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Identifiability of Classes of Input-Output Systems

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The identifiability of abstract classes of input-output systems from a finite set of input-output experiments is considered. Both exact and approximate identifiability are addressed. Here, "system" means a function from an input space to an output space. With only linear structure on the output space and on the class of unknown systems, it is shown that a finite-dimensional class of systems is always exactly identifiable, and the identified systems have the form of interpolations on the input-output data. With linear and topological structure on the input space, output space, and space of unknown systems we state conditions under which a class of unknown systems can be identified to within a specified tolerance by interpolative identification models.

1. INTRODUCTION

The word "identification," as used here, is to have the following meaning: from prior knowledge and observation of outputs produced by known inputs it is desired to construct a mathematical model for the unknown physical system that allows one to predict to within a specified tolerance the system output (or perhaps only certain attributes of the output) resulting from an arbitrarily chosen input belonging to a specified class. We shall be concerned in this paper with *identifiability*; i.e., we shall be concerned with characterizing abstractly situations in which identification is possible, rather than with obtaining identification algorithms. (Identifiability can of course be shown by exhibiting an identification procedure, but the procedure does not have to be practical.) Note that this definition immediately implies three things. First, there must be two classes of mathematical models available—one to represent the class that the unknown physical system belongs to, and another to represent the class that the identified system belongs to. These mathematical models will be re-

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ferred to simply as "systems"; when a distinction is necessary we shall use terms such as "unknown system models" and "identification models." In most identification literature these two models are tacitly taken to be the same; we shall find it convenient to keep them distinct. In particular, one would normally want the identification models to be simpler. Second, an input-output system is a *function* from inputs to outputs—it does not include the concept of state. In other words, it is not a "dynamical system." Third, inputs and outputs can be observed and inputs can perhaps be controlled.

One final remark which, though perhaps obvious, needs to be stressed. Identification is fundamentally an inverse problem. But it is an ill-posed inverse problem (except in trivial cases) because the data are unavoidably insufficient; hence, as is characteristic of such problems, the solution may not be unique or it may be too sensitive to changes in the data. The major significance of the work reported here is to characterize those situations in which these difficulties can be circumvented.

We shall now briefly outline the contents of the paper. Section 2 establishes the basic notation and definitions that will be used throughout the paper. In Section 3 we consider exact identifiability at a very general level with the input and output spaces being merely sets. We then impose the single requirement that the output space be linear, which allows us to consider linear spaces of mappings. The fundamental result that we shall prove is that any *finite-dimensional* space of system mappings can be exactly identified by an appropriate finite set of input-output tests. We shall also introduce the concept of *interpolative identification*, meaning the identified system must interpolate (in some sense) to the data points. Interpolative identifications play a central role in the theory developed here. In Section 4 we address approximate identifiability. We impose metric space structure on the input space, Banach space structure on the output space, and we normally require that the class of unknown systems be a subset of the Banach space of continuous mappings from input space to output space with bounded range and the uniform norm (hence we shall consider uniform approximation). We establish a fundamental upper bound on identification error, and specify conditions on the class of unknown systems which guarantee that any one of them can be approximated to within a specified tolerance by an interpolative identification. In Section 5 we present conclusions and comments. Section 6 consists of three appendixes containing proofs of theorems.

2. Preliminaries

In this section we state the basic definitions and notation that will be used throughout the paper.

DEFINITION 1. A *system* is a triple (Y, F, U) where U is a set, the *input space*, Y is a set, the *output space*, and F is a mapping from U into Y. $\mathcal{F}(U, Y)$ denotes the class of all mappings from U into Y. Subsets of $\mathcal{F}(U, Y)$ will normally be denoted by letters such as $\mathcal{G}$ and $\mathcal{H}$.

With this terminology and notation we can define the fundamental problem with which we are concerned.

DEFINITION 2. The *identification problem* is this: the unknown system mapping F belongs to a given class $\mathcal{G} \subseteq \mathcal{F}(U, Y)$, and on the basis of input-output data we are to choose an identification model $\hat{F}$ from another given class $\mathcal{H} \subseteq \mathcal{F}(U, Y)$ such that $\hat{F}$ approximates F in some prescribed sense. (Note that the unknown systems and the identification models always have the same input and output spaces.)

Specifying the class $\mathcal{G}$ of unknown systems is not a problem that can be treated mathematically, but mathematical convenience may influence the choice of U, Y, and $\mathcal{H}$. In particular, we sometimes specify $\mathcal{H}$ parametrically. The following notation, introduced by Root in [5], is used to deal with such situations.

DEFINITION 3. Given input and output spaces U and Y and a set X denoting a parameter space, let f be a mapping from $X \times U$ into Y. Then $\mathcal{S} = (Y, f, X, U)$ is a *compound system*. A single system mapping F is given by $F(\cdot) = f(x, \cdot)$ for each $x \in X$. Let $\mathcal{H}$ be the set of all $F : U \to Y$ so defined. The mapping $\psi : X \to \mathcal{H}$ defined by $\psi(x) = F$ is called the *natural mapping* of $\mathcal{S}$. Let $\bar{f} : \mathcal{H} \times U \to Y$ be defined by $\bar{f}(F, u) = F(u)$. The compound system $\bar{\mathcal{S}} = (Y, \bar{f}, \mathcal{H}, U)$ is called the *natural representation* of $\mathcal{S}$.

We begin at a very general level and impose more structure on the input and output spaces and the classes of system mappings as we proceed.

As we mentioned in the Introduction, the definition of system used here is not what one usually means by "dynamical system" since there is no explicit provision for state. Although U and Y can be spaces of functions of a real variable (time), different initial states, in what would normally be called one system, must be represented by different values of the parameter x; i.e., different initial states result here in a class of systems. The emphasis in this paper is on situations in which X is interpreted as a parameter space in the ordinary sense.

If $F \in \mathcal{H} \subseteq \mathcal{F}(U, Y)$, then of course we write

$$y = F(u), \qquad u \in U,\ y \in Y,$$

and u is considered as variable. But because we shall be discussing identification we often want to consider F as variable; we then write

$$y = u(F), \qquad F \in \mathcal{H},\ y \in Y.$$

We shall always use u to denote both the element of U and the mapping from $\mathcal{F}(U, Y)$ into Y induced by that element. Furthermore, if for a fixed F there is known a sequence $(u_1, y_1), \ldots, (u_N, y_N)$ of input-output pairs, we denote $(u_1, \ldots, u_N) \in U^N$ by u^N and $(y_1, \ldots, y_N) \in Y^N$ by y^N, and we write

$$u^N(F) = (F(u_1), \ldots, F(u_N)) = y^N.$$

Again, u^N denotes both an element of U^N and the corresponding mapping from $\mathcal{F}(U, Y)$ into Y^N.

We have defined the identification problem as that of choosing an identification model based on input-output data. We shall always require that the data consist of a finite number of input-output pairs, and so we make the following definition.

DEFINITION 4. The process of obtaining input-output pairs (u_n, y_n), $n = 1, \ldots, N$, is called a *sequence of N experiments* with *test set* $\{u_1, \ldots, u_N\}$. A sequence of N experiments is described by the equation

$$y^N = u^N(F), \qquad F \in \mathcal{H}$$

with fixed u^N.

If u^N is invertible on $u^N(\mathcal{H})$ then an exact identification of $F \in \mathcal{H}$ is given by

$$F = (u^N)^{-1}(y^N), \qquad y^N \in u^N(\mathcal{H}),$$

where the inverse is restricted to $u^N(\mathcal{H})$. Hence, as we said in the Introduction, system identification is fundamentally an inverse problem, but it is usually ill-posed because u^N is not invertible.

The definitions of input space, output space, and system given here are very general and are to apply in all cases. For example, the inputs and outputs may be functions of time and outputs may be causally related to inputs. Also, inputs and outputs may be stochastic; we shall not, however, discuss stochastic systems in this paper. At the level of generality presented so far, the only thing we can discuss is exact identifiability, which is the subject of the next section. When we discuss approximate identifiability in Section 4 we shall impose topological structure on U and linear and topological structure on Y and $\mathcal{H}$.

3. Exact Identifiability

The system identification problem described in Definition 2 requires two compound systems (or classes of systems): one to represent the class of unknown systems and one to represent the class of identification models. In this section we are concerned with exact identifiability, so we take these two classes to be the same. Hence we take $\bar{\mathcal{S}} = (Y, \bar{f}, \mathcal{H}, U)$ in natural form (Definition 3) for both the class of unknown systems and the class of identification models, and we wish to exactly identify $F \in \mathcal{H}$ by a sequence of N experiments.

We begin at a very general level with U and Y merely sets; we then shall impose the single requirement that Y be a linear space. Topological structure has no bearing on exact identification.

Although the identification "theory" at this level of generality is trivial, it is of interest and needs to be stated precisely; furthermore, the results of this section form the foundation of our discussion of approximate identifiability.

Let U^0 and Y be fixed sets. U^0 is to be the universal or biggest allowable input space; any input space U must be a subset of U^0. Likewise, $\mathcal{G}^0(U^0, Y) \subseteq \mathcal{F}(U^0, Y)$ is the universal space of systems; any admissible $\mathcal{H}$ must be a subset of $\mathcal{G}^0$. The restriction of $F \in \mathcal{G}^0$ to U is denoted by $F|U$, unless there is no confusion in using simply F. Note that since U and Y are just sets, U^N and Y^N denote the Cartesian products.

Definition 5. The pair $(U, \mathcal{H})$, with $U \subseteq U^0$, $\mathcal{H} \subseteq \mathcal{G}^0$ is *experimentally functionally determined* (henceforth just "determined") by $(u_1, \ldots, u_N)$, $u_i \in U^0$, if there is a function $\rho_{u_1,\ldots,u_N} : U \times [u^N(\mathcal{H})] \to Y$ satisfying the condition

$$\rho_{u_1,\ldots,u_N}(u, u^N(F)) = F(u) \tag{1}$$

for all $u \in U$ and $F \in \mathcal{H}$. The function $\rho = \rho_{u_1,\ldots,u_N}$ is called an *experimental model determination function* (henceforth just "determination function" or DF) for $(U, \mathcal{H})$ with test set $(u_1, \ldots, u_N)$.

Remarks. (a) The concept of the DF was first defined by Root in [6]. The definition may be paraphrased as follows: any $F \in \mathcal{H}|U$ can be exactly identified by the sequence $(u_1, F(u_1)), \ldots, (u_N, F(u_N))$ of experiments if and only if $(U, \mathcal{H})$ is determined by $(u_1, \ldots, u_N)$. Hence we transfer the discussion of exact identifiability onto a discussion of determination functions.

(b) U may always be extended to contain the test set since one can define $\rho(u_i, u^N(\mathcal{H})) = F(u_i)$ for $i = 1, \ldots, N$. We shall always assume that this is done.

(c) No DF can carry topological information about $\mathcal{H}$; but it can carry *linear* information, as we shall see.

Note that $u^N : \mathcal{G}^0 \to Y^N$; its restriction to $\mathcal{H}$ will be denoted by $u^N|\mathcal{H}$. But $\mathcal{H}$ consists of mappings F with domain U^0, with restrictions to U denoted by $F|U$. We shall also need to consider u^N restricted to $\mathcal{H}|U$; this will be denoted by $u^N_{\mathcal{H}}$. The total images in Y^N of $u^N|\mathcal{H}$ and $u^N_{\mathcal{H}}$ are the same (namely $u^N(\mathcal{H})$), but they are different mappings; in particular, $u^N_{\mathcal{H}}$ may be one-to-one when $u^N|\mathcal{H}$ is not. If $u^N_{\mathcal{H}}$ is one-to-one we let $(u^N_{\mathcal{H}})^{-1}$ denote its inverse on its range $u^N(\mathcal{H})$. (The distinction between $u^N_{\mathcal{H}}$ and $u^N|\mathcal{H}$ disappears if $U = U^0$.)

As we mentioned, identification is fundamentally the problem of inverting $u^N_{\mathcal{H}}$; if $u^N_{\mathcal{H}}$ is invertible on its range, then an exact identification of $F \in \mathcal{H}$ is given by $F = (u^N_{\mathcal{H}})^{-1}(u^N(F))$. The following lemma justifies Definition 5—it says that the existence of a DF is equivalent to the existence of $(u^N_{\mathcal{H}})^{-1}$.

LEMMA 1. (a) *If $(u_1, \ldots, u_N)$ determines $(U, \mathcal{H})$ with DF ρ, then $(u^N_{\mathcal{H}})^{-1}$ exists.*

(b) *Conversely, for given U and $\mathcal{H}$, if $(u^N_{\mathcal{H}})^{-1}$ exists then $(u_1, \ldots, u_N)$ determines $(U, \mathcal{H})$ with DF defined by*

$$\rho(u, y^N) = [(u^N_{\mathcal{H}})^{-1}(y^N)](u) \tag{2}$$

for $y^N = (y_1, \ldots, y_N) \in u^N(\mathcal{H})$.

Proof. (a) Every $F \in \mathcal{H}|U$ can be exactly identified using $(u_i, F(u_i))$, $i = 1, \ldots, N$, hence $u^N_{\mathcal{H}}$ is one-to-one.

(b) The function (2) trivially satisfies $\rho(u, u^N(F)) = F(u)$ for $u \in U$ and $F \in \mathcal{H}$ and so is a DF. ▌

We shall now prove a technical result that will be needed; it says that for a given test set there is a biggest U and a biggest $\mathcal{H}$ determined by the test set. If $(U, \mathcal{H})$ and $(U', \mathcal{H}')$ are both determined by $(u_1, \ldots, u_N)$ with DFs ρ and ρ', and if $U \subseteq U'$ and $\mathcal{H} \subseteq \mathcal{H}'$, then we write $(U, \mathcal{H}) < (U', \mathcal{H}')$. The relation $<$ gives a partial ordering on the set

$$\mathcal{D}(u_1, \ldots, u_N) = \{(U, \mathcal{H}) : (U, \mathcal{H}) \text{ is determined by } (u_1, \ldots, u_N)\}.$$

It follows from Definition 5 that

$$\rho_{u_1,\ldots,u_N}(u, y^N) = \rho'_{u_1,\ldots,u_N}(u, y^N) \tag{3}$$

for all $u \in U$ and all $y^N \in u^N(\mathcal{H})$.

THEOREM 1 (Root [6]). *The set* $\mathcal{D}(u_1, \ldots, u_N)$ *with partial ordering* $<$ *contains a maximal element* $(U^*, \mathcal{H}^*)$.

Proof. See Appendix 1.

We shall now impose the single requirement that Y be a linear space over a field $\mathbb{K}$. ($\mathbb{K}$ is either the real or complex field.) $\mathcal{F}(U^0, Y)$ is then a linear space over $\mathbb{K}$ with addition of mappings and scalar multiplication defined in the usual way. We require that the universal space $\mathcal{G}^0(U^0, Y)$ of mappings be a linear subspace of $\mathcal{F}(U^0, Y)$. Then the mappings $u : \mathcal{G}^0 \to Y$ and $u^N : \mathcal{G}^0 \to Y^N$ are linear. The set $\mathcal{H}$ may or may not be a linear subspace of $\mathcal{G}^0$. Since Y is a linear space, Y^N now denotes the direct sum.

The following lemma states an elementary linearity property of DFs.

LEMMA 2. *Let* $(U, \mathcal{H})$ *have DF* ρ *with test set* $(u_1, \ldots, u_N)$. *If* aF', bF'', *and* $aF' + bF''$ *all belong to* $\mathcal{H}$ *for* $F', F'' \in \mathcal{H}$ *and any* $a, b \in \mathbb{K}$, *then for all* $u \in U$

$$\rho(u, u^N(aF' + bF'')) = a\rho(u, u^N(F')) + b\rho(u, u^N(F'')).$$

Proof. By the definition of the DF,

$$\begin{aligned}\rho(u, u^N(aF' + bF'')) &= aF'(u) + bF''(u)\\ &= a\rho(u, u^N(F')) + b\rho(u, u^N(F'')). \quad \blacksquare\end{aligned}$$

If $(U, \mathcal{H})$ is determined by $(u_1, \ldots, u_N)$ then there is a maximal $\mathcal{H}^*$ (Theorem 1). But now that $\mathcal{G}^0$ is a linear space more is true if U is fixed: there is a maximal *linear* $\mathcal{H}^*$. (In fact any linear subspace $\mathcal{H}^*$ of $\mathcal{G}^0$ satisfying $\mathcal{H}^* \oplus \ker(u^N) = \mathcal{G}^0$ is maximal and linear, and such an $\mathcal{H}^*$ exists.)

If $\mathcal{H}$ is also a linear subspace of $\mathcal{G}^0$ then its image $\mathcal{E} = u^N(\mathcal{H})$ is necessarily a linear subspace of Y^N. (Of course the converse is not true in general.) If in addition Y is finite-dimensional (say $\dim Y = M$), then for any $(U, \mathcal{H})$ determined by $(u_1, \ldots, u_N)$, $\mathcal{H}$ must be finite-dimensional and $\dim \mathcal{H} \leq MN$. To see this, suppose without loss of generality that $\mathcal{E} = Y^N$ and let $\{e_1, \ldots, e_M\}$ be a basis for Y. Define mappings in $\mathcal{H}$ as

$$F_{nm}(u) = \rho(u, (0, \ldots, 0, y_n = e_m, 0, \ldots, 0)). \tag{4}$$

Let the element $y_n \in Y$ be written as $y_n = \sum_{m=1}^{M} \alpha_{nm} e_m$. Then by Lemma 2

$$\begin{aligned}\sum_{n=1}^{N}\sum_{m=1}^{M} \alpha_{nm}F_{nm}(u) &= \sum_{n=1}^{N}\sum_{m=1}^{M} \alpha_{nm}\rho(u, (0, \ldots, 0, y_n = e_m, 0, \ldots, 0)) \\ &= \sum_{n=1}^{N} \rho\left(u, \left(0, \ldots, 0, \sum_{m=1}^{M} \alpha_{nm}e_m, 0, \ldots, 0\right)\right) \\ &= \sum_{n=1}^{N} \rho(u, (0, \ldots, y_n, 0, \ldots, 0)) \qquad (5) \\ &= \rho(u, (y_1, \ldots, y_N)) \\ &= F(u),\end{aligned}$$

where $(y_1, \ldots, y_N) = u_{\mathcal{H}}^N(F)$. Hence $\mathcal{H}$ is spanned by the set $\{F_{nm}\}$.

Although exact identification is not possible in general, there is one not very surprising (in view of the above discussion) but important case in which it is always possible—the case in which $\mathcal{H}$ is a finite-dimensional subspace of $\mathcal{G}^0$. To prove this, we need an elementary lemma in linear algebra.

LEMMA 3. *If U is a set, Y a linear space, and $\{F_1, \ldots, F_N\}$ a linearly independent set in $\mathcal{F}(U, Y)$, then there exists a set $\{u_1, \ldots, u_N\} \subseteq U$ such that $\{u^N(F_1), \ldots, u^N(F_N)\}$ is a linearly independent set in Y^N.*

Proof. See Appendix 2.

THEOREM 2. *For any $U \subseteq U^0$ and $\mathcal{H}$ any finite-dimensional linear subspace of $\mathcal{G}^0$ there exists a test set $(u_1, \ldots, u_N)$ and a DF $\rho = \rho_{u_1,\ldots,u_N}$ for $(U, \mathcal{H})$. ρ is linear in $(y_1, \ldots, y_N) \in \mathcal{E} = u^N(\mathcal{H})$. N need never be larger than the dimension of $\mathcal{H}|U$.*

Proof. Let $N = \dim \mathcal{H}|U$ and let $\{F_1, \ldots, F_N\}$ be a basis. Choose $\{u_1, \ldots, u_N\} \subseteq U$ as in Lemma 3. For any $F \in \mathcal{H}|U$ write $F = b_1F_1 + \cdots + b_NF_N$ so that

$$u^N(F) = b_1u^N(F_1) + \cdots + b_Nu^N(F_N). \qquad (6)$$

If $u^N(F)$ were zero for some nonzero $F \in \text{span}\{F_1, \ldots, F_N\}$ then (6) would provide a trivial linear relation among the $u^N(F_i)$, in contradiction to Lemma 3. Hence $u_{\mathcal{H}}^N$ is invertible on its range. By Lemma 1

$$\rho(u, (y_1, \ldots, y_N)) = [(u_{\mathcal{H}}^N)^{-1}(y_1, \ldots, y_N)](u)$$

defines a DF for $(U, \mathcal{H})$, and this ρ is clearly linear in $(y_1, \ldots, y_N)$. ∎

Note that the theorem guarantees only that N need never exceed $\dim \mathcal{H}|U$; there may be a test set with fewer than N elements.

Note also that Y need not be finite-dimensional. If however Y is finite-dimensional, then the DF of Theorem 2 is an interpolation (in some sense) of the test data. To see this, let $M = \dim Y$. Given $(U, \mathcal{H})$, with $\mathcal{H}$ linear, suppose there is a test set $(u_1, \ldots, u_N)$ and DF ρ for $(U, \mathcal{H})$. Then, as we remarked following Lemma 2, $\dim \mathcal{H} \leq MN$; suppose $\dim \mathcal{H} = MN$. Let $\{e_1, \ldots, e_M\}$ be a basis for Y and let

$$F_{nm}(u) = \rho(u, (0, \ldots, 0, y_n = e_m, 0, \ldots, 0)) \tag{7}$$

as we have done previously in Eq. (4). Note that

$$F_{nm}(u_i) = \begin{cases} e_m & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases} \tag{8}$$

Let $\alpha^{(m)}(u_n)$ denote the mth coordinate of $F_{nm}(u_n) \in Y$ and let $\gamma_{nm}^{(k)}(u)$ denote the kth coordinate of $F_{nm}(u) \in Y$. Then by the linearity property of ρ (Lemma 2) we have for any $F \in \mathcal{H}$.

$$\begin{aligned} \rho(u, u^N(F)) = F(u) &= \sum_{n=1}^{N} \sum_{m=1}^{M} \alpha^{(m)}(u_n) F_{nm}(u) \\ &= \sum_{n=1}^{N} \sum_{m=1}^{M} \sum_{k=1}^{M} \alpha^{(m)}(u_n) \gamma_{nm}^{(k)}(u) e_k \\ &= \sum_{n=1}^{N} [e_1 \cdots e_M] \Gamma_n(u) \begin{bmatrix} \alpha^{(1)}(u_n) \\ \vdots \\ \alpha^{(M)}(u_N) \end{bmatrix}, \end{aligned} \tag{9}$$

where $\Gamma_n(u)$ is the $M \times M$ matrix $[\gamma_{nm}^{(k)}(u)]$, $k, m = 1, \ldots, M$. If the scalar coordinates $\gamma_{nm}^{(k)}(u)$ satisfy

$$\gamma_{nm}^{(k)}(u) = \begin{cases} \gamma_n(u) & \text{if } k = m \\ 0 & \text{if } k \neq m \end{cases} \tag{10}$$

(i.e., $\Gamma_n(u)$ is the diagonal matrix $\gamma_n(u)I$), then

$$\rho(u, u^N(F)) = \sum_{n=1}^{N} \gamma_n(u) F(u_n). \tag{11}$$

Thus in a fairly general case a DF is given as an interpolation of test data. Such DFs play a central role in the theory of approximate identifiability to be developed, so we make a formal definition.

DEFINITION 6. If a DF ρ is given by

$$\rho(u, u^N(F)) = \sum_{n=1}^{N} \gamma_n(u)F(u_n) \tag{12}$$

for some finite integer N where each $\gamma_n(u) = \gamma_n(u; u_1, \ldots, u_N)$, $n = 1, \ldots, N$, is a scalar function of u satisfying

(i) $\gamma_n(u_i) = \delta_{ni}$

(δ_{ni} is the Kronecker delta) then ρ is a *weak interpolation*. If in addition each γ_n satisfies

(ii) $0 \leq \gamma_n(u) \leq 1$, $u \in U$
(iii) $\sum_{n=1}^{N} \gamma_n(u) = 1$, $u \in U$

then ρ is an *interpolation*.

EXAMPLE. Under suitable conditions on the input and output spaces and on the space of kernel functions, Volterra polynomial systems are exactly identifiable by a weak interpolation.

A *Volterra polynomial* is a system representation of the form

$$\begin{aligned} y(t) &= [F(u)](t) \\ &= \sum_{p=0}^{P} \int_A \cdots \int_A x_p(t, s_1, \ldots, s_p)u(s_1) \cdots u(s_p)\, ds_1 \cdots ds_p, \qquad t \in T, \end{aligned} \tag{13}$$

where A and T are subsets of the real line. We assume that $u \in L_2(A)$ and $x_p \in L_2(T \times A^p)$ for $p = 0, 1, \ldots, P$; it follows that $y \in L_2(T)$. Volterra polynomials provide quite general classes of mappings that can be used as mathematical system models.

We shall require the following conditions.

(a) U is an L-dimensional closed linear subspace of $L_2(A)$.
(b) Y is an M-dimensional closed linear subspace of $L_2(T)$.
(c) Each of the kernels x_p is symmetric in $s_1, \ldots, s_p$.

The parameter space X (see Definition 3) can then be chosen to be a K-dimensional closed linear subspace of $\sum_{p=0}^{P} \oplus L_2(T \times A^p)$ where

$$K = M \sum_{p=0}^{P} \binom{L + p - 1}{p}.$$

Now the natural mapping ψ maps the linear space X into the linear space of bounded continuous mappings from U into Y. (Here bounded means that bounded sets in U are sent into bounded sets in Y.) It is easy to verify that ψ is linear. Let $\mathcal{H} = \psi(X)$ denote the class of Volterra polynomials of the form of Eq. (13), subject to the conditions (a)–(c); then dim $\mathcal{H} = K$ since ψ is linear, one-to-one, and onto $\mathcal{H}$. It follows from Theorem 2 that there is a test set $(u_1, \ldots, u_N)$, with $N \leq K$, that determines $(U, \mathcal{H})$. Furthermore, it can be shown that the resulting DF is a weak interpolation and that the test set can be chosen to satisfy

$$N = \frac{K}{M} = \sum_{p=0}^{P} \binom{L + p - 1}{p}.$$

The proof of this result, although fairly straightforward, is very long; complete details can be found in [3].

Note that we do not construct a test set and DF; we get their existence by invoking Theorem 2. Nevertheless, the result that finite-dimensional classes of Volterra polynomials are exactly identifiable by a weak interpolation is of interest because the generality of the Volterra polynomials makes them important system models; at least it is difficult to think of other algebras of operators which have all of the properties of the Volterra polynomials.

The approach taken in this paper, and in this Volterra system example in particular, is not the one normally taken in the identification literature. Usually, one would identify the Volterra kernels by observing a single input-output pair $(u(t), y(t))$ over a finite *time duration*, rather than by observing a finite sequence $(u_1, y_1), \ldots, (u_N, y_N)$ of experiments as we have done; but note that our finite sequence of experiments need not arise from N different input functions. In other words, our approach is more general in the sense that identification based on observing a single input-output pair over a finite time can be developed as a special case. That development is not, however, as trivial as one might at first think because causality and system memory introduce subtle problems which must be dealt with more carefully than we can do here.

4. Approximate Identifiability

For consideration of approximate identifiability we assume that $U = U^0$ is a metric space with distance d and Y is a Banach space with norm $\|\cdot\|$. Then Y^N is a Banach space when equipped with the norm.

$$\|y^N\| = \max_{i=1,\ldots,N} \|y_i\|.$$

$\mathcal{G}^0(U, Y)$ will refer to either $\mathcal{C}^\infty(U, Y)$, the Banach space of continuous mappings from U into Y with bounded range and norm $\|F\| = \sup_{u \in U} \|F(u)\|$, or the Banach space $\mathcal{B}^\infty(U, Y)$ of mappings from U into Y with bounded range and the same norm. Any space of mappings denoted by $\mathcal{H}$ or $\mathcal{G}$ is then at least a metric subspace of $\mathcal{G}^0$.

Several remarks are in order. (1) U is required only to be a metric space. This is because we shall want to consider bounded and even totally bounded input spaces, which of course cannot be linear spaces. We usually take U to be a metric subspace of some normal linear space. (2) The spaces $\mathcal{C}^\infty$ and $\mathcal{B}^\infty$ consist of mappings having *bounded range*. Usually a (nonlinear) mapping is said to be *bounded* if it sends bounded sets into bounded sets; we shall find it convenient to restrict the mappings to have bounded range. (3) The uniform norm has been chosen for $\mathcal{G}^0$ and so the kind of approximation has been fixed. In many cases uniform approximation seems appropriate, particularly in the cases of bounded or totally bounded input spaces. If U is not bounded, say if U is a normed linear space, then linear operators are not accommodated by the uniform norm. Other norms are available for such cases; for example, one could use the Lipschitz norm defined by

$$\|F\|_L = \sup_{u_1 \neq u_2} \frac{\|F(u_1) - F(u_2)\|}{\|u_1 - u_2\|}.$$

The space of mappings from U into Y with $F(0) = 0$ and with finite Lipschitz norm is then a Banach space (see [4]). For our purposes, however, the Lipschitz norm does not seem to provide an appropriate measure of identification error because it emphasizes too much the fine structure of the mappings.

With these preliminaries, we can now discuss approximations. The mapping $u^N : \mathcal{G}^0 \to Y^N$ is linear and its linear operator norm (if it exists) will be denoted by $|u^N|$:

$$|u^N| = \sup_{\|F\|=1} \|u^N(F)\| = \sup_{\|F\|=1} \max_{i=1,\dots,N} \|F(u_i)\|.$$

LEMMA 4. (a) *u^N is a continuous linear operator on $\mathcal{G}^0$ with $|u^N| = 1$.*

(b) *If $(U, \mathcal{H})$ is determined by $(u_1, \dots, u_N)$ then $u^N_{\mathcal{H}}$ has an inverse $(u^N_{\mathcal{H}})^{-1}$ on its range. If $u^N_{\mathcal{H}}$ has closed range then $(u^N_{\mathcal{H}})^{-1}$ is bounded.*

(c) *If $(U, \mathcal{H})$ is determined by $(u_1, \dots, u_N)$ and the DF is interpolative (Definition 6) then $u^N_{\mathcal{H}}$ is an isometry.*

Proof. (a)

$$|u^N| = \sup_{\|F\|=1} \max_{i=1,\dots,N} \|F(u_i)\| \leq \sup_{\|F\|=1} \|F\| = 1.$$

To prove the reverse inequality, let F_0 be the constant mapping $F_0(u) = z \in Y$ with $\|z\| = 1$; then

$$|u^N| = \sup_{\|F\|=1} \|u^N(F)\| \geq \|u^N(F_0)\| = \max_{i=1,\ldots,N} \|F_0(u_i)\| = 1.$$

(b) The existence of $(u_{\mathcal{H}}^N)^{-1}$ is Lemma 1; the boundedness is a well-known fact (see [2]).

(c) For each $F \in \mathcal{H}$ we have $\|u_{\mathcal{H}}^N(F)\| \leq |u_{\mathcal{H}}^N| \cdot \|F\| = \|F\|$. On the other hand, for each $F \in \mathcal{H}$ and $u \in U$,

$$\|F(u)\| = \left\| \sum_{i=1}^{N} \gamma_i(u) F(u_i) \right\| \leq \sum_{i=1}^{N} \gamma_i(u) \|F(u_i)\|$$

$$\leq \max_j \|F(u_j)\| \sum_{i=1}^{N} \gamma_i(u) = \max_j \|F(u_j)\| = \|u_{\mathcal{H}}^N(F)\|,$$

so $\|F\| \leq \|u_{\mathcal{H}}^N(F)\|$. Hence $\|F\| = \|u_{\mathcal{H}}^N(F)\|$. ∎

DEFINITION 7. For a given U and test set $(u_1, \ldots, u_N)$ a mapping $\phi : u^N(\mathcal{G}^0) \to \mathcal{G}^0$ provides an *ε-identification* for $(U, \mathcal{G})$, $\mathcal{G} \subseteq \mathcal{G}^0$, if

$$\|F - (\phi \circ u^N)(F)\| \leq \varepsilon$$

for all $F \in \mathcal{G}$. It is a *continuous ε-identification* if ϕ is continuous.

It is not required by this definition that any $F \in \mathcal{G}$ be exactly identified by an ε-identification for $(U, \mathcal{G})$; it may be however that ϕ does exactly identify $(U, \mathcal{H})$ for some $\mathcal{H} \subset \mathcal{G}$, and it is this situation that we want to consider with $\mathcal{H}$ required to be linear.

LEMMA 5 (Root [6]). *Let $\mathcal{H}$ be a linear subspace of $\mathcal{G}^0(U, Y)$, let $(U, \mathcal{H})$ be determined by $(u_1, \ldots, u_N)$, and let $u_{\mathcal{H}}^N$ have closed range. Suppose that $F', F'' \in \mathcal{G}^0$ satisfy two conditions*:

(i) $\inf_{F\in\mathcal{H}} \|F' - F\| < \varepsilon$ *and* $\inf_{F\in\mathcal{H}} \|F'' - F\| < \varepsilon$,

(ii) $u^N(F') = u^N(F'') \in u^N(\mathcal{H})$.

Then

$$\|F' - F''\| \leq 2\varepsilon(1 + |(u_{\mathcal{H}}^N)^{-1}|). \tag{14}$$

Proof. There exist $F_1, F_2 \in \mathcal{H}$ such that $\|F' - F_1\| \leq \varepsilon$ and $\|F'' - F_2\| \leq \varepsilon$. Then, since $|u^N| = 1$ (Lemma 4),

$$\|u^N(F') - u^N(F_1)\| \leq \varepsilon,$$

$$\|u^N(F'') - u^N(F_2)\| \leq \varepsilon.$$

Therefore, since $u^N(F') = u^N(F'')$,

$$\|u^N(F_1) - u^N(F_2)\| \leq 2\varepsilon.$$

Now $(u^N_{\mathscr{H}})^{-1}$ exists as a bounded operator because $u^N_{\mathscr{H}}$ has closed range (Lemma 4), and $u^N(F_1)$ and $u^N(F_2)$ both lie in the domain of $(u^N_{\mathscr{H}})^{-1}$. Hence

$$\|F_1 - F_2\| = \|(u^N_{\mathscr{H}})^{-1}[u^N(F_1) - u^N(F_2)]\| \leq 2\varepsilon|(u^N_{\mathscr{H}})^{-1}|.$$

The conclusion follows from the triangle inequality. ∎

COROLLARY 1. *Let $(U, \mathscr{H})$ and F' satisfy the conditions of the lemma. Then*

$$\|F' - [(u^N_{\mathscr{H}})^{-1} \circ u^N](F')\| \leq \varepsilon(1 + |(u^N_{\mathscr{H}})^{-1}|). \tag{15}$$

COROLLARY 2. *If the conditions of the lemma are satisfied and if in addition the DF is interpolative, then the bounds become*

$$\|F' - F''\| \leq \varepsilon \tag{16}$$

and

$$\|F' - [(u^N_{\mathscr{H}})^{-1} \circ u^N](F')\| \leq 2\varepsilon \tag{17}$$

because $u^N_{\mathscr{H}}$ is an isometry by Lemma 4.

A trivial example shows that these bounds cannot be improved. Consider a case in which the system mappings are ordinary real functions: let $U = [0, 1]$, $Y = \mathbb{R}$, and $\mathscr{G}^0 = \mathscr{B}^\infty(U, Y)$. For a given $(u_1, ..., u_N)$ in U, let $\rho : U \times Y^N \to Y$ be an interpolative DF:

$$\rho(u, y^N) = \sum_{n=1}^{N} \gamma_n(u) y_n, \qquad y^N \in Y^N,$$

where $\gamma_n(u_i) = \delta_{ni}$, $0 \leq \gamma_n(u) \leq 1$, and $\sum_{n=1}^N \gamma_n(u) = 1$. Then define $F(u) = \rho(u, y^N)$ and let $\mathscr{H}$ be the linear subspace of $\mathscr{G}^0$ consisting of all such F. Suppose we choose the interpolation functionals in the following way. For each u_n let I_n be a left-closed right-open interval containing u_n (but no other u_j) such that the I_n are disjoint and cover U. (The last interval I_N is closed at both ends.) Then define

$$\gamma_n(u) = \begin{cases} 1 & \text{if } u \in I_n \\ 0 & \text{if } u \in I_n^c. \end{cases}$$

With these γ_n's, each F is a staircase function and $\mathcal{H}$ is the linear space of all staircase functions on [0, 1]. Now for an arbitrarily small positive δ consider a function $F' \in \mathcal{B}^\infty$ satisfying

$$F'(u_n) = 2\varepsilon - \delta, \qquad n = 1, \ldots, N,$$

and

$$F'(u) = 0 \qquad \text{for some } u \in [0, 1].$$

Then

$$\inf_{F \in \mathcal{H}} \|F' - F\| = \varepsilon - \frac{1}{2}\delta.$$

(In fact $F(u) = \varepsilon - \delta/2$ is the function in $\mathcal{H}$ satisfying $\|F' - F\| = \varepsilon - \delta/2$.) Also,

$$u^N(F') = (2\varepsilon - \delta, \ldots, 2\varepsilon - \delta) \in u^N(\mathcal{H}).$$

Hence

$$\|F' - [(u_{\mathcal{H}}^N)^{-1} \circ u^N](F')\| = 2\varepsilon - \delta.$$

Since δ is arbitrarily small, this example shows that the bound in (15) is the best possible.

We can now connect these bounds with the facts established in Section 3 about exact identifiability and bring them to bear on the problem of approximate identifiability. Suppose that $U = U^0$ is fixed and it is desired to identify an unknown F belonging to a specified $\mathcal{G} \subseteq \mathcal{G}^0$. The idea, roughly, is to approximate $\mathcal{G}$ by a finite-dimensional linear $\mathcal{H}$, obtain a DF ρ for $\mathcal{H}$, and use ρ to give an approximate identification of the unknown F. But we must say this more carefully. Suppose that $\mathcal{H}$ has been chosen; then there exists a test set $(u_1, \ldots, u_N)$ and a DF ρ for $(U, \mathcal{H})$ (Theorem 2). Let $\eta = \varepsilon(1 + |(u_{\mathcal{H}}^N)^{-1}|)$ and define

$$\mathcal{H}^\varepsilon = \{F \in \mathcal{G}^0 : d(F, \mathcal{H}) < \varepsilon\}.$$

Then Corollary 1 says this: *if u^N carries each $F \in \mathcal{H}^\varepsilon$ into the range of $u_{\mathcal{H}}^N$, then an identification is provided for any $F \in \mathcal{H}^\varepsilon$ with error not exceeding η.* If $\mathcal{G} \subseteq \mathcal{H}^\varepsilon$, the same can be said for each $F \in \mathcal{G}$. There are three difficulties. The first is to assure that $u^N(\mathcal{G}) \subseteq u^N(\mathcal{H})$. This is not serious because if $\mathcal{H}$ is maximal for fixed U and given test set, then $u^N(\mathcal{G}^0) =$

$u^N(\mathcal{H})$. The second, and more serious, difficulty is finding a suitable test set given a finite-dimensional linear $\mathcal{H}$. Furthermore, since $|(u^N_{\mathcal{H}})^{-1}|$ is not known until after the test set is chosen, the relation between ε and η is not known until *after* $\mathcal{H}$ and the test set are fixed. One would like $\mathcal{H}$ to be ε-dense in $\mathcal{G}$, but ε is not known a priori. The final difficulty is this: even if ε were known it would be hard to specify a finite-dimensional linear $\mathcal{H}$ that would be ε-dense in $\mathcal{G}$— presuming it exists. Apparently the thing to do is start the problem from the other end. That is, rather than starting with $(U, \mathcal{H})$ and attempting to find a test set and a function ρ such that ρ is a DF for $(U, \mathcal{H})$, we proceed as follows. With U fixed one starts with a test set $(u_1, \ldots, u_N)$ (probably chosen to be representative in some sense) and then chooses a function $\rho : U \times Y^N \rightarrow Y$ satisfying

$$\rho(u_n, (y_1, \ldots, y_N)) = y_n, \qquad (y_1, \ldots, y_N) \in \mathcal{E} \subseteq Y^N. \tag{18}$$

Then define $\hat{F} : U \rightarrow Y$ by

$$\hat{F}(u) = \rho(u, (y_1, \ldots, y_N)), \qquad (y_1, \ldots, y_N) \in \mathcal{E}, u \in U, \tag{19}$$

and let $\mathcal{H}$ be the set of all such $\hat{F}$. It follows then that

$$\rho(u, (\hat{F}(u_1), \ldots, \hat{F}(u_N))) = \hat{F}(u),$$

and so ρ is a DF for $(U, \mathcal{H})$ with test set $(u_1, \ldots, u_N)$. An interpolative ρ is a good choice for then $u^N_{\mathcal{H}}$ is an isometry (Lemma 4). If $\mathcal{E}$ is a linear subspace of Y^N then $\mathcal{H}$ is a finite-dimensional subspace of $\mathcal{G}^0$. Then by Corollary 2 every $F \in \mathcal{H}^\varepsilon$ can be identified to within 2ε. Note that we still have the difficulty of getting $\mathcal{H}$ to satisfy $\mathcal{H}^\varepsilon \supseteq \mathcal{G}$. In some general cases this can be done, and it is those cases that we shall now consider. We begin by imposing more conditions on the interpolation functionals of Definition 6.

DEFINITION 8. Let $\{\gamma_1, \ldots, \gamma_N\}$ be a set of interpolation functionals with respect to the test set $(u_1, \ldots, u_N)$; i.e., they are real-valued functions on U satisfying conditions (i), (ii), (iii) of Definition 6. They are called *continuous* interpolation functionals if

(iv) each γ_n is continuous on U,

and, for $\delta > 0$ they are called δ-local interpolation functionals if

(v) $\gamma_n(u) = 0$ whenever $d(u, u_n) \geq \delta$, $n = 1, \ldots, N$.

One construction of continuous δ-local interpolation functionals is given in [5]; a simpler construction is given in [7]. Note that U must be totally bounded to support δ-local interpolation functionals.

Continuous interpolation functionals can be used to construct ε-identifications. We begin with the following theorem.

THEOREM 3. *Let Y be finite-dimensional. Let U be a compact metric space and $\mathcal{G}$ a compact metric subspace of $\mathcal{C}^{\alpha}(U, Y)$. For a given $\varepsilon > 0$ there exists a test set $(u_1, \ldots, u_N)$ and a set of continuous local interpolation functionals $\{\gamma_1, \ldots, \gamma_N\}$ such that $\hat{F} \in \mathcal{C}^{\alpha}$ defined by*

$$\hat{F}(u) = \sum_{n=1}^{N} \gamma_n(u; u_1, \ldots, u_N) F(u_n) \tag{20}$$

satisfies $\|F - \hat{F}\| \leq \varepsilon$ for all $F \in \mathcal{G}$.

Proof. This theorem is a special case of Proposition I.7 of [5]; the proof is omitted.

Hence if $\mathcal{H}$ is defined as the linear subspace of $\mathcal{C}^{\alpha}$ comprising all $\hat{F}$ defined by Eq. (20) as F ranges over $\mathcal{C}^{\alpha}$, then

$$\rho(u, u^N(F)) = \sum_{n=1}^{N} \gamma_n(u) F(u_n) \tag{21}$$

is a DF for $(U, \mathcal{H})$, and uniformly good approximation is guaranteed not only for $\mathcal{G}$ but also for $\mathcal{H}^{\varepsilon} \supseteq \mathcal{G}$.

If $M = \dim Y$ then the parameters for the identification model defined by Eq. (20) are the MN coordinates of the $F(u_n)$; i.e., the model is finitely parametrized. The model is also linear in the parameters (Lemma 2).

Proposition I.7 in [5] is more general because Y is allowed to be an infinite-dimensional Banach space; but the identification model is not necessarily an interpolation and $\mathcal{H}$ is not necessarily linear. If, however, we are willing to sacrifice the condition that the interpolation functionals be δ-local, then we can show that the conclusion of Theorem 3 holds for noncompact U and infinite-dimensional Y.

THEOREM 4. *Let U be a metric space and Y a Banach space. Let $\mathcal{G}$ be totally bounded in $\mathcal{C}^{\alpha}(U, Y)$ and let*

$$\mathcal{G}(U) \stackrel{\text{def}}{=} \{F(u) : F \in \mathcal{G}, u \in U\}$$

be totally bounded in Y. For a given $\varepsilon > 0$ there exists a test set $(u_1, \ldots, u_N)$ and a set of continuous interpolation functionals $\{\gamma_1, \ldots, \gamma_N\}$ such that $\hat{F} \in \mathcal{C}^{\alpha}$ defined by

$$\hat{F}(u) = \sum_{n=1}^{N} \gamma_n(u; u_1, \ldots, u_N) F(u_n) \tag{22}$$

satisfies $\|F - \hat{F}\| \leq \varepsilon$ *for all* $F \in \mathscr{G}$. *If* Y *is finite-dimensional then the identification model* $\hat{F}$ *is finitely parametrized.*

Proof. See Appendix 3.

Remarks. (a) The interpolation functionals are not δ-local, but they are continuous.

(b) This theorem gives an interpolative identification even when Y is infinite-dimensional; when Y is finite-dimensional, then of course the identification model is finitely parametrized. In a practical problem one would naturally insist that the model be finitely parametrized. This theorem could be modified to provide a finitely parametrized model for infinite-dimensional Y, but the model would not be interpolative. We shall not pursue this further here.

(c) Theorem 4 is in fact a generalization of Theorem 3—the hypothesis in Theorem 4 that $\mathscr{G}(U)$ be totally bounded is not additional because it is implied by the hypotheses of Theorem 3. The construction of the interpolation functionals in Theorem 4, however, seems to be less amenable to calculation than that given in the original proof of Theorem 3.

(d) Theorems 3 and 4 give conditions guaranteeing the existence of a *simultaneous* interpolation and approximation; it is of course a natural requirement that the identification model should interpolate to the unknown system at the test inputs. There does not appear to be an exact parallel in the classical theory of approximation of functions, although there are some theorems with a similar flavor. In particular, we mention a theorem of Walsh: let S be a closed bounded set in the complex plane and let $z_1, \ldots, z_N$ be distinct points of S. If f is defined on S and is uniformly approximable there by polynomials, then f is uniformly approximable by polynomials p that also satisfy $p(z_n) = f(z_n)$ for $n = 1, \ldots, N$. (A proof of this theorem is given in [1].) This and Weierstrass's Theorem imply that any $f \in \mathscr{C}[a, b]$ is uniformly approximable by interpolative polynomials. But our Theorem 3 and 4 are different: they say that there is *some set* $(u_1, \ldots, u_N)$ such that we can interpolate and approximate *every* F *in the class* $\mathscr{G}$.

5. Conclusion

In this paper we have considered the identifiability of abstract classes of input-output systems based on a finite sequence of input-output experiments. We shall briefly summarize and organize our results.

We began with exact identiability. A system F belonging to a class $\mathscr{H}$ is exactly identifiable if and only if the mapping $u_{\mathscr{H}}^N$ is invertible; the identification is given by

$$F = (u^N_{\mathcal{H}})^{-1}(u^N(F)).$$

We defined the concept of determination function so that we can also say that every $F \in \mathcal{H}$ is exactly identifiable if and only if a DF ρ exists for the pair $(U, \mathcal{H})$. The fundamental result we proved is that if Y is a linear space and if $\mathcal{H}$ is a finite-dimensional linear space of systems, then every F in $\mathcal{H}$ is exactly identifiable by a finite sequence of test inputs. (The *same* test set applies to each F.) Furthermore, if Y is also finite-dimensional then the DF can be a weak interpolation. In particular, the identification of finite-dimensional subspaces of Volterra polynomials with finite-dimensional input and output spaces can be given by a weak interpolation.

We next addressed approximate identifiability by imposing topological structure—U is a metric space, Y a Banach space, and any class of systems is a subset of $\mathcal{C}^\infty(U, Y)$ or $\mathcal{B}^\infty(U, Y)$. We concentrated on interpolative then $u^N_{\mathcal{H}}$ is an isometry. The main result is Theorem 4 which gives conditions on U, Y, and $\mathcal{G}$ guaranteeing the existence of a test set $(u_1, \ldots, u_N)$ such that every $F \in \mathcal{G}$ can be identified to within a specified tolerance by the interpolation

$$\hat{F}(u) = \sum_{n=1}^{N} \gamma_n(u) F(u_n).$$

We therefore provide an approximate identification for a class $\mathcal{G}$ as follows. For a given test set $(u_1, \ldots, u_N)$, the function

$$\rho(u, u^N(F)) = \hat{F}(u) = \sum_{n=1}^{N} \gamma_n(u) F(u_n),$$

with the appropriate choice of interpolation functionals $\{\gamma_1, \ldots, \gamma_N\}$, is an interpolative DF for $(U, \mathcal{H})$ where $\mathcal{H}$ is the linear space of all such $\hat{F}$ as F ranges over all of $\mathcal{C}^\infty(U, Y)$. Then, provided $\mathcal{G}$ is compact in $\mathcal{C}^\infty$ and $\mathcal{G}(U)$ is compact in Y, Theorem 4 says that $\mathcal{G} \subseteq \mathcal{H}^\varepsilon$. Corollary 2 of Lemma 5 then says that any $F \in \mathcal{H}^\varepsilon$ can be identified to within 2ε; hence the same is true of any $F \in \mathcal{G}$. Note that the identification models are continuous interpolations belonging to a linear space of systems. If Y is finite-dimensional then Theorem 3 gives us continuous *local* interpolations belonging to a finite-dimensional $\mathcal{H}$.

We shall end with a few concluding remarks.

(i) We have considered only interpolative identifications, for two major reasons. One is that it simply is natural to require that the identified system should interpolate to the test data; the other reason is Lemma 4.

(ii) We have not considered random noise in the observations of the inputs or outputs. We get approximations in Section 4 because of modeling error, not because of noise.

(iii) More generally, we have not included stochastic systems, but neither have we specifically excluded them. Our results depend only on the topological and linear properties of U, Y, and $\mathcal{G}$ and so they still hold whether or not inputs and outputs have a stochastic interpretation. Of course, to say anything about stochastic systems beyond obvious generalities requires more structure than we have imposed here.

(iv) As we have mentioned previously, "system," as used in this paper, does not have the usual interpretation of "dynamical system." Clearly there is a connection between the identification of input-output systems from a finite number of input-output experiments and the identification of dynamical systems from a single input-output experiment performed over a finite interval of time. To clearly display that connection, however, requires a great deal of definition and detail which could not be included here.

APPENDIX 1: PROOF OF THEOREM 1

Let $\mathcal{D}' = \{(U', \mathcal{H}') : (U', \mathcal{H}') > (U, \mathcal{H})\} \subseteq \mathcal{D}$ and let $\mathcal{M}$ be an ordered subset of $\mathcal{D}'$, say $\mathcal{M} = \{(U_\alpha, \mathcal{H}_\alpha)\}_\alpha$. Put $\overline{U} = \bigcup_\alpha U_\alpha$ and $\overline{\mathcal{H}} = \bigcup_\alpha \mathcal{H}_\alpha$. Note that $\overline{U} \subseteq U^0$ and $\overline{\mathcal{H}} \subseteq \mathcal{G}^0$. A DF $\bar{\rho}$ for $(\overline{U}, \overline{\mathcal{H}})$ may be defined as follows. If $u \in \overline{U}$, then $u \in \mathcal{U}_\alpha$ or some α, and if $(y_1, \ldots, y_N) \in u^N(\overline{\mathcal{H}})$ then $(y_1, \ldots, y_N) \in u^N(\mathcal{H}_\beta)$ for some β (because $u^N(\overline{\mathcal{H}}) = u^N(\bigcup \mathcal{H}_\alpha) = \bigcup u^N(\mathcal{H}_\alpha)$). Either $(U_\alpha, \mathcal{H}_\alpha) < (U_\beta, \mathcal{H}_\beta)$ or the other way round since $\mathcal{M}$ is an ordered set. Suppose that $(U_\alpha, \mathcal{H}_\alpha) < (U_\beta, \mathcal{H}_\beta)$. Then $u \in U_\beta$. Define

$$\bar{\rho}(u, (y_1, \ldots, y_N)) = \rho_\beta(u, (y_1, \ldots, y_N)),$$

where ρ_β is the DF for $(U_\beta, \mathcal{H}_\beta)$. By the consistency of the DFs described by Eq. (3) this defines $\bar{\rho}$ on $U \times [u^N(\overline{\mathcal{H}})]$. $\bar{\rho}$ is a DF for $(\overline{U}, \overline{\mathcal{H}})$, and $(\overline{U}, \overline{\mathcal{H}})$ is an upper bound for $\mathcal{M}$. Hence by Zorn's Lemma $\mathcal{D}'$ contains a maximal element, and the proof is complete.

APPENDIX 2: PROOF OF LEMMA 3

(a) Since $\{F_1, \ldots, F_N\}$ is a linearly independent set, none of the F_i's is zero. So there is $u_1 \in U$ such that $F_1(u_1) \neq 0$.

(b) Consider $F_1(u_1)$ and $F_2(u_1)$ in Y.

Case (i). They are linearly dependent in Y. (If $Y = \mathbb{R}$ this is the only case.) Then there are scalars a_1, a_2 not both zero such that $a_1F_1(u_1) + a_2F_2(u_1) = 0$; a_1 and a_2 are unique to within a scalar multiple. There must then be a $u_2 \in U$ such that $a_1F_1(u_2) + a_2F_2(u_2) \neq 0$. (If $a_1F_1(u) + a_2F_2(u) = 0$ for all u then $a_1F_1 + a_2F_2 = 0$, contradicting linear independence.) We shall now show that $u^2(F_1) = (F_1(u_1), F_1(u_2))$ and $u^2(F_2) = (F_2(u_1), F_2(u_2))$ are linearly independent in Y^2. Consider a non-trivial relation

$$c_1u^2(F_1) + c_2u^2(F_2) = 0 \tag{23}$$

or

$$c_1F_1(u_1) + c_2F_2(u_1) = 0 \tag{24}$$

$$c_1F_1(u_2) + c_2F_2(u_2) = 0. \tag{25}$$

Equation (24) implies that $c_1 = ka_1$ and $c_2 = ka_2$ since a_1 and a_2 are unique up to a multiplier. But then Eq. (25) cannot hold because of the way in which u_2 was chosen. Hence $c_1 = c_2 = 0$ and $u^2(F_1)$ and $u^2(F_2)$ are linearly independent.

Case (ii). $F_1(u_1)$ and $F_2(u_1)$ are linearly independent in Y. Choose any $u_2 \neq u_1$ in U. Consider again the linear relation (23). Equation (24) implies $c_1 = c_2 = 0$, and so $u^2(F_1)$ and $u^2(F_2)$ are linearly independent in Y^2.

(c) General inductive step. Suppose that $\{u_1, \ldots, u_n\}$, $n < N$, have been chosen so that $u^n(F_1), \ldots, u^n(F_n)$ are linearly independent in Y^n.

Case (i). $u^n(F_1), \ldots, u^n(F_n), u^n(F_{n+1})$ are linearly dependent in Y^n. Then there are scalars $a_1, \ldots, a_{n+1}$ not all zero such that $\sum_{i=1}^{n+1} a_iu^n(F_i) = 0$. The a_i's are unique up to a scalar multiple. Now if for every $u \in U$ we have $\sum_{i=1}^{n+1} a_iF_i(u) = 0$, then $\sum_{i=1}^{n+1} a_iF_i = 0$, which contradicts the independence of the F_i's. Hence there is $u_{n+1} \in U$ such that $\sum_{i=1}^{n+1} a_iF_i(u_{n+1}) \neq 0$. Consider

$$c_1u^{n+1}(F_1) + \cdots + c_{n+1}u^{n+1}(F_{n+1}) = 0 \tag{26}$$

or

$$\begin{aligned} c_1F_1(u_1) + \cdots + c_{n+1}F_{n+1}(u_1) &= 0 \\ &\vdots \\ c_1F_1(u_n) + \cdots + c_{n+1}F_{n+1}(u_n) &= 0 \\ c_1F_1(u_{n+1}) + \cdots + c_{n+1}F_{n+1}(u_{n+1}) &= 0. \end{aligned} \tag{27}$$

The first n equations of (27) imply $c_i = ka_i$ for $i = 1, \ldots, n + 1$. But then the last equation cannot hold because of the choice of u_{n+1}. Hence $c_1 = \cdots = c_{n+1} = 0$ and $u^{n+1}(F_1), \ldots, u^{n+1}(F_{n+1})$ are linearly independent in Y^{n+1}.

Case (ii). $u^n(F_1), \ldots, u^n(F_n), u^n(F_{n+1})$ are linearly independent in Y^n. Choose any u_{n+1} distinct from $u_1, \ldots, u_n$. Then for any linear relation (26), the first n equations of (27) imply $c_1 = \cdots = c_{n+1} = 0$ and so $u^{n+1}(F_1), \ldots, u^{n+1}(F_{n+1})$ are linearly independent in Y^{n+1}. This completes the proof.

APPENDIX 3: PROOF OF THEOREM 4

Fix $\varepsilon > 0$. Let $\{F_1, \ldots, F_K\}$ be an $\varepsilon/4$-*net for* $\mathcal{G}^-$; i.e., for any $F \in \mathcal{G}$ there is a k such that $\|F - F_k\| \leq \varepsilon/4$. ($\mathcal{G}^-$ denotes the closure of $\mathcal{G}$.) Let $\{B_1, \ldots, B_L\}$ be open $\varepsilon/4$-balls covering $\mathcal{G}(U)^-$, and put $\tilde{B}_l^k = F_k^{-1}(B_l)$ for $l = 1, \ldots, L$, $k = 1, \ldots, K$. Each $\tilde{B}_l^k$ is open in U since F_k is continuous. Consider the nonempty sets among $\tilde{B}_{i_1}^1 \cap \tilde{B}_{i_2}^2 \cap \cdots \cap \tilde{B}_{i_K}^K$, $i_1, \ldots, i_K = 1, \ldots, L$. These sets are open and cover U (in fact, for each k any $u \in U$ is in $F_k^{-1}(B_l)$ for some l). Enumerate the nonempty $D_n = \tilde{B}_{i_1}^1 \cap \tilde{B}_{i_2}^2 \cap \cdots \cap \tilde{B}_{i_K}^K$, $n = 1, \ldots, N$, in such a way that no D_n is contained in a union of preceding ones. (If a candidate for D_n is contained in a union of $D_1, \ldots, D_{n-1}$, discard it.)

Now choose any $u_1 \in D_1$. Since D_2 is not a subset of D_1 and D_2 is not empty, we can choose $u_2 \in D_2$ with $u_2 \neq u_1$. Similarly, since D_3 is not contained in $D_1 \cup D_2$, we can choose $u_3 \in D_3$ distinct from u_1 and u_2; in general we can choose $u_n \in D_n$ with $u_n \notin D_1 \cup D_2 \cup \cdots \cup D_{n-1}$. There is thus obtained a set $\{u_1, \ldots, u_n\}$ of distinct points with $u_n \in D_n$. Put

$$C_n = D_n \setminus \bigcup_{i \neq n} \{u_i\}, \qquad n = 1, \ldots, N.$$

Each C_n is nonvacuous (because $u_n \in C_n$) and the C_n's provide an open cover for U (because the D_n's provide an open cover).

To define the interpolation functionals γ_n first define for $n = 1, \ldots, N$ and all $u \in U$,

$$f_n(u) = \frac{d(u, C_n^c)}{d(u, u_n) + d(u, C_n^c)}.$$

This is well defined because $d(u, u_n) + d(u, C_n^c) > 0$ for all $u \in U$. Now define

$$\gamma_n(u) = \frac{f_n(u)}{\sum_{i=1}^N f_i(u)}.$$

It is easy to verify that the γ_n satisfy (i) $0 \le \gamma_n(u) \le 1$ for all u and n, (ii) $\sum_{n=1}^{N} \gamma_n(u) = 1$ for all u, (iii) $u \in C_n^c$ implies $\gamma_n(u) = 0$, (iv) $u = u_n$ implies $\gamma_n(u) = 1$, and (v) γ_n is continuous on U. In other words the γ_n are continuous interpolation functionals.

Now define the approximation of $F \in \mathcal{G}$:

$$\hat{F}(u) = \sum_{n=1}^{N} \gamma_n(u; u_1, \dots, u_N) F(u_n), \qquad u \in U.$$

It remains to show that $\|F - \hat{F}\| \le \varepsilon$. First note that for any $k \in \{1, \dots, K\}$

$$\|F(u) - \hat{F}(u)\| \le \|F(u) - F_k(u)\| + \|F_k(u) - \hat{F}_k(u)\| + \|\hat{F}_k(u) - \hat{F}(u)\|.$$

Consider each of the three terms.

(1) Choose k so that $\|F - F_k\| \le \varepsilon/4$; then $\|F(u) - F_k(u)\| \le \varepsilon/4$.

(2) Let

$$\begin{aligned} \|F_k(u) - \hat{F}_k(u)\| &= \|F_k(u) - \sum_{n=1}^{N} \gamma_n(u) F_k(u_n)\| \\ &= \|\sum_{n=1}^{N} \gamma_n(u)[F_k(u) - F_k(u_n)]\| \\ &\le \sum_{n=1}^{N} \gamma_n(u) \|F_k(u) - F_k(u_n)\|. \end{aligned}$$

Let $C_{n_1}, \dots, C_{n_P}$ (where $P \le N$) be those C_n's with $u \in C_n$. Then

$$\sum_{n=1}^{N} \gamma_n(u)\|F_k(u) - F_k(u_n)\| = \sum_{p=1}^{P} \gamma_{n_p}(u)\|F_k(u) - F_k(u_{n_p})\|$$

because $\gamma_n(u) = 0$ for the other indices. But

$$\|F_k(u) - F_k(u_{n_p})\| \le \varepsilon/2.$$

In fact

$$u \in C_{n_p} \subseteq D_{n_p} = \tilde{B}^1_{i_1} \cap \tilde{B}^2_{i_2} \cap \cdots \cap \tilde{B}^K_{i_K}$$

for some $i_1, \dots, i_K$. Also $u_{n_p} \in C_{n_p}$ and so $u, u_{n_p} \in F_k^{-1}(B_{i_1})$ in particular. Hence $F_k(u) \in B_{i_1}$ and $F_k(u_{n_p}) \in B_{i_1}$; but B_{i_1} is an open $\varepsilon/4$-ball in Y. Hence

$$\|F_k(u) - \hat{F}_k(u)\| \leq \varepsilon/2.$$

(3) Let

$$\begin{aligned}\|\hat{F}_k(u) - \hat{F}_k(u)\| &= \left\|\sum_{n=1}^{N} \gamma_n(u)[F_k(u_n) - F(u_n)]\right\| \\ &\leq \sum_{n=1}^{N} \gamma_n(u)\|F_k(u_n) - F(u_n)\| \\ &\leq \|F_k - F\| \leq \varepsilon/4.\end{aligned}$$

Hence,

$$\|F - \hat{F}\| \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon,$$

and the theorem is proved.

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