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Quasilinear elliptic inclusions of hemivariational type: Extremality and compactness of the solution set

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Abstract

We consider the Dirichlet boundary value problem for an elliptic inclusion governed by a quasilinear elliptic operator of Leray–Lions type and a multivalued term which is given by the difference of Clarke's generalized gradient of some locally Lipschitz function and the subdifferential of some convex function. Problems of this kind arise, e.g., in mechanical models described by nonconvex and nonsmooth energy functionals that result from nonmonotone, multivalued constitutive laws. Our main goal is to characterize the solution set of the problem under consideration. In particular we are going to prove that the solution set possesses extremal elements with respect to the underlying natural partial ordering of functions, and that the solution set is compact. The main tools used in the proofs are abstract results on pseudomonotone operators, truncation, and special test function techniques, Zorn's lemma as well as tools from nonsmooth analysis.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. In this paper we consider the Dirichlet problem for the following elliptic inclusion:

$$Au + \partial j(\cdot, u) - \partial\beta(\cdot, u) \ni f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

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where A is a second order quasilinear differential operator in divergence form of Leray–Lions type given by

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\big(x, \nabla u(x)\big) \quad \text{with } \nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right),$$

and the function $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is assumed to be the primitive of some locally bounded and Borel measurable function $g : \Omega \times \mathbb{R} \to \mathbb{R}$, i.e.,

$$j(x, s) = \int_0^s g(x, \tau)\, d\tau. \tag{1.2}$$

Thus $j(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and Clarke's generalized gradient $\partial j(x, \cdot) : \mathbb{R} \to 2^{\mathbb{R}} \setminus \emptyset$ of j with respect to its second argument exists which is defined by

$$\partial j(x, s) := \big\{\zeta \in \mathbb{R} \mid j^0(x, s; r) \geqslant \zeta r,\ \forall r \in \mathbb{R}\big\}, \tag{1.3}$$

where $j^0(x, s; r)$ denotes the generalized directional derivative of j at s in the direction r given by

$$j^0(x, s; r) = \limsup_{y \to s,\, t \downarrow 0} \frac{j(x, y + tr) - j(x, y)}{t},$$

cf., e.g., [8, Chapter 2]. The function $\beta : \Omega \times \mathbb{R} \to \mathbb{R}$ is assumed to be the primitive of some Borel measurable function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ which is monotone nondecreasing in its second variable, i.e.,

$$\beta(x, s) = \int_0^s h(x, \tau)\, d\tau. \tag{1.4}$$

Thus $\beta(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is convex with $\partial\beta(x, \cdot) : \mathbb{R} \to 2^{\mathbb{R}} \setminus \emptyset$ denoting the usual subdifferential of β with respect to its second argument, and one has

$$\partial\beta(x, s) = \big[\underline{h}(x, s), \bar{h}(x, s)\big], \tag{1.5}$$

where $\underline{h}$ and $\bar{h}$ denote the left-sided and right-sided limits of h, respectively, with respect to the second argument.

Differential inclusions of the form (1.1) have attracted increasing attention over the last decade mainly due to its many applications in mechanics and engineering, cf., e.g., [11,12]. This type of inclusions arise, e.g., in mechanical problems when nonconvex, nonsmooth energy functionals (so-called superpotentials) occur, which result from nonmonotone, multivalued constitutive laws, such as, for example, certain contact and friction problems, cf., e.g., [9,11,12]. The special case in which the function $s \mapsto j(\cdot, s)$ is a convex function too, leads to a multivalued term that is given by the generalized gradient of so called d.c.-functions (difference of convex functions). Elliptic and parabolic inclusions with generalized Clarke's gradient of d.c.-functions have been treated, e.g., in [1–5] under the assumption that appropriately defined super- and subsolutions are available.

Our main goal is to prove extremality and compactness results of the solution set of (1.1) when $j(x,\cdot):\mathbb{R}\to\mathbb{R}$ is only locally Lipschitz, and without assuming super- and subsolutions.

The plan of the paper is as follows. In Section 2 we give the basic notions and hypotheses, and formulate the main result. In Section 3 we prove the existence of a priori bounds of (1.1), and in Section 4 we review the elliptic counterpart of an extremality result of some auxiliary hemivariational inequality recently obtained by the authors in the parabolic case. The proof of our main result is given in Section 5. In Section 6 we consider as a special case an inclusion of the form (1.1) with A given by the p-Laplacian. The main tools used in the proofs are abstract results on pseudomonotone operators, truncation, and special test function techniques, Zorn's lemma as well as tools from nonsmooth analysis.

2. Notation, hypotheses, and main result

Let $V = W^{1,p}(\Omega)$ and $V_0 = W_0^{1,p}(\Omega)$, $1 < p < \infty$, denote the usual Sobolev spaces, and V^* and V_0^* their corresponding dual spaces, respectively.

We assume $f \in V_0^*$ and impose the following hypotheses of Leray–Lions type on the coefficient functions a_i, $i = 1,\dots,N$, of the operator A.

(A1) Each $a_i:\Omega\times\mathbb{R}^N\to\mathbb{R}$ satisfies the Carathéodory conditions, i.e., $a_i(x,\xi)$ is measurable in $x\in\Omega$ for all $\xi\in\mathbb{R}^N$ and continuous in ξ for almost all $x\in\Omega$. There exist a constant $c_0>0$ and a function $k_0\in L^q(\Omega)$, $1/p+1/q=1$, such that

$$\bigl|a_i(x,\xi)\bigr| \leqslant k_0(x) + c_0|\xi|^{p-1}$$

for a.e. $x\in\Omega$ and for all $\xi\in\mathbb{R}^N$.

(A2) $\sum_{i=1}^N (a_i(x,\xi)-a_i(x,\xi'))(\xi_i-\xi_i') > 0$ for a.e. $x\in\Omega$ and for all $\xi,\xi'\in\mathbb{R}^N$ with $\xi\neq\xi'$.

(A3) $\sum_{i=1}^N a_i(x,\xi)\xi_i \geqslant \nu|\xi|^p - k_1(x)$ for a.e. $x\in\Omega$ and for all $\xi\in\mathbb{R}^N$ with some constant $\nu>0$ and some function $k_1\in L^1(\Omega)$.

As a consequence of (A1) and (A2) the semilinear form a associated with the operator A by

$$\langle Au,\varphi\rangle := a(u,\varphi) = \int_\Omega \sum_{i=1}^N a_i(x,\nabla u)\frac{\partial\varphi}{\partial x_i}\,dx, \quad \forall\varphi\in V_0,$$

is well defined for any $u\in V$, and the operator $A:V_0\to V_0^*$ is continuous, bounded, and monotone. Here $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V_0 and V_0^*. A partial ordering in $L^p(\Omega)$ is defined by $u\leqslant w$ if and only if $w-u$ belongs to the positive cone $L_+^p(\Omega)$ of all nonnegative elements of $L^p(\Omega)$. This induces a corresponding partial ordering also in the subspaces $V_0\subset V$ of $L^p(\Omega)$. We define the notion of weak solution of problem (1.1) as follows.

Definition 2.1. A function $u \in V_0$ is a *solution* of the BVP (1.1) if there are functions $\eta \in L^q(\Omega)$ and $\gamma \in L^q(\Omega)$ such that the following holds:

(i) $\eta(x) \in \partial j(x, u(x))$ and $\gamma(x) \in \partial\beta(x, u(x))$ for a.e. $x \in \Omega$,
(ii) $\langle Au, \varphi\rangle + \int_\Omega (\eta(x) - \gamma(x))\varphi(x)\,dx = \langle f, \varphi\rangle$, $\forall \varphi \in V_0$.

As for the function g related with j by (1.2) and the function h related with β by (1.4) we assume the following hypotheses.

(H1) The function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies
  (i) g is Borel measurable in $\Omega \times \mathbb{R}$, and $g(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is locally bounded;
  (ii) There exists a constant $c_1 \geqslant 0$ such that
$$g(x, s_1) \leqslant g(x, s_2) + c_1 (s_2 - s_1)^{p-1}$$
  for a.e. $x \in \Omega$ and for all s_1, s_2 with $s_1 < s_2$;
  (iii) There is a function $k_2 \in L^q_+(\Omega)$ and a constant $\mu_1 \geqslant 0$ such that
$$\big|g(x, s)\big| \leqslant k_2(x) + \mu_1|s|^{p-1}$$
  for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$.

(H2) The function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is Borel measurable, monotone nondecreasing in its second argument, and satisfies with some function $k_3 \in L^q_+(\Omega)$ and with some constant $\mu_2 \geqslant 0$ the growth condition
$$\big|h(x, s)\big| \leqslant k_3(x) + \mu_2|s|^{p-1}$$
for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$.

(H3) Let $c_F > 0$ denote the best constant in Poincaré–Friedrichs inequality and denote $\mu := \mu_1 + \mu_2$, where μ_1 and μ_2 are the nonnegative constants of (H1) and (H2), respectively. Then the positive constant ν of (A3) is related with μ and c_F by
$$c_F\mu < \nu.$$

Remark. Condition (H1)(ii) implies that Clarke's gradient of the function $j : \Omega \times \mathbb{R} \to \mathbb{R}$ fulfills the following condition: For $\eta_i \in \partial j(x, s_i)$, $i = 1, 2$, one has
$$\eta_1 \leqslant \eta_2 + c_1(s_2 - s_1)^{p-1}$$
for a.e. $x \in \Omega$ and for all s_1, s_2 with $s_1 < s_2$. This condition is mainly used in the proof of Theorem 4.1 which we recall in Section 4.

Definition 2.2. A solution u^* of (1.1) is called the *greatest solution* if for any solution u of (1.1) we have $u \leqslant u^*$. Similarly, u_* is the *least solution* if for any solution u one has $u_* \leqslant u$. The least and greatest solutions of the BVP (1.1) are called the *extremal* ones.

The main result of the present paper is given by the following theorem.

Theorem 2.1. *Let hypotheses* (A1)–(A3) *and* (H1)–(H3) *be satisfied. Then the BVP* (1.1) *possesses extremal solutions and the solution set of all solutions of* (1.1) *is a compact subset in* V_0.

The proof of Theorem 2.1 requires several preliminary results which are of interest in its own and which will be provided in Sections 3 and 4. We will assume throughout the rest of the paper that the hypotheses of Theorem 2.1 are satisfied.

3. A priori bounds

In this section we shall prove the existence of a priori bounds for the solutions of (1.1) which are crucial in the proof of our main result. To this end we consider the following auxiliary BVP:

$$\text{Find } u \in V_0\colon \quad Au = f + k + \mu|u|^{p-1} \quad \text{in } V_0^*, \tag{3.1}$$

$$\text{Find } u \in V_0\colon \quad Au = f - k - \mu|u|^{p-1} \quad \text{in } V_0^*, \tag{3.2}$$

where $k \in L_+^q(\Omega)$ is given by $k(x) = k_2(x) + k_3(x)$ and $\mu = \mu_1 + \mu_2$.

Lemma 3.1. *There exist solutions of the BVP* (3.1) *and* (3.2) *and their respective solution sets are bounded in* V_0.

Proof. We prove the existence and boundedness of the solution set for the BVP (3.1) only, since the same arguments can be applied for the BVP (3.2). Let P denote the Nemytskij operator related with the function $s \mapsto \mu|s|^{p-1}$, then $P : L^p(\Omega) \to L^q(\Omega) \subset V_0^*$ is continuous and bounded, and due to the compact embedding $V_0 \subset L^p(\Omega)$ it follows that $P : V_0 \to V_0^*$ is compact. Thus due to the property of A the operator $A - P : V_0 \to V_0^*$ is a bounded, continuous, and pseudomonotone operator. Rewriting the BVP (3.1) in the form

$$u \in V_0\colon \quad (A - P)u = f + k \quad \text{in } V_0^*, \tag{3.3}$$

and noting that $f + k \in V_0^*$, there exist solutions of (3.3) provided $A - P$ is coercive, i.e., the following holds:

$$\frac{\langle (A - P)u, u\rangle}{\|u\|_{V_0}} \to \infty \quad \text{as } \|u\|_{V_0} \to \infty. \tag{3.4}$$

Let $c_F > 0$ denote the best constant in Poincaré–Friedrichs inequality, i.e., in

$$\|u\|_{L^p(\Omega)} \leqslant c_F \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in V_0,$$

then by means of (A3) we obtain

$$\begin{aligned} \big\langle (A - P)u, u\big\rangle &\geqslant \nu\|\nabla u\|^p_{L^p(\Omega)} - \|k_1\|_{L^1(\Omega)} - \mu\|u\|^p_{L^p(\Omega)} \\ &\geqslant (\nu - c_F\mu)\|\nabla u\|^p_{L^p(\Omega)} - \|k_1\|_{L^1(\Omega)}, \end{aligned} \tag{3.5}$$

which proves the coercivity due to (H3) and the fact that $\|u\| = \|\nabla u\|_{L^p(\Omega)}$ defines an equivalent norm in V_0. The coercivity argument applies also to get the boundedness of the solution set of (3.1). To this end let u be any solution of the BVP (3.1), then from (3.3) and (3.5) one gets

$$(\nu - c_F\mu)\|\nabla u\|^p_{L^p(\Omega)} - \|k_1\|_{L^1(\Omega)} \leqslant \big\langle (A - P)u, u\big\rangle \leqslant \big(\|f\|_{V_0^*} + \|k\|_{L^q(\Omega)}\big)\|u\|_{V_0}, \tag{3.6}$$

which proves the assertion. □

For the proof of the next result we recall the notion of *directedness* of a partially ordered set.

Definition 3.1. Let $(\mathcal{P}, \leqslant)$ be a partially ordered set. A subset $\mathcal{C}$ of $\mathcal{P}$ is said to be *upward directed* if for each pair $x, y \in \mathcal{C}$ there is $z \in \mathcal{C}$ such that $x \leqslant z$ and $y \leqslant z$, and $\mathcal{C}$ is *downward directed* if for each pair $x, y \in \mathcal{C}$ there is $w \in \mathcal{C}$ such that $w \leqslant x$ and $w \leqslant y$. If $\mathcal{C}$ is both upward and downward directed it is called *directed*.

Lemma 3.2. *The solution sets of the BVP* (3.1) *and* (3.2), *respectively, are directed sets.*

Proof. We are going to prove the assertion for the BVP (3.1) only, since analogous arguments apply for the BVP (3.2). Let us denote by $\mathcal{S}$ the solution set of the BVP (3.1). Then $\mathcal{S} \neq \emptyset$ in view of Lemma 3.1. Given $u_1, u_2 \in \mathcal{S}$, then $\underline{u} := \max(u_1, u_2) \in V_0$ is a subsolution of the BVP (3.1), see, e.g., [6, Lemma 6.1.3]. Define the following truncation operator T:

$$(Tu)(x) = \begin{cases} u(x) & \text{if } \underline{u}(x) \leqslant u(x), \\ \underline{u}(x) & \text{if } u(x,t) < \underline{u}(x). \end{cases}$$

It is well known that $T : V_0 \to V_0$ is a continuous and bounded operator, cf., e.g., [6, Chapter C.4]. Consider the auxiliary BVP,

$$\text{Find } u \in V_0\text{:} \quad Au = f + k + \mu|Tu|^{p-1} \quad \text{in } V_0^*. \tag{3.7}$$

The same arguments as in the proof of Lemma 3.1 apply to ensure the existence of solutions of the BVP (3.7). We shall show that any solution u of the BVP (3.7) satisfies $u \geqslant \underline{u}$, and hence due to $Tu = u$ it follows that u is a solution of the BVP (3.1) which exceeds the given solutions u_1 and u_2. This proves $\mathcal{S}$ is upward directed. Let u be any solution of (3.7) and recall that $\underline{u}$ is a subsolution of (3.1), i.e., we have

$$\langle A\underline{u}, \varphi\rangle \leqslant \langle f, \varphi\rangle + \mu \int_\Omega \big(k + |\underline{u}|^{p-1}\big)\varphi\, dx, \quad \forall \varphi \in V_0 \cap L_+^p(\Omega), \tag{3.8}$$

and u is any solution of (3.7), i.e.,

$$\langle Au, \varphi\rangle = \langle f, \varphi\rangle + \mu \int_\Omega \big(k + |Tu|^{p-1}\big)\varphi\, dx, \quad \forall \varphi \in V_0. \tag{3.9}$$

Taking as special nonnegative test function $\varphi = (\underline{u} - u)^+ := \max(\underline{u} - u, 0) \in V_0 \cap L_+^p(\Omega)$, we obtain by subtracting (3.9) from (3.8) the inequality

$$\begin{aligned} &\int_\Omega \sum_{i=1}^N \big(a_i(x, \nabla\underline{u}) - a_i(x, \nabla u)\big) \frac{\partial(\underline{u} - u)^+}{\partial x_i}\, dx \\ &\quad \leqslant \mu \int_\Omega \big(|\underline{u}|^{p-1} - |Tu|^{p-1}\big)(\underline{u} - u)^+\, dx \\ &\quad = \mu \int_{\{\underline{u} > u\}} \big(|\underline{u}|^{p-1} - |\underline{u}|^{p-1}\big)(\underline{u} - u)\, dx = 0, \end{aligned} \tag{3.10}$$

where $\{\underline{u} > u\} := \{x \in \Omega \mid \underline{u}(x) > u(x)\}$. By means of (A2) we deduce from (3.10) that $\nabla(\underline{u} - u)^+ = 0$, and thus $(\underline{u} - u)^+ = 0$ which yields $\underline{u} \leqslant u$. This completes the proof for $\mathcal{S}$ being upward directed. Noting that for any solutions $u_1, u_2 \in \mathcal{S}$ the function $\bar{u} := \min(u_1, u_2)$ is a supersolution of the BVP (3.1), see, e.g., [6, Lemma 6.1.3], we can show in a similar way that $\mathcal{S}$ is also downward directed, and thus the directedness of $\mathcal{S}$. □

Lemma 3.3. *The BVP* (3.1) *and* (3.2) *have extremal solutions.*

Proof. Again the proof will be given for the BVP (3.1) only, since for the BVP (3.2) it can be done similarly. Moreover, we will concentrate on the existence of the greatest solution of the BVP (3.1), because the existence of the least solution follows by obvious dual reasoning. Let $\mathcal{S}$ denote the solution set of the BVP (3.1). First we shall show the existence of a maximal element of $\mathcal{S}$ with respect to the underlying partial ordering by means of Zorn's lemma. To this end let $\mathcal{C} \subset \mathcal{S}$ be any well-ordered chain which is bounded in V_0 by Lemma 3.1, and thus, in particular, also bounded in $L^p(\Omega)$. Then there exists an increasing sequence (u_n) of $\mathcal{C}$ which converges strongly in $L^p(\Omega)$ and weakly in V_0 to $w := \sup(\mathcal{C})$. We claim that w belongs to $\mathcal{S}$. From (3.1) we immediately get

$$\langle Au_n, u_n - w\rangle = \langle f, u_n - w\rangle + \mu \int_\Omega \big(k + |u_n|^{p-1}\big)(u_n - w)\,dx. \tag{3.11}$$

Taking the convergence property of (u_n) and its boundedness into account we obtain from (3.11),

$$\limsup_{n\to\infty} \langle Au_n, u_n - w\rangle \leqslant 0, \tag{3.12}$$

which implies $u_n \to w$ strongly in V_0 due to due to the (S_+)-condition satisfied by the operator A, cf., e.g., [6, Theorem D.2.1]. Thus we may pass to the limit in (3.1) as $n \to \infty$, i.e., in

$$u_n \in V_0\colon \quad Au_n = f + k + \mu\,|u_n|^{p-1} \quad \text{in } V_0^*,$$

which proves that $w \in \mathcal{S}$. Thus $\mathcal{C}$ possesses an upper bound in $\mathcal{C}$, so that Zorn's lemma can be applied, which ensures the existence of a maximal element $\bar{w}$. Because $\mathcal{S}$ is, in particular, upward directed the maximal element is unique and must be the greatest one. Thus $\bar{w}$ is the greatest solution. □

By means of Lemmas 3.1–3.3 we are now able to derive a priori bounds of the original BVP (1.1).

Lemma 3.4. *Let $\bar{w}$ be the greatest solution of the BVP* (3.1) *and $\underline{w}$ be the least solution of the BVP* (3.2) *according to Lemma* 3.3. *Then any solution u of the BVP* (1.1) *is contained in $[\underline{w}, \bar{w}]$.*

Proof. Let u be any solution of (1.1), i.e., we have by Definition 2.1,

$$\langle Au, \varphi\rangle + \int_\Omega \big(\eta(x) - \gamma(x)\big)\varphi(x)\,dx = \langle f, \varphi\rangle, \quad \forall \varphi \in V_0, \tag{3.13}$$

where $\eta(x) \in \partial j(x, u(x))$ and $\gamma(x) \in \partial\beta(x, u(x))$ for a.e. $x \in \Omega$. In view of the growth conditions of (H1) and (H2) we have

$$\left|\eta(x)\right| \leqslant k_2(x) + \mu_1\left|u(x)\right|^{p-1}, \qquad \left|\gamma(x)\right| \leqslant k_3(x) + \mu_2\left|u(x)\right|^{p-1}. \tag{3.14}$$

From (3.13) and (3.14) we see that u is a subsolution of the BVP (3.1). Now the same arguments as in the proof Lemma 3.2 apply which show that there exist solutions of the BVP (3.1) that are greater than u. However, $\bar{w}$ is the greatest solution of (3.1), and thus it exceeds u which proves that $\bar{w}$ is an upper bound of any solution of the original problem (1.1). The proof for $\underline{w}$ to be a lower bound is carried out in a similar way. □

4. Auxiliary hemivariational inequality

In this section we consider the following subproblem of the BVP (1.1):

$$Au + \partial j(\cdot, u) \ni f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{4.1}$$

where $f \in V_0^*$ is a given element. One can show that any solution of problem (4.1) is a solution of the hemivariational inequality

$$\langle Au, \varphi - u\rangle + \int_\Omega j^0(\cdot\,, u; \varphi - u)\,dx \geqslant \langle f, \varphi - u\rangle, \quad \forall\varphi \in V_0. \tag{4.2}$$

We recall an existence and extremality result for the BVP (4.1) in terms of appropriately defined super- and subsolutions which may be considered as the elliptic counterpart of a result recently obtained by the authors in [7] in the parabolic case. To this end we first provide a generalization of the notion of super- and subsolution known for singlevalued equations to the hemivariational inequality (4.1).

Definition 4.1. A function $\bar{u} \in V$ is called a *supersolution* of the BVP (4.1) if there is a function $\bar{v} \in L^q(\Omega)$ such that

(i) $\bar{u} \geqslant 0$ on $\partial\Omega$,
(ii) $\bar{v}(x) \in \partial j(x, \bar{u}(x))$ for a.e. $x \in \Omega$,
(iii) $\langle A\bar{u}, \varphi\rangle + \int_\Omega \bar{v}(x)\varphi(x)\,dx \geqslant \langle f, \varphi\rangle$, $\forall\varphi \in V_0 \cap L_+^p(\Omega)$.

Similarly, a function $\underline{u} \in V$ is a *subsolution* of the BVP (4.1) if the reversed inequalities hold in Definition 4.1 with $\bar{u}$ and $\bar{v}$ replaced by $\underline{u}$ and $\underline{v}$, respectively.

The following existence and extremality result can be deduced from [7].

Theorem 4.1. *Let $\underline{u}$ and $\bar{u}$ be sub- and supersolutions of* (4.1), *respectively, satisfying $\underline{u} \leqslant \bar{u}$. Then the BVP* (4.1) *has extremal solutions within the order interval* $[\underline{u}, \bar{u}]$.

5. Proof of Theorem 2.1

In this section we are going to prove our main result. The proof is inspired by an idea of the first author used in [3,5] to treat boundary hemivariational inequalities of the d.c.-type. Our proof will be given in two steps.

Proof. (a) Existence of extremal solutions of (1.1).

Lemma 3.4 provides a priori bounds $\bar{w}$ and $\underline{w}$ satisfying $\underline{w} \leqslant \bar{w}$, where $\bar{w}$ is the greatest solution of the BVP (3.1) and $\underline{w}$ is the least solution of the BVP (3.2). We are going to prove that (1.1) possesses extremal solutions within the order interval $[\underline{w}, \bar{w}]$, which proves the existence of extremal solutions of (1.1). Let us concentrate on the existence of the greatest solution, because the existence of the least solution can be shown similarly.

We recall that the subdifferential $\partial\beta(x, s)$ is generated by the function $h : \Omega \times \mathbb{R} \to \mathbb{R}$, which is monotone nondecreasing in its second argument via

$$\partial\beta(x, s) = \left[\underline{h}(x, s), \bar{h}(x, s)\right]$$

with $s \mapsto \underline{h}(x, s)$ and $s \mapsto \bar{h}(x, s)$ being the left- and right-sided limits, respectively, of $s \mapsto h(x, s)$. Denote by $\underline{H}$ and $\bar{H}$ the Nemytskij operator associated with $\underline{h}$ and $\bar{h}$, respectively. By hypothesis (H2) the operators $\underline{H}, \bar{H} : L^p(\Omega) \to L^q(\Omega)$ are well defined, monotone nondecreasing, but not necessarily continuous. Consider the following hemivariational inequality:

$$u \in V_0: \quad Au + \partial j(\cdot, u) \ni f + \bar{H}(u). \tag{5.1}$$

Our goal is to show that (5.1) has the greatest solution u^* within $[\underline{w}, \bar{w}]$, and that u^* is the greatest solution of the original problem (1.1). To this end let us consider first the following hemivariational inequality with given right-hand side:

$$u \in V_0: \quad Au + \partial j(\cdot, u) \ni f + \bar{H}(\bar{w}). \tag{5.2}$$

By (H1)(iii) and (H2), and taking into account that $\bar{w}$ is the greatest solution of (3.1), we get for any $\bar{\eta} \in \partial j(\cdot, \bar{w})$ the estimate

$$A\bar{w} + \bar{\eta} = f + k + \mu|\bar{w}|^{p-1} + \bar{\eta} \geqslant f + k_1 + \mu_1|\bar{w}|^{p-1} \geqslant f + \bar{H}(\bar{w}),$$

which proves that $\bar{w}$ is a supersolution of (5.2). Analogously one shows that $\underline{w}$ is a subsolution of (5.2). Thus by applying Theorem 4.1 with the right-hand side $f + \bar{H}(\bar{w}) \in V_0^*$ there exist extremal solutions of (5.2) within the interval $[\underline{w}, \bar{w}]$. Let u_1 denote the greatest solution of (5.2) within $[\underline{w}, \bar{w}]$, and consider next the hemivariational inequality

$$u \in V_0: \quad Au + \partial j(\cdot, u) \ni f + \bar{H}(u_1). \tag{5.3}$$

By the monotonicity of $\bar{H}$ we have $\bar{H}(u_1) \leqslant \bar{H}(\bar{w})$, and thus u_1 is a supersolution for (5.3). One readily verifies that $\underline{w}$ is a subsolution for (5.3) as well. Again by applying Theorem 4.1 there exist extremal solutions of (5.3) within $[\underline{w}, u_1]$. In this way we are able to define by induction the following iteration process: Let $u_0 := \bar{w}$ and define by $u_{n+1} \in V_0$ the greatest solution of

$$u \in V_0: \quad Au + \partial j(\cdot, u) \ni f + \bar{H}(u_n) \tag{5.4}$$

within $[\underline{w}, u_n]$. Due to $u_{n+1} \in [\underline{w}, u_n]$ this iteration yields a monotone nonincreasing sequence (u_n) that satisfies

$$\underline{w} \leqslant \cdots \leqslant u_{n+1} \leqslant u_n \leqslant \cdots \leqslant u_1 \leqslant u_0 := \bar{w} \tag{5.5}$$

and

$$Au_{n+1} + v_{n+1} = f + \bar{H}(u_n) \quad \text{in } V_0^*, \tag{5.6}$$

where $v_{n+1} \in \partial j(\cdot, u_{n+1})$ and $v_{n+1} \in L^q(\Omega)$. Since the sequence (u_n) can easily be seen to be bounded in V_0, and because $(v_n) \subset L^q(\Omega)$ is bounded as well, we obtain the following convergence properties:

(i) $u_n \rightharpoonup u^*$ in V_0,
(ii) $u_n \to u^*$ in $L^p(\Omega)$,
(iii) $v_n \rightharpoonup v^*$ in $L^q(\Omega)$ (for some subsequence which is again denoted by (v_n)),

where in (iii) we have $v^* \in \partial j(\cdot, u^*)$. The boundedness of $(\bar{H}(u_n))$ in $L^q(\Omega)$ and the convergence properties (i)–(iii) imply

$$\limsup_{n\to\infty} \langle Au_n, u_n - u^* \rangle \leqslant 0,$$

so that we obtain

(iv) $Au_n \rightharpoonup Au^*$ in V_0^*.

The function $s \mapsto \bar{h}(x, s)$ related with the Nemytskij operator $\bar{H}$ is monotone nondecreasing and right-sided continuous, so that by means of Lebesgue's dominated convergence theorem and due to the a.e. monotone pointwise convergence of the sequence (u_n) according to (5.5) we get

$$\int_\Omega \bar{H}(u_n)\varphi\, dx \to \int_\Omega \bar{H}(u^*)\varphi\, dx \tag{5.7}$$

for all $\varphi \in L^p(\Omega)$. The convergence properties (i)–(iv) above and (5.7) allow us to pass to the limit in (5.6) as $n \to \infty$, which shows that u^* is a solution of the BVP (5.1) within $[\underline{w}, \bar{w}]$. Moreover, u^* is the greatest solution of (5.1) within $[\underline{w}, \bar{w}]$. To verify this let $u \in [\underline{w}, \bar{w}]$ be any solution of (5.1). Then u is, in particular, a lower solution of (5.1). Replacing in the iteration above $\underline{w}$ by u we see that $u \leqslant u_n \leqslant \bar{w}$ holds for all n. Thus we get $u \leqslant u^*$, i.e., u^* is the greatest solution of (5.1) in $[\underline{w}, \bar{w}]$. Now defining $\gamma^* := \bar{H}(u^*)$, obviously one has $\gamma^*(x) \in \partial\beta(x, u^*(x))$ for a.e. $x \in \Omega$, and thus u^* satisfies

$$Au^* + v^* - \gamma^* = f \quad \text{in } V_0^*,$$

which means that u^* is a solution of the original problem (1.1) as well.

Finally, to prove that u^* is the greatest solution of (1.1) take any solution $\tilde{u}$ of (1.1), which by definition satisfies

$$A\tilde{u} + \tilde{\eta} - \tilde{\gamma} = f,$$

where $\tilde{\eta} \in \partial j(\cdot, \tilde{u})$ and $\tilde{\gamma} \in \partial\beta(\cdot, \tilde{u}) \subset [\underline{H}(\tilde{u}), \bar{H}(\tilde{u})]$. Since $\tilde{\gamma} \leqslant \bar{H}(\tilde{u})$ we see that $\tilde{u} \leqslant \bar{w}$ is a subsolution of the hemivariational inequality (5.1). By the same iteration procedure introduced above with $\underline{w}$ replaced by $\tilde{u}$ we get $\tilde{u} \leqslant u_n \leqslant \bar{w}$ which implies $\tilde{u} \leqslant u^*$, and thus u^* must be the greatest solution of the original problem (1.1). The existence of the least solution u_* can be shown by obvious dual reasoning which completes the proof of the extremality result. We remark the interesting fact that γ^* related with the greatest solution u^* is thus given by $\gamma^* = \max\{\partial\beta(\cdot, u^*)\}$.

(b) Compactness of the solution set.

We denote by $\mathcal{T}$ the set of all solutions of the BVP (1.1). Then $\mathcal{T} \subset [u_*, u^*]$, where u_* and u^* is the least and the greatest solution of (1.1). Let $(u_n) \subset \mathcal{T}$ be any sequence. Then (u_n) is bounded in V_0 and one has the following convergence properties for some subsequence denoted by (u_k):

(i) $u_k \rightharpoonup u$ in V_0,
(ii) $u_k \to u$ in $L^p(\Omega)$,
(iii) $\eta_k \rightharpoonup \eta$ and $\gamma_k \rightharpoonup \gamma$ in $L^q(\Omega)$,

where $\eta_k \in \partial j(\cdot, u_k)$ and $\gamma_k \in \partial\beta(\cdot, u_k)$, and we have

$$Au_k + \eta_k - \gamma_k = f \quad \text{in } V_0^*. \tag{5.8}$$

The compact embedding $L^p(\Omega) \subset V_0$ implies the compact embedding $L^q(\Omega) \subset V_0^*$ which yields

$$\eta_k \to \eta, \quad \gamma_k \to \gamma \quad \text{in } V_0^*, \tag{5.9}$$

where $\eta \in \partial j(\cdot, u)$ and $\gamma \in \partial\beta(\cdot, u)$. Due to (5.9) from (5.8) we get

$$\langle Au_k, u_k - u\rangle = \langle f - \eta_k + \gamma_k, u_k - u\rangle \to 0,$$

so that in view of the pseudomonotonicity of A we get $Au_k \rightharpoonup Au$ in V_0^* as $k \to \infty$. Passing to the limit as $k \to \infty$ in (5.8) yields

$$Au + \eta - \gamma = f \quad \text{in } V_0^*,$$

and thus $u \in \mathcal{T}$. Finally, by applying the (S_+)-property of A we get the strong convergence $u_k \to u$ in V_0 which completes the compactness proof. □

6. Special case and remarks

As a special case of (1.1) we consider the following BVP:

$$u \in V_0: \quad -\Delta_p u + \partial j(\cdot, u) - \partial\beta(\cdot, u) \ni f, \tag{6.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $1 < p < \infty$ denotes the p-Laplacian. Obviously, $-\Delta_p$ satisfies (A1)–(A3). The variational characterization of the first Dirichlet eigenvalue λ_1 of $-\Delta_p$ which is positive and given by

$$\lambda_1 = \inf_{0 \neq u \in V_0} \frac{\int_\Omega |\nabla u|^p\,dx}{\int_\Omega |u|^p\,dx}$$

(see [10]), enables us to sharpen condition (H3) as follows:

(H4) Let $\mu := \mu_1 + \mu_2 < \lambda_1$ be satisfied.

The following result is an immediate consequence of the general result obtained in the preceding sections.

Theorem 6.1. *Under the conditions* (H1), (H2), *and* (H4) *the BVP* (6.1) *has extremal solutions and the solution set is compact in* V_0.

Remarks. (i) Our main result, Theorem 2.1, can be extended to more general Leray–Lions operators A such as, e.g.,

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\big(x, u(x), \nabla u(x)\big) + a_0\big(x, u(x), \nabla u(x)\big).$$

Only for the sake of simplifying our presentation, and in order to emphasize the main idea we have taken a nonlinear, monotone operator A.

(ii) In case that one assumes the existence of an ordered pair $\underline{w} \leqslant \bar{w}$ that satisfies the following inequalities:

$$\bar{w} \in V: \quad A\bar{w} + \bar{\eta} \geqslant f + \bar{H}(\bar{w}), \quad \text{where } \bar{\eta} \in \partial j(\cdot, \bar{w}),$$

and

$$\underline{w} \in V: \quad A\underline{w} + \underline{\eta} \leqslant f + \underline{H}(\underline{w}), \quad \text{where } \underline{\eta} \in \partial j(\cdot, \underline{w}),$$

then the results obtained in this paper apply to prove extremality and compactness of the solution set contained in the interval $[\underline{w}, \bar{w}]$. In this case hypothesis (H3) can be dropped, and only local growth conditions of g and β with respect to the interval $[\underline{w}, \bar{w}]$ are required.

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