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Renormalization of supersymmetric gauge theories on orbifolds: Brane gauge couplings and higher derivative operators

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Abstract

We consider supersymmetric gauge theories coupled to hypermultiplets on five- and six-dimensional orbifolds and determine the bulk and local fixed point renormalizations of the gauge couplings. We infer from a component analysis that the hypermultiplet does not induce renormalization of the brane gauge couplings on the five-dimensional orbifold $S^1/\mathbb{Z}_2$. This is not due to supersymmetry, since the bosonic and fermionic contributions cancel separately. We extend this investigation to $T^2/\mathbb{Z}_N$ orbifolds using supergraph techniques in six dimensions. On general $\mathbb{Z}_N$ orbifolds the gauge couplings do renormalize at the fixed points, except for the $\mathbb{Z}_2$ fixed points of even ordered orbifolds. To cancel the bulk one-loop divergences a dimension six higher derivative operator is needed, in addition to the standard bulk gauge kinetic term.

1. Introduction and summary

The investigation of theories of extra dimensions has been an active field of research initiated by [1,2]. Most of the phenomenological activity has focused on five-dimensional (5D) models, in particular models on simple orbifolds like $S^1/\mathbb{Z}_2$ or $S^1/\mathbb{Z}_2 \times \mathbb{Z}'_2$ [3–5]. An important issue of such investigations was the running of the 4D gauge coupling in extra dimensions and possible gauge coupling unification [6,7]. A complication is that the gauge couplings are sensitive to the ultra-violet (UV) completion of the theory [8]. In this Letter we study the gauge coupling running by calculating the self-energy in extra dimensions. In particular, we investigate the renormalization of bulk and fixed point gauge operators in supersymmetric (SUSY) field theories on 5D and 6D orbifolds.

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As a warm up, we start our analysis with a single complex scalar coupled to a gauge field in the bulk of $S^1/\mathbb{Z}_2$. To cancel the divergences of the scalar loop both bulk and brane localized counter terms are needed for the gauge field. This result is an example of the generic fact that on an orbifold both bulk and fixed point localized operators renormalize [9–11]. However, such localized counter terms are not always required: a charged bulk fermion does not require counter terms for the gauge field at the orbifold fixed points. The absence of brane gauge counter terms persists in SUSY models, because the contributions of the complex scalars of the hypermultiplet also cancel.

This raises the question, whether this is an accident of the simple $S^1/\mathbb{Z}_2$ orbifold or holds more generically for $T^2/\mathbb{Z}_N$ orbifolds in 6D SUSY theories. We investigate this question by computing the one-loop self-energy for the vector multiplet in 6D. To this end we set up a 6D extension of $\mathcal{N}=1$ supergraphs based on representing 6D SUSY theories by $\mathcal{N}=1$ 4D superfields [12–14]. We find that for generic $\mathbb{Z}_N$ orbifolds the gauge couplings at almost all fixed points do renormalize due to bulk hyper multiplets. There is no contradiction with the 5D $S^1/\mathbb{Z}_2$ result, because $\mathbb{Z}_2$ fixed points of even ordered orbifolds (and therefore $\mathbb{Z}_2$ orbifolds in particular) are the only fixed points that do not receive any gauge coupling renormalization.

Since we compute the full one-loop gauge multiplet self-energy, we can determine the bulk renormalization of the gauge multiplet. We find that a dimension six higher derivative term for the gauge multiplet is generated. (Higher derivative counter terms are also needed in 5D orbifold models if brane localized interactions for bulk fields are considered [15].) Such higher derivative theories may have remarkable UV properties [16]: The higher derivative operators act as regulators that make many loop graphs finite. Higher derivative hypermultiplet operators do not seem to be allowed by gauge and 6D Lorentz invariance combined. (All gauge coupling corrections at one loop would be finite if they were present.)

Let us close with a few comments on the context and possible extensions of our work. In 6D the constraints of anomalies are very severe [17,18], but since we were only interested in the gauge coupling running, we do not take these constraints into account. Moreover, we restrict ourselves to Abelian theories only; in a future publication [19] we investigate non-Abelian theories and work out the details of the threshold corrections we identify. Our investigation is restricted to one-loop corrections only. However, we expect that the results in fact hold to all orders in perturbation theory up to infra-red (IR) effects. Both at the fixed points and in the bulk holomorphicity arguments [20–24] of $\mathcal{N}=1$ SUSY field theories in 4D apply.

The outline is as follows: in Section 2 we study the running of local gauge couplings due to scalars and fermions on $S^1/\mathbb{Z}_2$. In Section 3 we perform a manifestly SUSY one-loop computation of the gauge multiplet self-energy on generic $T^2/\mathbb{Z}_N$ in 6D. We determine the bulk and fixed point renormalizations of the gauge coupling and identify a higher derivative operator in the bulk. In Appendix A we describe the regularization of the divergent integral encountered in 4, 5 and 6D.

2. Bulk and fixed point localized corrections on $S^1/\mathbb{Z}_2$

2.1. Scalar on $S^1/\mathbb{Z}_2$

We begin our analysis with a complex scalar $\tilde{\phi}$ coupled to a U(1) gauge field $\tilde{A}_M$ in 5D compactified on $S^1/\mathbb{Z}_2$. The coordinate y of the covering circle S^1 is periodic $y \sim y + 2\pi R$. The $\mathbb{Z}_2$ reflection acts on these fields as

$$\tilde{\phi}(-y) = Z\tilde{\phi}(y), \qquad \tilde{A}_\mu(-y) = \tilde{A}_\mu(y), \qquad \tilde{A}_5(-y) = -\tilde{A}_5(y), \tag{1}$$

where we have suppressed the 4D coordinate x^μ. To be able to trace the dependence on the orbifold boundary conditions, we keep the parity eigenvalue $Z = \pm$ of the scalar $\tilde{\phi}$ arbitrary. In many studies of the orbifold $S^1/\mathbb{Z}_2$ the fields are expanded into even and odd mode functions. For sufficiently simple orbifolds this is a useful procedure, but since we want to extend our analysis eventually to more complicated orbifolds, we choose instead to obtain orbifold compatible fields from fields defined on the covering space [9]. For example, let ϕ be a complex scalar on the covering circle. By employing an orbifold projector we obtain a field $\tilde{\phi}$ satisfying (1) as

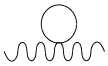
$$\tilde{\phi}(y) = \frac{1}{2}\big(\phi(y) + Z\phi(-y)\big). \tag{2}$$

Extensions to other fields are obvious. We define orbifold compatible functional differentiation as

$$\tilde{\delta}_{21} = \frac{\delta \tilde{J}_2}{\delta \tilde{J}_1} = \frac{1}{2}\big(\delta^5(y_2 - y_1) + Z\delta^5(y_2 + y_1)\big), \tag{3}$$

where $\tilde{J}$ is the source coupled to $\tilde{\phi}$. Here and throughout the Letter we only indicate the internal coordinate(s) explicitly where the orbifolding is non-trivial, i.e., $\delta^5(y_2 \pm y_1) = \delta^4(x_2 - x_1)\delta(y_2 \pm y_1)$.

We used this method to obtain the gauge field self-energy at one loop due to the complex scalar $\tilde{\phi}$ with charge q. There is a tadpole (seagull) diagram



$$= q^2 \int \big(\mathrm{d}^5 X\big)_{12}\, \tilde{A}_1^M \tilde{A}_1^N \eta_{MN} \tilde{\delta}_{21} \frac{1}{(\Box_5 - m^2)_2} \tilde{\delta}_{21}, \tag{4}$$

and a genuine self-energy diagram

$$= q^2 \int \big(\mathrm{d}^5 X\big)_{12} \tilde{A}_1^M \tilde{A}_2^N \bigg(\frac{1}{(\Box_5 - m^2)_2} \tilde{\delta}_{21} \frac{\partial_{1M}\partial_{2N}}{(\Box_5 - m^2)_2} \tilde{\delta}_{21} - \frac{\partial_{1M}}{(\Box_5 - m^2)_2} \tilde{\delta}_{21} \frac{\partial_{2N}}{(\Box_5 - m^2)_2} \tilde{\delta}_{21} \bigg). \tag{5}$$

Here $(\mathrm{d}^5 X)_{12}$ denotes the integration over the coordinates $X_1^M = (x_1^\mu, y_1)$ and $X_2^M = (x_2^\mu, y_2)$, and partial differentiation w.r.t. X_2^M is indicated by $\partial_{2M} = \partial/\partial X_2^M$. The spacetime metric η_{MN} uses the mostly plus convention, and the 4D and 5D kinetic operators read $\Box = \partial^\mu \partial_\mu$ and $\Box_5 = \Box + \partial_5^2$, respectively. Notice that all terms in both expressions contain two orbifold delta functions $\tilde{\delta}_{21}$, i.e., the orbifold projector is inserted twice. Since a projector squared is the projector again, one of them can be replaced by a conventional delta function $\tilde{\delta}_{21} \to \delta_{21} = \delta^4(x_2 - x_1)\delta(y_2 - y_1)$. This can be confirmed explicitly by inserting (3) for one of the orbifold delta functions and perform a change of coordinates $y_2 \to -y_2$. The leftover $\tilde{\delta}_{21}$ consists of two parts, see (3): the first part, $\frac{1}{2}\delta_{21}$, gives rise to contributions in 5D compactified on a circle, with an additional normalization factor of $\frac{1}{2}$. The second part of the orbifold delta function reads $\frac{1}{2} Z \delta^4(x_2 - x_1)\delta(y_2 + y_1)$. If there were no derivatives, integration over y_2 would lead to the fixed point delta function $\delta(2y_1)$ and hence to localization at the orbifold fixed points. In the presence of the y derivatives in the propagators the amplitude acquires non-local contributions which are sourced by the fixed points. However, the counter terms needed to cancel the divergences are local. As the 4D-localized parts are proportional to the factor Z, it follows that for two complex scalars of opposite parities (and equal or opposite charges) all localized contributions cancel identically.

2.2. Fermion on $S^1/\mathbb{Z}_2$

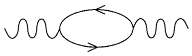
Next we move to a Dirac fermion ψ on the same orbifold, which satisfies the boundary conditions

$$\tilde{\psi}(-y) = \gamma_5 \tilde{\psi}(y), \qquad \tilde{\bar{\psi}}(-y) = \tilde{\bar{\psi}}(y)(-\gamma_5), \tag{6}$$

so that the kinetic terms are invariant. The functional derivative w.r.t. the source $\tilde{J}$ for $\tilde{\bar{\psi}}$ reads

$$\tilde{\delta}_{21} = \frac{\delta \tilde{J}_2}{\delta \tilde{J}_1} = \frac{1}{2}\big(\delta^5(y_2 - y_1) - \gamma_5 \delta^5(y_2 + y_1)\big), \tag{7}$$

where we again suppressed the 4D coordinate dependence in the delta function. Functional differentiation w.r.t. the source $\tilde{\bar{J}}$ for $\tilde{\psi}$ defines $\tilde{\bar{\delta}}$ in a similar fashion; it is obtained from $\tilde{\delta}$ by replacing $\gamma_5 \to -\gamma_5$. Using similar steps as in the scalar calculation, we can evaluate the photon self-energy due to the fermion: all orbifold projectors in the loop can be removed except for one



$$= \Sigma_F = \frac{q^2}{2} \int (\mathrm{d}X)_{12}\, \mathrm{tr}\big[\not{A}_1 (\not{\partial} + m)_1^{-1} \delta_{12} \not{A}_2 (\not{\partial} + m)_2^{-1} \tilde{\delta}_{21}\big]. \tag{8}$$

Here the trace is over the four component spinor indices and $\not{A} = A^M \gamma_M$. Again we see from the expression for the delta function for the fermion (7) that the amplitude consists of 5D and 4D localized parts. In fact, the localized part vanishes.

To see this, we expand the localized part in momentum space:

$$\begin{aligned} \Sigma_{F4\mathrm{D}} = &-\frac{q^2}{4} \int \frac{\mathrm{d}^4 p\, \mathrm{d}^4 k}{(2\pi)^8} \frac{1}{(2\pi R)^2} \sum_{n_1, n_2 \in \mathbb{Z}/R} \frac{1}{[p^2 + n_3^2 + m^2][(p+k)^2 + n_4^2 + m^2]} \\ &\times \big\{ A^\mu(k, n_1) A^\nu(-k, n_2)\, \mathrm{tr}\big[\gamma_5 \gamma_\mu (\not{p} + n_3 \gamma_5 + im) \gamma_\nu (\not{p} + \not{k} + n_4 \gamma_5 + im)\big] \\ &+ A^5(k, n_1) A^5(-k, n_2)\, \mathrm{tr}\big[\gamma_5^2 (\not{p} + n_3 \gamma_5 + im) \gamma_5 (\not{p} + \not{k} + n_4 \gamma_5 + im)\big] \big\}. \end{aligned} \tag{9}$$

Here p, k are 4D (loop) momenta. The loop Kaluza–Klein (KK) momenta $2n_3 = n_2 - n_1$ and $2n_4 = -n_2 - n_1$ are expressed in terms of those of the external photons. As these are localized contributions, the KK number is not preserved: n_2 need not be equal to $-n_1$. Instead, n_1 and n_2 are either both even or both odd, hence there is no mixing between A^5 and A^μ. The presence of γ_5 in these expressions shows that all traces vanish identically except for $\mathrm{tr}[\gamma_5 \gamma_\mu \not{p} \gamma_\nu \not{k}]$. By employing a Feynman parameterization of the propagators, the loop integral implies that $p_\rho \sim k_\rho$, and therefore also this trace vanishes.

2.3. Hypermultiplet gauge coupling renormalization on $S^1/\mathbb{Z}_2$

We use the previous results to get some feeling for the localization of gauge couplings in SUSY theories: the two chiral multiplets inside a hypermultiplet have opposite $\mathbb{Z}_2$ boundary conditions. From Section 2.1 we know that two scalars with opposite boundary conditions do not give localized gauge coupling contributions. And in Section 2.2 we reached the same conclusion for a Dirac fermion, i.e., two chiral fermions with opposite charges and boundary conditions. This implies that the hypermultiplet will not lead to any localized gauge coupling renormalization.

We have confirmed that no brane localized gauge counter terms are needed by performing an explicit supergraph calculation of the VV-, $\bar{S}S$- and SV-selfenergies that are given in Fig. 1. (Details will be presented in the next section in 6D.) The 5D bulk gauge coupling renormalizes as

$$\frac{1}{g_R^2} = \frac{1}{g^2} - \frac{2q^2}{(4\pi)^2} |m|, \tag{10}$$

where the subscript R refers to the renormalized gauge coupling. This result is compatible with the results obtained by Witten [25] and used by Seiberg et al. [26–28] to analyze SUSY gauge theory in non-compact 5D.

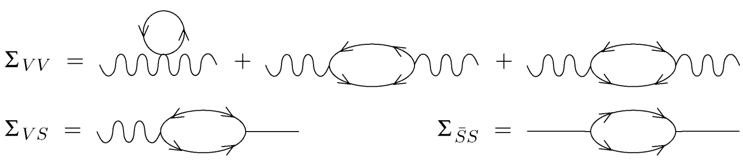


Fig. 1. The gauge self-energy supergraphs are drawn. The wavy and straight lines indicate the superfields V, S and $\bar{S}$. The lines with double arrows depict the hypermultiplet propagators (17).

3. Supersymmetric gauge theories with matter on 6D orbifolds

We investigate SUSY theories on an arbitrary 6D orbifold $T^2/\mathbb{Z}_N$. The field content we consider is a charged hypermultiplet coupled to a gauge multiplet. We employ $\mathcal{N}=1$ 4D superfields to describe these multiplets [12,13,29,30], and the superspace conventions of Wess and Bagger [31]. The gauge multiplet contains a vector multiplet V and a chiral multiplet S. The hypermultiplet consists of two chiral multiplets $\Phi_\pm$ that are charged oppositely. The superfields are made orbifold compatible using methods similar to (2). In order to keep the notation simple, we have dropped the twiddles on them.

We employ complex coordinates $z=\frac{1}{2}(x_5-ix_6)$ and $\bar{z}=\frac{1}{2}(x_5+ix_6)$, so that we find for the derivatives: $\partial=\partial_5+i\partial_6$, $\bar\partial=\partial_5-i\partial_6$ and $\partial\bar\partial=\partial_5^2+\partial_6^2$. (The reduction to 5D is straightforward: set $z=\bar{z}=\frac{1}{2}y$, $R_5=R$ and $\partial=\bar\partial=\partial_5$.) The periodicity conditions of the torus T^2, $z\sim z+\pi R_1\sim z+\pi e^{i\vartheta}R_2$, define the "winding mode" lattice Λ_W. This lattice, the KK lattice Λ_K and the volumes of their fundamental domains are collected in Table 1. An orbifold $T^2/\mathbb{Z}_N$ is obtained by requiring that the field theory on the covering torus T^2 is invariant under the $\mathbb{Z}_N$ rotation:

$$z\to e^{-i\phi}z,\qquad \bar{z}\to e^{i\phi}\bar{z},\qquad \partial\to e^{i\phi}\partial,\qquad \bar\partial\to e^{-i\phi}\bar\partial,\tag{11}$$

where the phase ϕ is such that $e^{iN\phi}=1$. In order for this $\mathbb{Z}_N$ orbifold action to be compatible with the lattice, conditions on the radii R_5, R_6 and phase ϑ may apply. (For example for a $\mathbb{Z}_3$ orbifold $R_5=R_6=R$ and $\vartheta=\phi=2\pi/3$.) The superfields V, S, Φ_+ and Φ_- transform as

$$V\to V,\qquad S\to e^{i\phi}S,\qquad \Phi_\pm\to e^{ia_\pm\phi}\Phi_\pm.\tag{12}$$

Only for the hypermultiplet we have an arbitrary integer $0\leqslant a_+\leqslant N-1$ since $a_-=N-1-a_+$. (Note that this is compatible with the $\mathbb{Z}_2$ case: there one chiral multiplet is even and the other is odd.) As in Section 2.1, we define the orbifold delta function as

$$\tilde\delta_{21}^{(a)}=\frac{1}{N}\sum_{b=0}^{N-1}e^{iba\phi}\delta\big(z_2-e^{ib\phi}z_1\big),\tag{13}$$

where $\delta(z_2-z_1)=\delta^2(z_2-z_1)\delta^4(x_2-x_1)\delta^4(\theta_2-\theta_1)$, for a superfield that transforms with a phase $e^{ia\phi}$. With this formalism we can set up a supergraph formalism [31–33] for orbifold theories.

We close this introductory section with an exposition of the relevant Lagrangians written in terms of $\mathcal{N}=1$ superfields. The gauge invariant bulk vector multiplet Lagrangian can be written as

$$\mathcal{L}_{\text{gauge}}=\frac{1}{2g^2N}\int d^2\theta\, W^\alpha W_\alpha+\frac{1}{g^2N}\int d^4\theta\,(\partial V\bar\partial V+\bar{S}S-\sqrt{2}\bar\partial VS-\sqrt{2}\partial V\bar{S}),\tag{14}$$

where $W^\alpha=-\frac{1}{4}\bar{D}^2D_\alpha V$ is the 4D superfield strength, and $1/g^2$ is the mass dimension two gauge coupling. The factor $1/N$ in the Lagrangian (14) is included, because we perform all our calculations on the covering space of the $T^2/\mathbb{Z}_N$ orbifold. In addition, for orbifolds we can have fixed point localized 4D gauge actions of the form

$$\mathcal{L}_{\text{gauge}}^{\text{fix}}=\sum_{b=1}^{N-1}\frac{1}{2g_b^2N}\int d^2\theta\, W^\alpha W_\alpha\delta^2\big(\big(1-e^{ib\phi}\big)z\big),\tag{15}$$

Table 1
This table summarizes our notation for the circle and the torus: $\text{Vol}_W=(2\pi)^{D-d}\,\text{Vol}$ with $D-d=1,2$, respectively, and $\text{Vol}\cdot\text{Vol}_K=1$

	Vol	Vol_K	Λ_W	Λ_K
S^1	R	$\frac{1}{R}$	$2\pi R\mathbb{Z}$	$\frac{1}{R}\mathbb{Z}$
T^2	$R_5R_6\sin\vartheta$	$\frac{1}{R_5R_6\sin\vartheta}$	$\pi(R_5\mathbb{Z}+e^{i\vartheta}R_6\mathbb{Z})$	$\frac{i}{\sin\vartheta}\big(\frac{e^{-i\vartheta}}{R_5}\mathbb{Z}+\frac{1}{R_6}\mathbb{Z}\big)$

where the gauge couplings $1/g_b^2$ are dimensionless. Note that they have a non-standard normalization and that $g_{N-b} = g_b$. The gauge invariant Lagrangian for the hypermultiplet with charge q reads

$$\mathcal{L}_{\text{hyper}} = \frac{1}{N}\int \mathrm{d}^2\theta\, \Phi_-(\partial + \sqrt{2}qS)\Phi_+ + \text{h.c.} + \frac{1}{N}\int \mathrm{d}^4\theta\, \bar{\Phi}_\pm e^{\pm 2qV}\Phi_\pm, \tag{16}$$

where in the last term summation over $+$ and $-$ is implied. The Hermitian conjugation acts on the chiral superfields as well as on the holomorphic derivative ∂. In 6D the hypermultiplet is massless [34,35], while in 5D it can have a real mass $m(y) = m\epsilon(y)$, with $\epsilon(y)$ the step function on S^1, which can be thought of as the vacuum expectation value of the real part of S.

3.1. Bulk and fixed point localized gauge selfenergies on $T^2/\mathbb{Z}_N$

We investigate the renormalization of (localized) gauge couplings on 6D orbifolds. As we consider an Abelian theory, only the hypermultiplet loops lead to gauge coupling renormalization. The propagators of chiral components of the hypermultiplet read

$$\bar{\Phi}_\pm \longrightarrow \Phi_\pm = \frac{1}{\Box_6}, \qquad \Phi_+ \longleftrightarrow \Phi_- = \frac{\bar{\partial}}{\Box_6}\frac{D^2}{-4\Box}, \tag{17}$$

where $\Box_6 = \Box + \partial\bar{\partial}$. (In 5D these propagators may contain the mass m of the hypermultiplet.) We computed the one-loop self energy diagrams for external superfields VV, VS and $\bar{S}S$, given in Fig. 1, on the $T^2/\mathbb{Z}_N$ orbifold. The tadpole graph cancels gauge non-invariant contributions from the other two graphs of Σ_{VV}. By including the superfields V, S and $\bar{S}$ in the amplitudes, the sum of the supergraphs $\Sigma = \Sigma_{VV} + \Sigma_{VS} + \Sigma_{V\bar{S}} + \Sigma_{\bar{S}S}$ becomes

$$\begin{aligned}\Sigma = \frac{2q^2}{N}\sum_{b=0}^{N-1}\int \left(\mathrm{d}^6X\right)_{12}\mathrm{d}^4\theta\, \mathcal{P}_b(X_2, X_1)\cos\left(a_+ + \frac{1}{2}\right)b\phi\left\{\cos\left(\frac{1}{2}b\phi\right)V_2\frac{(D^\alpha\bar{D}^2D_\alpha)_1}{8}V_1\right.\\ \left. + \bar{\partial}_2V_2\partial_1V_1 + \bar{S}_2S_1 - \sqrt{2}\bar{\partial}_2V_2S_1 - \sqrt{2}\bar{S}_2\partial_1V_1\right\}.\end{aligned} \tag{18}$$

We have replaced the two orbifolded delta functions (13) that appear in these graphs by one, and written that one out explicitly. In addition, we have performed a change of coordinates $z_1 \to e^{-\frac{i}{2}b\phi}z_1$ and symmetrized the result explicitly under $b \to -b$, by defining

$$\mathcal{P}_b(X_2, X_1) = \frac{1}{(\Box_6 - m^2)_2}\delta^6\left(z_2 - e^{-\frac{i}{2}b\phi}z_1\right)\frac{1}{(\Box_6 - m^2)_2}\delta^6\left(z_2 - e^{\frac{i}{2}b\phi}z_1\right), \tag{19}$$

which satisfies: $\mathcal{P}_{-b} = \mathcal{P}_b$. Here we have introduced an IR regulator mass m to identify the quadratic divergences in the dimensional reduction (DR) scheme. (In 5D m denotes the mass of the hypermultiplet.) In the delta functions we have only indicated the compact coordinates explicitly, as only there one encounters the phase $\exp(\pm\frac{i}{2}b\phi)$. We can read off from (18) whether the combination of self-energy diagrams of Fig. 1 has localized contributions. The contribution $b = 0$ gives the bulk amplitude. The contributions $b \neq 0$, sourced by the fixed points, depend on the orbifold:

- For the $\mathbb{Z}_2$ orbifold and the $\mathbb{Z}_2$ sector ($b = N/2$) of even ordered $\mathbb{Z}_N$ orbifolds we find no localized contributions, independently of the hypermultiplet twist eigenvalue a_+, since $\cos(a_+ + \frac{1}{2})\pi = 0$.
- However, for a generic $\mathbb{Z}_N$ orbifold with $N > 2$ we find contributions sourced by the fixed points for the sectors $b = \pm 1, \ldots, \pm[(N-1)/2]$.

This confirms and extends the results of Section 2 based on a component analysis on $S^1/\mathbb{Z}_2$.

3.2. Higher derivative counter terms and renormalized gauge couplings

After having distinguished bulk and localized fixed point contributions, we determine the counter terms required by this theory. The bulk contribution, $b=0$, is proportional to the 6D momentum integral

$$\int \frac{\mathrm{d}^D P}{(2\pi)^D} \Delta^m_{PK} = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{\mathrm{Vol}_W} \sum_{n\in\Lambda_K} \frac{1}{p^2+|n|^2+m^2} \frac{1}{(p-k)^2+|n-l|^2+m^2}. \tag{20}$$

The sum is over the 2D KK lattice Λ_K, see Table 1. The dimensionally regularized $D=2+d=6-2\epsilon$ integral is defined to include the factor $1/\mu^{d-4}$ so as to keep the mass dimension canonical throughout the regularization process. In Appendix A some steps are given to show that (20) can be represented as

$$\int \frac{\mathrm{d}^D P}{(2\pi)^D} \Delta^m_{PK} = \frac{\mu^2}{(4\pi)^{\frac{D}{2}}} \int_0^1 \mathrm{d}s \int_0^\infty \frac{\mathrm{d}t}{t^{\frac{d}{2}}} e^{-t\{s(1-s)(k^2+|l|^2)+m^2\}/\mu^2} \theta_W \begin{bmatrix} 0 \\ -sl \end{bmatrix} \left(\frac{i\mu^2}{2t}\right). \tag{21}$$

The Jacobi theta function $\theta_W\left[\begin{smallmatrix} 0 \\ -sl \end{smallmatrix}\right]$ associated with the winding mode lattice Λ_W (defined in (A.2)) is obtained after a Poisson resummation.

This expression contains a lot of information: from the expression of $\theta_W\left[\begin{smallmatrix} 0 \\ -sl \end{smallmatrix}\right]$ given in (A.2), it follows that $\theta_W\left[\begin{smallmatrix} 0 \\ -sl \end{smallmatrix}\right] \to 1$ in the UV ($t\to 0$), since all terms in the winding mode sum are exponentially suppressed. Therefore, to determine the counter terms we can put $\theta_W\left[\begin{smallmatrix} 0 \\ -sl \end{smallmatrix}\right]$ equal to 1. This shows that the bulk counter terms respect the 6D Lorentz invariance, since the external momenta appear in the combination $K^2=k^2+|l|^2$ only. The difference $\theta_W\left[\begin{smallmatrix} 0 \\ -sl \end{smallmatrix}\right]-1$ encodes the threshold corrections due to the (Poisson resummed) KK modes. Such threshold corrections have been studied for external zero modes ($l=0$) in the effective field theory limit of string theory [36,37] and extra dimension models [38]. Our result shows that for non-zero mode KK states the threshold corrections will be different from those for the zero modes. (This is related to non-local corrections to KK masses studied in Ref. [39].) The counter terms are determined by I_2^{div} given in (A.6) of Appendix A. The divergence proportional to K^2 in (A.6) requires the higher derivative counter term with a dimensionless coupling $1/h^2$:

$$\mathcal{L}^{\mathrm{hd}}_{\mathrm{gauge}} = -\frac{1}{2h^2N}\int \mathrm{d}^2\theta\, W^\alpha \Box_6 W_\alpha - \frac{1}{h^2N}\int \mathrm{d}^4\theta\, (\partial V \Box_6 \bar\partial V + \bar S \Box_6 S - \sqrt{2}\bar\partial V \Box_6 S - \sqrt{2}\partial V \Box_6 \bar S). \tag{22}$$

To conclude the Letter we compute the renormalized gauge couplings. Here we only give the parts of the couplings which do not depend on the external KK momenta. In addition we neglect the finite threshold correction due to the resummed KK states. In a complete treatment the brane localized kinetic terms should be taken into account [40]. For the sake of brevity we ignore all these complications; in a future publication we return to them in detail [19]. The renormalizations of bulk gauge couplings g and h, defined in (14) and (22) respectively, are given by

$$\frac{1}{g_R^2} = \frac{1}{g^2} + \frac{2q^2}{(4\pi)^3} m^2 \left[1+\ln\left(\frac{\mu^2}{m^2}\right)\right], \qquad \frac{1}{h_R^2} = \frac{1}{h^2} - \frac{1}{6}\frac{2q^2}{(4\pi)^3} \ln\left(\frac{\mu^2}{m^2}\right), \tag{23}$$

in the $\overline{\mathrm{DR}}$ scheme. (The 5D result is discussed in Section 2.3.) The coupling h renormalizes as anticipated by [16].

The localized contributions with $b\neq 0$ can be analyzed in a similar fashion. Neither of the KK loop momenta n_1, n_2 are free since they are fixed by the external KK momenta l_1, l_2 as

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \frac{-i}{2\sin\frac{1}{2}b\phi} \begin{pmatrix} 1 & -e^{-\frac{i}{2}b\phi} \\ -1 & e^{\frac{i}{2}b\phi} \end{pmatrix} \begin{pmatrix} l_1 \\ -l_2 \end{pmatrix}. \tag{24}$$

(This generalizes the violation of the KK-momenta that we encountered in Section 2.2.) Therefore the divergences can only come from the 4D momentum p in the loop

$$\int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{p^2+m^2+|n_1|^2} \frac{1}{(p-k)^2+m^2+|n_2|^2}. \tag{25}$$

The divergent part I_0^{div} (given in (A.6)) of this integral is independent of the external KK numbers l_1 and l_2 up to finite renormalizations which are ignored here. The running of the fixed point gauge couplings g_b, given in (15), reads

$$\frac{1}{(g_b^2)_R} = \frac{1}{g_b^2} - \frac{2q^2}{(4\pi)^2} \cos\left(a_+ + \frac{1}{2}\right) b\phi \cos\frac{1}{2} b\phi \ln\left(\frac{\mu^2}{m^2}\right). \tag{26}$$

Finally, we note that in the limit where we take the IR regulator m to zero, h_R and $(g_b)_R$ suffer from logarithmic IR singularities, and the coupling g_R becomes equal to its tree level value. All these statements of course ignore important finite volume effects that lead to finite KK number dependent renormalizations and will have to be discussed in [19].

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Appendix A. Regularization of the common scalar integrals

We extract the divergent parts of the integral (20) in 4, 5, and 6D. Using a Schwinger proper time reparameterization t and a Feynman parameter s this integral can be expressed as

$$\int \frac{\mathrm{d}^D P}{(2\pi)^D} \Delta^m_{PK} = \frac{1}{(4\pi)^{\frac{d}{2}} \mathrm{Vol}_W} \int_0^1 \mathrm{d}s \int_0^\infty \frac{\mathrm{d}t}{t^{\frac{d}{2}-1}} e^{-t\{s(1-s)(k^2+|l|^2)+m^2\}/\mu^2} \theta_K\begin{bmatrix} sl \\ 0 \end{bmatrix}\left(\frac{2it}{\mu^2}\right). \tag{A.1}$$

Here we have introduced the Jacobi theta functions for the KK and winding mode lattices

$$\theta_K\begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\tau) = \sum_{n\in\Lambda_K} e^{i\frac{\tau}{2}|n-\alpha|^2 - i(\bar{n}-\bar{\alpha})\bar{\beta} - i(n-\alpha)\beta}, \qquad \theta_W\begin{bmatrix} \beta \\ \alpha \end{bmatrix}(\tau) = \sum_{w\in\Lambda_W} e^{2i\tau|w-\beta|^2 - i(\bar{w}-\bar{\beta})\bar{\alpha} - i(w-\beta)\alpha}, \tag{A.2}$$

see Table 1. They are related to each other via a Poisson resummation

$$\theta_K\begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\tau) = \left(\frac{2\pi}{-i\tau}\right)^{\frac{D-d}{2}} \mathrm{Vol}\,\theta_W\begin{bmatrix} \beta \\ -\alpha \end{bmatrix}\left(\frac{-1}{\tau}\right). \tag{A.3}$$

Applying this relation to (A.1) we obtain the formula given in (21) in the main text. To determine the divergent parts of (21), we define the integral expression

$$I_{D-d}\big(K^2, m^2\big) = \frac{\mu^{D-d}}{(4\pi)^{\frac{D}{2}}} \int_0^1 \mathrm{d}s \int_0^\infty \frac{\mathrm{d}t}{t^{\frac{D}{2}-1}} e^{-t\{s(1-s)K^2+m^2\}/\mu^2}. \tag{A.4}$$

Provided that $d \in \mathbb{C}$ is suitably chosen, this expression is convergent and can be cast into the form

$$I_{D-d} = \frac{1}{(4\pi)^2} \left(\frac{m^2}{4\pi}\right)^{\frac{D-d}{2}} \left(4\pi \frac{\mu^2}{m^2}\right)^{2-\frac{d}{2}} \sum_{n\geqslant 0} (-)^n \frac{\Gamma(n+2-\frac{D}{2})n!}{(2n+1)!} \left(\frac{K^2}{m^2}\right)^n. \tag{A.5}$$

The terms with $0 \leqslant n \leqslant \frac{D}{2} - 2$ correspond to the terms in the Taylor expansion of (A.4) in K^2 with divergent coefficients, if we had not analytically continued $D \in \mathbb{C}$. We refer to these terms by the notation $I^{\mathrm{div}}_{D-d}(K^2, m^2)$. Explicitly, we have in $D - d = 0, 1, 2$ extra dimensions and with $d = 4 - 2\epsilon$:

$$\begin{aligned} I^{\mathrm{div}}_0 &= \frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon} - \gamma + \ln\left(4\pi \frac{\mu^2}{m^2}\right)\right), \qquad I^{\mathrm{div}}_1 = -\frac{1}{(4\pi)^2}|m|, \\ I^{\mathrm{div}}_2 &= -\frac{1}{(4\pi)^3}\left[m^2 + \left(\frac{1}{\epsilon} - \gamma + \ln\left(4\pi\frac{\mu^2}{m^2}\right)\right)\left(m^2 + \frac{1}{6}K^2\right)\right], \end{aligned} \tag{A.6}$$

where γ is the Euler constant. The case $D - d = 0$ gives the familiar 4D expression.

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