), \|Af\| = 1\}. \tag{8}$$

Remarks 7. The $[\ \mid\ ]_\sigma$ introduced above are a special case of a general abstract notion concerning vector spaces, characterized by the properties (7), intimately linked with (and also characterized in terms of) semi-inner-products. Furthermore, it could be shown that in all the situations we deal with here, we could choose a general pseudo-inner-product $[f \mid g]$ which is linear in f and, thus, is a semi-inner-product. (Actually in some more complicated situations it turns out that the pseudo-inner-products $[\ \mid\ ]$ are sometimes more convenient to work with than semi-inner-products). However, none of these facts is directly needed here, nor will be proved or used here. The meaning of $N(A)$ above is similar to that of the numerical ranges studied in [1, 5], references which the interested reader may consult.

In terms of numerical corange, Theorem 5 can now be reformulated (as can Theorem 6) as follows: "A is codissipative if and only if $N(A)$ is in the half-plane $\{z \in \mathbb{C}: \operatorname{Re} z \geqslant 0\}$, for an appropriate pseudo-inner-product."

THEOREM 8. *Suppose* $D(A) \supset C_{00}(\Omega, \mathbb{C})$. *Suppose* $\exists$ *a number* $\theta \in \mathbb{R}$, $0 < \theta \leqslant \pi/2$, *such that* $\forall\lambda < 0$ *the distance from* λ *to* $N(A)$, $d(\lambda, N(A))$, *is* $\geqslant |\lambda| \sin\theta$. *Then* A *precogenerates an* L_0*-resolvent family* $\{R_\lambda\}_{\lambda>0}$ *with* $\|\lambda R_\lambda\| \leqslant (\sin\theta)^{-1}$ $\forall\lambda > 0$.

For the proof of Theorem 8 we need the following lemma.

LEMMA 9. *Suppose* A *satisfies* $d(\lambda, N(A)) \geqslant \sin\theta\,|\lambda|$ $\forall\lambda < 0$, *where* θ *is a real constant*, $0 < \theta \leqslant \pi/2$. *Then* $\forall f \in D(A)$,

$$(\sin\theta)^{-1}\|f + \lambda Af\| \geqslant \|\lambda Af\| \qquad \forall\lambda > 0. \tag{9}$$

If $(1 + \lambda_0 A)^{-1} \in B(X)$ *for one* $\lambda_0 > 0$, *then* $(1 + \lambda A)^{-1}$, $A(1 + \lambda A)^{-1} \in B(X)$ *for all* $\lambda > 0$, *and* $\|(1 + \lambda A)^{-1}\| \leqslant 1 + 1/\sin\theta$, $\|A(1 + \lambda A)^{-1}\| \leqslant 1/(\sin\theta)\lambda$, $\forall\lambda > 0$. *If* $D(A)$ *is dense in* X, *then* A *is closeable.*

Proof. Let $f \in D(A)$, $\|Af\| = 1$, $\lambda > 0$. Then by (7), $\|f + \lambda Af\| \geqslant |[f \mid Af] + \lambda| \geqslant d(-\lambda, N(A)) \geqslant (\sin\theta)\lambda$. Replacing f, by $f/\|Af\|$ for the general case $\|Af\| > 0$, we obtain (9). It follows at once that if $(1 + \lambda A)^{-1} \in B(X)$ for a $\lambda > 0$, then $A(1 + \lambda A)^{-1} \in B(X)$ and $\|A(1 + \lambda A)^{-1}\| \leqslant (\sin\theta)^{-1}\lambda^{-1}$. Suppose now that $(1 + \lambda_0 A)^{-1} \in B(X)$ for some $\lambda_0 > 0$. If $(1 + \lambda A)(1 + \lambda_0 A)^{-1}$ has an inverse in $B(X)$ then $(1 + \lambda A)^{-1} = (1 + \lambda_0 A)^{-1}[(1 + \lambda A)(1 + \lambda_0 A)^{-1}]^{-1} \in B(X)$, and since $(1 + \lambda A)(1 + \lambda_0 A)^{-1} = 1 + (\lambda - \lambda_0) A(1 + \lambda_0 A)^{-1}$, the desired inverse is in $B(X)$ whenever $|\lambda - \lambda_0|\,\|A(1 + \lambda_0 A)^{-1}\| < 1$, thus, when $|\lambda - \lambda_0| < \lambda_0 \sin\theta$. A standard continuation argument permits easily to derive from the latter that $\forall\lambda > 0$, $(1 + \lambda A)^{-1}$, $A(1 + \lambda A)^{-1} \in B(X)$. Also (9) yields at once $\|(1 + \lambda A)^{-1}\| \leqslant 1 + (1/\sin\theta)$. Finally, in view of (9) the statement concerning closeability of A follows from a result in [3]; for the sake of completeness we repeat the proof here. Suppose, thus, that $f_n \in D(A)$, $\|f_n\| \to 0$, $\|Af_n - g\| \to 0$ for some $g \in X$. For $\lambda > 0$, $h \in D(A)$, by (9), setting $M = (\sin\theta)^{-1}$,

$$M\|f_n + \lambda h + \lambda A(f_n + \lambda h)\| \geqslant \lambda\|A(f_n + \lambda h)\|;$$

hence, $M\|h + g + \lambda Ah\| \geqslant \|g + \lambda Ah\|$. For $\lambda \to 0$ one has $M\|h + g\| \geqslant \|g\|$, and since $D(A)$ is dense $g = 0$. Q.E.D.

Proof of Theorem 8. The appropriate analog of (i), for "codissipative" replaced by "(9)," is also proved in [3], and in fact the necessary ingredients for that proof are in Lemma 9. Thus, one proceeds as in the proof of Theorem 6, and again uses A_ψ. For $z \in \mathbb{C}$ let "arg z" denote the usual determination of the "argument" (whereby arg $1 = 0$, $\arg(-i) = -\pi/2$). The hypothesis of Theorem 8 implies that $N(\bar{A}) \subset G = \{z \in \mathbb{C}\colon |\arg z| \leqslant \pi - \theta\} \cup \{0\}$. Considering the $[\ |\ ]$ which defines $N(\bar{A})$ and a given $f \in X$, one has if $\| \bar{A}\psi f \| = 1$, $[\psi f \mid \bar{A}\psi f] = \psi(x^{\sigma}_{\psi f, \bar{A}\psi f})[f \mid A_\psi f]_{\sigma'} \in G$, for a σ', and we see that $N_{\sigma'}(A_\psi) \subset G$ and, therefore, the same estimates that hold for A hold also for A_ψ, in particular (9), and $\|(1 + A_\psi)^{-1}\| \leqslant 1 + (\sin\theta)^{-1}$; the rest then goes as for Theorem 6. Q.E.D.

Clearly an example in which one encounters the situation covered by Theorem 8 is that of $A = \tilde{A}p$, where $\tilde{A}$ is a codissipative operator with $D(\tilde{A}) \supset C_{00}(\Omega, \mathbb{C})$, and $\tilde{A}p$ denotes the operator $f \to \tilde{A}(pf)$ where $p \in C(\Omega, \mathbb{C})$, $p(x) \neq 0$, $|\arg p(x)| \leqslant (\pi/2) - \theta$, $(0 < \theta \leqslant \pi/2)$, $\forall x \in \Omega$.

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