On Rotationally Symmetric Mean Curvature Flow

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1. INTRODUCTION

The evolutionary motion of surfaces by their mean curvature has been studied by several authors from different points of view. A measure theoretic approach is studied by K. A. Brakke [B]; the classical parametric problem is studied by G. Huisken [H1]. Non-parametric mean curvature evolution with boundary conditions is treated in [H2]. For a detailed discussion see [H3].

We restrict our attention to the case in which the initial surface has rotational symmetry. We shall show that under suitable initial conditions the solution degenerates in the sense that its curvature develops a singularity at exactly one point. This confirms numerical results of the first author [D] for general surfaces in \( \mathbb{R}^3 \). See also the work of M. A. Grayson [G1, p. 286, G2]. The proofs rely on the parabolic maximum principle.

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2. PRELIMINARIES

Consider a compact two-dimensional surface \( S(0) \) without boundary smoothly embedded in \( \mathbb{R}^3 \) which is represented locally by a diffeomorphism.
$u_0: \Omega \to \mathbb{R}^3$ where $\Omega \subset \mathbb{R}^2$, $u_0(\Omega) \subset S(0)$. Then one looks for maps $u(t, \cdot): \Omega \to \mathbb{R}^3$ ($t > 0$) such that for some $T > 0$

$$\frac{\partial u}{\partial t} - A_{S(t)}u = 0 \quad \text{in} \quad (0, T) \times \Omega \quad (1)$$

and

$$u(0, \cdot) = u_0.$$  

Here $A_{S(t)}$ represents the Beltrami operator on $S(t)$ where $S(t)$ is given by $u(t, \Omega) \subset S(t)$. Since

$$u(t, \cdot) = \text{id}_{S(t)} \quad \text{on} \quad S(t),$$

we have

$$-A_{S(t)}u(t, \cdot) = 2H(t, \cdot)n(t, \cdot) \quad (2)$$

and (1) reduces to

$$\frac{\partial u}{\partial t} = -2Hn,$$

where $H$ is the mean curvature of $S(t)$ and $n$ is the outer unit normal.

The most simple example for a solution of (1) is the shrinking sphere $S(t) = \{ x \in \mathbb{R}^3 \mid |x| = R(t) \}$ with radius $R(t) = (R(0)^2 - 4t)^{1/2}$ and initial radius $R(0)$.

In [H1] it was proved that for a uniformly convex initial surface $S(0)$ the system (1) possesses a smooth solution on a finite time interval $(0, T)$ and $S(t)$ converges "spherelike" to a single point as $t \to T$.

If one looks at non-parametric surfaces $u = u(t, x_1, x_2)$, then (1) becomes

$$\frac{\partial u}{\partial t} - (1 + |\nabla u|^2)^{1/2} \nabla \cdot \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0 \quad (3)$$

in $\Omega \subset \mathbb{R}^2$. Of course (3) has to be complemented by suitable boundary conditions of Dirichlet or Neumann type. In the non-parametric case (3) $u(t, \cdot)$ converges to a solution of the minimal surface equation as $t \to \infty$ if one imposes a Dirichlet boundary condition $u = u_0$ on $\partial \Omega$, $\partial \Omega$ having non-negative mean curvature, and $u(0, \cdot) = u_0 \in C^{2,\alpha}(\bar{\Omega})$. For the proof we refer to [H2].

If in (3) $u$ is rotational symmetric with respect to the $x_1$-axis, the equation can be rewritten in the form

$$\frac{\partial u}{\partial t} - \frac{u_{x_1, x_1}}{1 + u_{x_1}^2} = -\frac{1}{u} \quad \text{in} \quad (0, T) \times \Omega \quad (4)$$
where $u = u(t, x_1)$. A particular solution of (4) under Neumann boundary conditions is the shrinking cylinder $u(t, x) = (u_0^2 - 2t)^{1/2}$ which collapses at time $T = u_0^2/2$ to a line. G. Huisken has shown in [H3, Section 5] that any solution of (4), if its curvature develops a singularity, collapses like $(T - t)^{1/2}$ as long as the initial surface $u_0$ has non-positive curvature; i.e., as long as

$$\frac{u_{0xx}}{1 + u_{0x}^2} \frac{1}{u_0} \leq 0$$

holds. This result could also be obtained by a modification of arguments in [FK], where the equation

$$u_t - Au = -\frac{1}{u^2}$$

was studied. The phenomenon that $u$ approaches zero in finite time is known as quenching, pinching, or necking. In [AK] one can find a result on single point quenching for solutions of (6), which can be modified to treat (4).

3. THE NEUMANN PROBLEM

Consider Eq. (4) under boundary conditions

$$u_x(t, 0) = u_x(t, a) = 0$$

or equivalently for a periodically deformed infinite cylinder.

**Lemma 1.** (a) Let $u(0, x) = u_0(x) \in C^0([0, a])$. Then $u$ develops a singularity in finite time $T > 0$.

(b) Let $u_0 \in C^1([0, a])$ and suppose that

$$u_{0x} \geq 0.$$  

Then $u_x(t, x) \geq 0$ for $x \in [0, a]$, $t \in [0, T]$.

(c) Let $u_0 \in C^2([0, a])$ satisfy (5); i.e., the initial surface has non-positive mean curvature

$$H = \frac{1}{2} (1 + u_{0x}^2)^{-1/2} \left( \frac{u_{0xx}}{1 + u_{0x}^2} - \frac{1}{u_0} \right).$$

Then $u_t \leq 0$ for $x \in [0, a]$, $t \in [0, T]$. 

Proof. To prove (a) we compare $u$ with a shrinking cylinder of initial radius $\max_{[0,a]} u_0$ as in [H3].

Let us now assume that $u_{0x} \geq 0$. We set $v = u_x$ and differentiate Eq. (4) with respect to $x$. Then

$$v_t - \frac{1}{1 + u_x^2} v_{xx} - \frac{2u_x u_{xx}}{(1 + u_x^2)^2} v_x - \frac{1}{u^2} v = 0. \quad (9)$$

Now if $v$ attains a negative minimum this must occur on the parabolic boundary. But due to (7) $v = 0$ for $x = 0, a$ and (8) implies $v \geq 0$ for $t = 0$. Thus (b) is proved.

The proof of part (c) is similar if we set $v = u_{xt}$, differentiate (4) with respect to $t$, and arrive at Eq. (9). But since $v_x = u_{xt} = 0$ for $x = 0, a$ any positive maximum can only occur initially. The latter is excluded by (5), however.

The following Lemma gives a condition on the shape of the initial surface $u_0$ which ensures single point necking. Condition (10) can be rewritten as

$$\left(\sqrt{1 + u_{0x}^2} H\right)_x \geq 0.$$

Lemma 2. Suppose in addition to (5) and (8) that

$$\left(\frac{u_{0xx}}{1 + u_{0x}^2} - \frac{1}{u_0/x}\right) \geq 0. \quad (10)$$

Then $u_{xt} \geq 0$ for $x \in (0, a), t > 0$.

Proof. $u_{xt} \geq 0$ on the parabolic boundary. We note

$$u_{xt} = \frac{u_{xxx}}{1 + u_x^2} - \frac{2u_x^2 u_{xx}}{(1 + u_x^2)^2} + \frac{u_x^2}{u^2}$$

and set $v = u_{xt}$. Then

$$v_t = \frac{u_{xxt}}{1 + u_x^2} - \frac{2u_x u_{xxy} u_{xt}}{(1 + u_x^2)^2} - \frac{2u_x^2 u_{xx} u_{xt} + 4u_x u_{x} u_{xx} u_{xt}}{(1 + u_x^2)^2}$$

$$- \frac{8u_x u_{xx}^2 u_{xt}}{(1 + u_x^2)^3} + \frac{u_{xt}}{u^2} - \frac{2u_x u_t}{u^3}$$
and
\[ v_t - \frac{1}{1 + u_x^2} v_{xx} + \frac{4u_x u_{xx}}{(1 + u_x^2)^2} v_x + \left( \frac{2u_{xxx} u_x + 2u_x^2}{(1 + u_x^2)^2} - \frac{8u_{xxx} u_x^2}{(1 + u_x^2)^3} - \frac{1}{u^3} \right) v = -\frac{2u_x u_t}{u^3} \geq 0 \]
by Lemma 1 parts (b) and (c). Consequently \( v \geq 0 \).

Thus we have proved:

**Theorem 3.** Under assumptions (5), (8), (10), and \( u_{0x} \neq 0 \) there exists a finite \( T > 0 \) such that \( \lim_{t \to T} u(t, 0) = 0 \) and \( \lim_{t \to T} u(t, x) > 0 \) for \( x > 0 \).

**Example.** Let \( a = \pi/2 \) and \( u_0(x) = b \sin^2 x + c \) with \( b \geq 0 \), \( c > 0 \). Then condition (5) amounts to \( 2bc \leq 1 \). \( 4(b + c)^2 (1 + 2b^2) \leq 1 \) is sufficient for (10).

### 4. Other Boundary Conditions

If we consider Eq. (4) under mixed boundary conditions
\[ u_x(t, 0) = \gamma, \quad u(t, a) = \gamma, \quad (11) \]
we can equivalently treat the pure Dirichlet boundary conditions
\[ u(t, -a) = u(t, a) = \gamma \quad (12) \]
on the reflected domain \( x \in (-a, a), t > 0 \) and with even initial data
\[ u(0, x) = u_0(x) = u_0(-x). \quad (13) \]
Contrary to the pure Neumann case there can exist stationary solutions under suitable conditions. These are the catenoids which are given by
\[ u(x) = \frac{1}{\alpha} \cosh(\alpha x) \quad (14) \]
with \( \alpha > 0 \), \( x \in (-a, a) \). Note that (14) is compatible with (11) only if \( \gamma/a \) is sufficiently large.

**Lemma 4.** (a) Suppose that \( \gamma/a < 2/\pi \). Then a solution of (4), (12), and (13) develops a singularity in finite time.
(b) Suppose that \( \gamma/a < \min, \cosh s/|s| \) and that the assumptions (5), (8), and (10) are satisfied. Then a solution of (4), (12), and (13) develops a singularity in finite time.

Proof. (a) Otherwise the solution exists and \( u \) is positive for every finite time. We multiply (4) by \( u \) and integrate with respect to \( x \) to obtain

\[
\frac{\partial}{\partial t} \int_{-a}^{a} u^2/2 \, dx = \int_{-a}^{a} u(\arctg u_x)_x - 2a = - \int_{-a}^{a} u_x \arctan u_x \, dx + 2\gamma \arctan u_x(a) - 2a
\]

\[
\leq \gamma \pi - 2a.
\]

Therefore \( \|u(t, \cdot)\|_{L^2(-a, a)} \) becomes negative in finite time, a contradiction.

(b) For the proof we have to inspect the proofs of Lemmata 1 parts (b) and (c) and 2 and to adjust them to Dirichlet boundary conditions to conclude that \( u_x \geq 0 \). This and the fact that \( u_x > 0 \) after an initial time step implies that

\[
u(t, x) \geq c(\delta) > 0,
\]

\[
u_x(t, x) \leq C(\delta) < \infty
\]

for \( t \geq t_0 \) and \( x \geq \delta > 0 \) and that \( u(t, x) \) converges to a stationary solution

\[
u_{\infty}(x) = c_1 \cosh \left( \frac{x}{c_1 + c_2} \right)
\]

for \( x \geq \delta > 0 \) with positive constants \( c_1 \) and \( c_2 \). In fact, to see the boundedness of \( u_x \) one observes that

\[
\left( -\log \frac{u}{\sqrt{1 + u_x^2}} \right)_x = u_x u_{xx} \leq 0.
\]

Now we distinguish two cases: Case (1) \( \lim_{t \to \infty} u(t, 0) > 0 \). In this case \( u \) tends to a stationary solution \( u_{\infty} \) on \([0, a]\) with \( u'(0) = 0, u_{\infty}(a) = \gamma \) which leads to a contradiction. Case (2) \( \lim_{t \to \infty} u(t, 0) = 0 \). Then there exists a time \( t_1 \) such that for all \( t > t_1 \)

\[
u(t, 0) < \frac{c_1}{2} \cosh c_2,
\]

while for every \( x > 0 \)

\[
u(t, x) \geq c_1 \cosh c_2.
\]
Since $u$ is continuous for finite $t$, this is a contradiction. The Liapunov-functional associated with the flow $t \to \infty$ is given by the surface area

$$V(t) = 2\pi \int_{-\alpha}^{\alpha} u \sqrt{1 + u_x^2} \, dx.$$

Thus we have shown

**Theorem 5.** Under assumptions (5), (8), and (10) and the assumptions of Lemma 4 the solution of (4), (11), and (13) or (4), (12), and (13) goes to zero as $t \to$ some $T < \infty$ only in the origin $x = 0$. 

![Fig. 1. Seven time steps of mean curvature evolution of a cylinder with fixed boundary.](image)
Proof. We have to inspect the proofs of Lemmata 1 parts (b) and (c) and 2 and to adjust them to Dirichlet boundary conditions. This is straightforward.

Example. The assumptions of Theorem 5 are satisfied by \( u_0(x) = cx^2 + d \) with suitable \( c \geq 0, \ d > 0 \) such that \( 2cd \leq 1 \).

5. Numerical Example

Using the algorithm EVO from [D] we calculated an example for Dirichlet boundary conditions. We did not make use of the rotational symmetry here. We took \( a = 1.0, \ u_0(x) \equiv 1 \). For numerical stability the discrete time step \( \tau \) was chosen according to the radius \( r \) of the neck developing at the origin,

\[ \tau = r^2/2. \]

See Fig. 1.

References