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Comparison of the asymptotic stability properties for two multirate strategies☆

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Abstract

This paper contains a comparison of the asymptotic stability properties for two multirate strategies. For each strategy, the asymptotic stability regions are presented for a 2×2 test problem and the differences between the results are discussed. The considered multirate schemes use Rosenbrock type methods as the main time integration method and have one level of temporal local refinement. Some remarks on the relevance of the results for 2×2 test problems are presented.

MSC: 65L05; 65L06; 65L50; 65M06; 65M20

Keywords: Multirate time stepping; Local time stepping; Ordinary differential equations; Asymptotic stability

1. Introduction

Many practical applications give rise to systems of ordinary differential equations (ODEs) with different time scales which are localized over the components. To solve such systems multirate time stepping strategies are considered. These strategies integrate the slow components with large time steps and the fast components with small time steps. In this paper we will focus on two strategies: the *recursive refinement* strategy proposed in [4,7] and the *compound step* strategy used in [1,3,9,10]. We will analyze these multirate approaches for solving systems of ODEs

$$w'(t) = F(t, w(t)), \quad w(0) = w_0, \tag{1.1}$$

with $w_0 \in \mathbb{R}^m$.

In the *recursive refinement* strategy, given a global time step τ, a tentative approximation at the new time level is computed first. For those components, where the error estimator indicates that smaller steps would be needed, the computation is redone with a smaller time step $\frac{1}{2}\tau$. At this refinement stage, the values at the intermediate time levels of components which are not refined might be needed. These values can be calculated by using interpolation or a dense output formula. During a single global time step the refinement procedure can be recursively continued until the local errors for all components are below a given tolerance, hence the name '*recursive*'. In our comparison in this paper we consider only the most simple case with one level of refinement.

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In the *compound step* strategy (sometimes also called *mixed compound-fast* [10]) the macro-step τ (for the slow components) and the first micro-step of a smaller size (for the active components) are computed simultaneously. Again, the values at the intermediate time levels of the slow components can be obtained by interpolation or dense output. This strategy may require values at the macro-step time level of the fast components. These values can be obtained by extrapolation. The integration is followed by a sequence of micro-steps for the fast components, until the time integration is synchronized with the slow components. In this paper in the compound step strategy also only micro-steps of size $\frac{1}{2}\tau$ are considered for the comparison with the recursive refinement strategy.

The values at the macro-step time level for the active components are calculated twice in the recursive refinement strategy, the first time during the global step and the second time during the refinement step. The compound step strategy avoids this extra work, however, the partitioning in slow and fast components for this strategy has to be done in advance before solving the system. With the recursive refinement strategy, implicit relations of the same structure as with single-rate time stepping are obtained. The refinement step just leads to a system of smaller size. With the compound step strategy the compound step has a somewhat more complicated structure.

In this paper we consider multirate schemes for systems with two levels of activity, slow and fast. It should be noted, however, that with the recursive refinement strategy it is easy to extend these schemes to multirate schemes with more levels of activity; for example, the multirate time stepping strategy presented in [7] can be used. With the compound step strategy handling more levels of activity is not easy.

In this paper we study and compare asymptotic stability of these two multirate strategies for linear problems in $\mathbb{R}^2$. Our particular interest is to see how the extrapolation of the fast components affects the asymptotic stability of the scheme. A time integration method is called asymptotically stable if its amplification matrix S satisfies $\|S^n\| \to 0$ when $n \to \infty$. A method is asymptotically stable if and only if all eigenvalues of S are inside the unit disk. Asymptotic stability does not guarantee stability, but it can help us with understanding the instability of some schemes. We also discuss the relevance of the results for the simple test equation in $\mathbb{R}^2$ for some interesting higher-dimensional systems.

The contents of this paper is as follows. In Section 2 we introduce the Rosenbrock ROS1 and ROS2 methods which will be used as our basic numerical integration methods. In Section 3 we describe the 2×2 test problem for which the asymptotic stability domains are determined. The two multirate versions of ROS1 and ROS2 will be analyzed in Sections 4 and 5. Some remarks on the relevance of the results for the 2×2 test problem are presented in Section 6. Section 7 is devoted to a property of the eigenvalues of the partitioned Rosenbrock methods. Finally, Section 8 contains the conclusions.

2. Numerical integration methods ROS1 and ROS2

As the basic methods for the multirate schemes in this paper we use two Rosenbrock methods [5]. The first method is a one-stage method, called in this paper ROS1, which for non-autonomous systems $w'(t) = F(t, w(t))$ is given by

$$\begin{aligned} w_n &= w_{n-1} + k_1, \\ (I - \gamma\tau J)k_1 &= \tau F(t_{n-1}, w_{n-1}) + \gamma\tau^2 F_t(t_{n-1}, w_{n-1}), \end{aligned} \tag{2.1}$$

where w_n denotes the approximation to $w(t_n)$ and $J \approx F_w(t_{n-1}, w_{n-1})$. The method is of order two if $\gamma = \frac{1}{2}$. Otherwise the order is one. The method is A-stable for any $\gamma \geqslant \frac{1}{2}$ and L-stable for $\gamma = 1$. In this paper we use $\gamma = \frac{1}{2}$.

The second method is the two-stage second-order method, to which we will refer to as ROS2,

$$\begin{aligned} w_n &= w_{n-1} + \tfrac{3}{2}\bar{k}_1 + \tfrac{1}{2}\bar{k}_2, \\ (I - \gamma\tau J)\bar{k}_1 &= \tau F(t_{n-1}, w_{n-1}) + \gamma\tau^2 F_t(t_{n-1}, w_{n-1}), \\ (I - \gamma\tau J)\bar{k}_2 &= \tau F(t_n, w_{n-1} + \bar{k}_1) - \gamma\tau^2 F_t(t_{n-1}, w_{n-1}) - 2\bar{k}_1, \end{aligned} \tag{2.2}$$

where $J \approx F_w(t_{n-1}, w_{n-1})$. The method is also linearly implicit (to compute the internal vectors $\bar{k}_1$ and $\bar{k}_2$, a system of linear algebraic equations is to be solved), and it is of order two for any choice of the parameter γ and for any choice of the matrix J. Furthermore, the method is A-stable for $\gamma \geqslant \frac{1}{4}$ and it is L-stable if $\gamma = 1 \pm \frac{1}{2}\sqrt{2}$. In this paper we use $\gamma = 1 - \frac{1}{2}\sqrt{2}$.

Other possible values of the parameter γ were also considered ($\gamma=1$ for ROS1; $\gamma=\frac{1}{2}$ and $\gamma=1+\frac{1}{2}\sqrt{2}$ for ROS2). These values gave similar results and conclusions.

2.1. Interpolation and extrapolation

For given approximations $w_{n-1}\approx w(t_{n-1})$, $w_n\approx w(t_n)$, the multirate schemes will require an intermediate value $w_I(t_{n-1/2})\approx w(t_{n-1/2})$. In [4] it was shown that for the multirate scheme based on the ROS1 method (with $\gamma=\frac{1}{2}$) and linear interpolation, stiffness may lead to an order reduction. For a special linear parabolic problem order 1.5 was obtained. Numerical experiments with the ROS2 method led to the same conclusion. Nevertheless, for many problems order reduction will not be observed. Therefore, we consider in this paper along with linear interpolation

$$w_I(t_{n-1/2})=\tfrac{1}{2}(w_{n-1}+w_n), \tag{2.3}$$

also forward quadratic interpolation

$$w_I(t_{n-1/2})=\tfrac{3}{4}w_{n-1}+\tfrac{1}{4}w_n+\tfrac{1}{4}\tau F(t_{n-1},w_{n-1}), \tag{2.4}$$

and backward quadratic interpolation

$$w_I(t_{n-1/2})=\tfrac{1}{4}w_{n-1}+\tfrac{3}{4}w_n-\tfrac{1}{4}\tau F(t_n,w_n). \tag{2.5}$$

With the ROS2 method we could also use what we call 'embedded' quadratic interpolation, which uses the stages values of the method and avoids explicit evaluations of F:

$$w_I(t_{n-1/2})=w_{n-1}+\frac{1}{8(1-2\gamma)}(5-12\gamma)\bar{k}_1+\frac{1}{8(1-2\gamma)}(1-4\gamma)\bar{k}_2. \tag{2.6}$$

This interpolation mimics the quadratic interpolation based on $w(t_{n-1})$, $w(t_n)$ and $w'(t_{n-1}+\gamma\tau)$,

$$w_I(t_{n-1/2})=\frac{1}{4(1-2\gamma)}((3-4\gamma)w_{n-1}+(1-4\gamma)w_n+\tau F(t_{n-1+\gamma},w_{n-1+\gamma})).$$

For linear problems and $\gamma=1\pm\frac{1}{2}\sqrt{2}$ the interpolation (2.6) coincides with (2.5). In the case of ROS1 with $\gamma=\frac{1}{2}$, backward quadratic interpolation is equivalent to the forward quadratic interpolation.

For the compound step strategy also extrapolation is needed: $w_{\mathrm{E}}(t_n)\approx w(t_n)$. Again, we consider three types of extrapolation: linear

$$w_{\mathrm{E}}(t_n)=2w_{n-1/2}-w_{n-1}, \tag{2.7}$$

forward quadratic

$$w_{\mathrm{E}}(t_n)=4w_{n-1/2}-3w_{n-1}-\tau F(t_{n-1},w_{n-1}), \tag{2.8}$$

and backward quadratic

$$w_{\mathrm{E}}(t_n)=w_{n-1}+\tau F(t_{n-1/2},w_{n-1/2}). \tag{2.9}$$

Usually, for the compound step strategy, extra- and interpolations are done via internal stages.

3. The linear test problem in $\mathbb{R}^2$

Usually, linear stability analysis of an integration method is based on the scalar Dahlquist test equation $w'(t)=\lambda w(t)$, $\lambda\in\mathbb{C}$. For multirate methods the scalar problem cannot be used. Instead we consider a similar test problem, a linear 2×2 system

$$w'(t)=Aw(t),\quad w=\begin{pmatrix}u\\ v\end{pmatrix},\quad A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}. \tag{3.1}$$

We denote

$$Z = \tau A, \quad z_{ij} = \tau a_{ij}. \tag{3.2}$$

We will assume that the first component u of the system is fast and the second component v is slow. Thus, to perform the time integration from t_{n-1} to $t_n = t_{n-1} + \tau$ we will complete two time steps of size $\frac{1}{2}\tau$ for the first component and one time step of size τ for the second component.

We denote

$$\kappa = \frac{a_{22}}{a_{11}}, \quad \beta = \frac{a_{12}a_{21}}{a_{11}a_{22}}. \tag{3.3}$$

It will be assumed that

$$a_{11} < 0 \quad \text{and} \quad a_{22} < 0. \tag{3.4}$$

Then, both eigenvalues of the matrix A have a negative real part if and only if $\det(A) > 0$. This condition can also be written as

$$\beta < 1. \tag{3.5}$$

We can regard κ as a measure for the stiffness of the system, and β indicates the coupling between the fast and slow part of the system. For this two-dimensional test equation we will consider asymptotic stability whereby it is required that the eigenvalues of the amplification matrix of the multirate method are less than one in modulus. Instead of $z_{11} \leqslant 0$ and $\beta < 1$ it is convenient to use the quantities

$$\xi = \frac{z_{11}}{1 - z_{11}}, \quad \eta = \frac{\beta}{2 - \beta}, \tag{3.6}$$

which are bounded between -1 and 0, and -1 and 1, respectively.

4. Asymptotic stability for multirate ROS1

4.1. Recursive refinement strategy

In our recursive strategy, first we take the global step

$$\overline{w}_n = w_{n-1} + k_1,$$

$$(I - \gamma Z)k_1 = Z w_{n-1}, \tag{4.1}$$

from which we also obtain an approximation $v_{\mathrm{I}}(t_{n-1/2})$ for the second component at the intermediate time level $t_{n-1/2}$ by interpolation.

We continue with the first update step for the first component by solving the subproblem

$$u'(t) = a_{11}u(t) + a_{12}v_{\mathrm{I}}(t),$$

where the interpolant v_{I} is now considered as a time-dependent source term. We get

$$u_{n-1/2} = u_{n-1} + \tilde{k}_1,$$

$$(1 - \tfrac{1}{2}\gamma z_{11})\tilde{k}_1 = \tfrac{1}{2}(z_{11}u_{n-1} + z_{12}v_{n-1}) + \tfrac{1}{4}\gamma z_{12}\tau v_{\mathrm{I}}'(t_{n-1}), \tag{4.2}$$

where the time derivative term is approximated by

$$\tau v_{\mathrm{I}}'(t_{n-1}) = \overline{v}_n - v_{n-1} \tag{4.3}$$

without loosing the second order of the method.

At this point we have an numerical approximation of the solution at time $t_{n-1/2}$,

$$w_{n-1/2} = \begin{pmatrix} u_{n-1/2} \\ v_{\mathrm{I}}(t_{n-1/2}) \end{pmatrix}. \tag{4.4}$$

We proceed with the second update step

$$u_n = u_{n-1/2} + \hat{k}_1,$$
$$(1 - \tfrac{1}{2}\gamma z_{11})\hat{k}_1 = \tfrac{1}{2}(z_{11}u_{n-1/2} + z_{12}v_{\mathrm{I}}(t_{n-1/2})) + \tfrac{1}{4}\gamma z_{12}\tau v_{\mathrm{I}}'(t_{n-1/2}), \tag{4.5}$$

where, again, we approximate

$$\tau v_{\mathrm{I}}'(t_{n-1/2}) = \overline{v}_n - v_{n-1}, \tag{4.6}$$

without loosing the second order of the method. The final numerical value of the solution at time t_n is now given by

$$w_n = \begin{pmatrix} u_n \\ \overline{v}_n \end{pmatrix}. \tag{4.7}$$

4.2. Compound step strategy

In the compound step strategy, the first micro-step for the first component

$$u_{n-1/2} = u_{n-1} + k_1,$$
$$(1 - \tfrac{1}{2}\gamma z_{11})k_1 = \tfrac{1}{2}(z_{11}u_{n-1} + z_{12}v_{n-1}) + \tfrac{1}{4}\gamma z_{12}\tau v_{\mathrm{I}}'(t_{n-1}) \tag{4.8}$$

and the time step for the second component

$$v_n = v_{n-1} + \hat{k}_1,$$
$$(1 - \gamma z_{22})\hat{k}_1 = (z_{21}u_{n-1} + z_{22}v_{n-1}) + \gamma z_{21}\tau u_{\mathrm{I}}'(t_{n-1}) \tag{4.9}$$

are computed at the same time. Then we continue with the second micro-step for the first component

$$u_n = u_{n-1/2} + \tilde{k}_1,$$
$$(1 - \tfrac{1}{2}\gamma z_{11})\tilde{k}_1 = \tfrac{1}{2}(z_{11}u_{n-1/2} + z_{12}v_{\mathrm{I}}(t_{n-1/2})) + \tfrac{1}{4}\gamma z_{12}\tau v_{\mathrm{I}}'(t_{n-1/2}). \tag{4.10}$$

The time derivative terms are approximated by

$$\tau u_{\mathrm{I}}'(t_{n-1}) = 2(u_{n-1/2} - u_{n-1}), \tag{4.11}$$

$$\tau v_{\mathrm{I}}'(t_{n-1}) = v_n - v_{n-1}, \tag{4.12}$$

$$\tau v_{\mathrm{I}}'(t_{n-1/2}) = v_n - v_{n-1}. \tag{4.13}$$

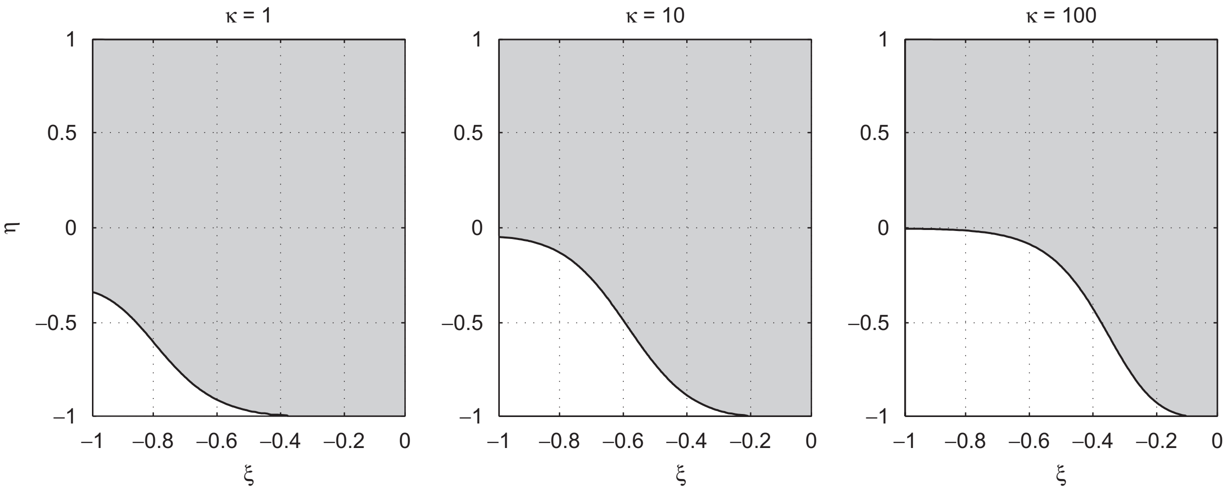

Fig. 1. Recursive refinement, ROS1 with linear interpolation. Asymptotic stability domains (gray areas) for $\kappa = 1, 10, 100$.

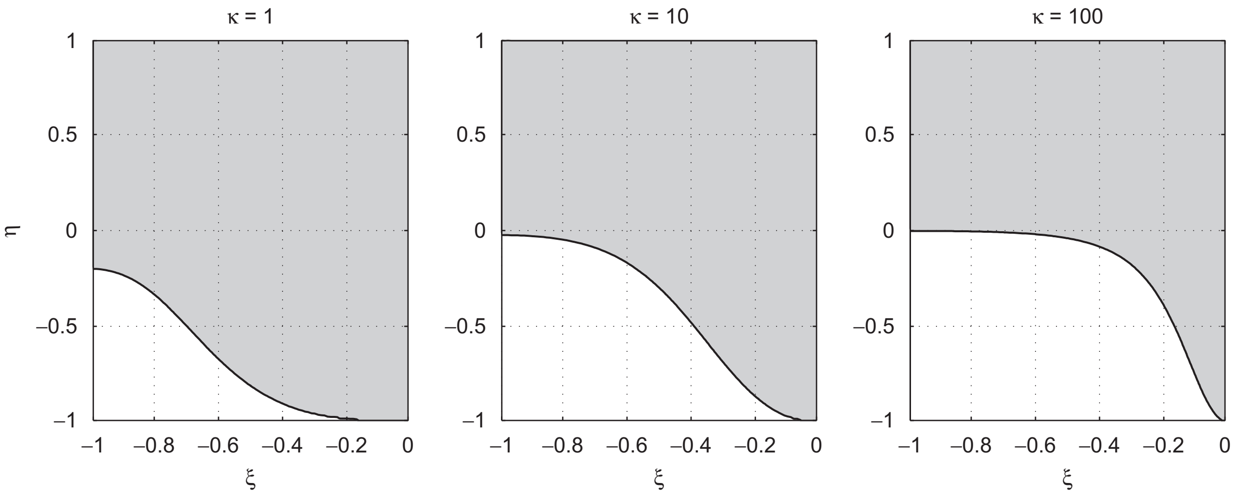

Fig. 2. Compound step, ROS1 with linear interpolation. Asymptotic stability domains (gray areas) for $\kappa = 1, 10, 100$.

Since these approximations are used for the $\tau^2 F_t$ term in (2.1), it follows that the order of the method does not change by Eqs. (4.11)–(4.13). Relations

$$2(u_{n-1/2} - u_{n-1}) = 2k_1\,,$$

$$v_n - v_{n-1} = \hat{k}_1,$$

used for Eqs. (4.11)–(4.13), reveal the joint computation of k_1 and $\hat{k}_1$ in Eqs. (4.8)–(4.9).

4.3. Results

Both considered strategies can be written in the form of partitioned Rosenbrock methods (see for example [2]). Therefore the eigenvalues of the amplification matrix of the multirate schemes depend just on three parameters κ, η and ξ (see Section 7). The domains of asymptotic stability are shown in Figs. 1–4 for both strategies and all considered

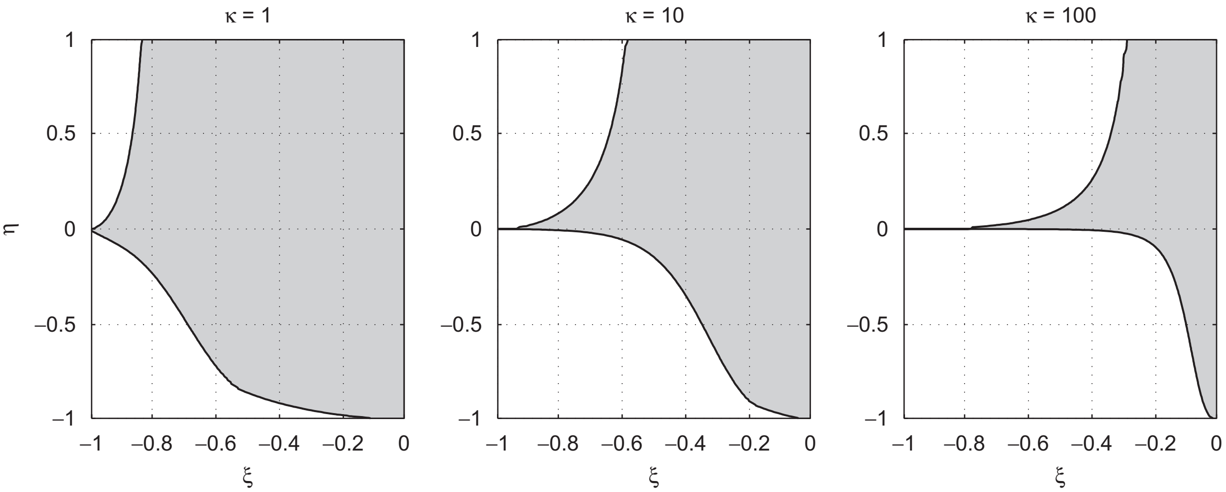

Fig. 3. Recursive refinement, ROS1 with forward quadratic interpolation. Asymptotic stability domains (gray areas) for $\kappa = 1, 10, 100$.

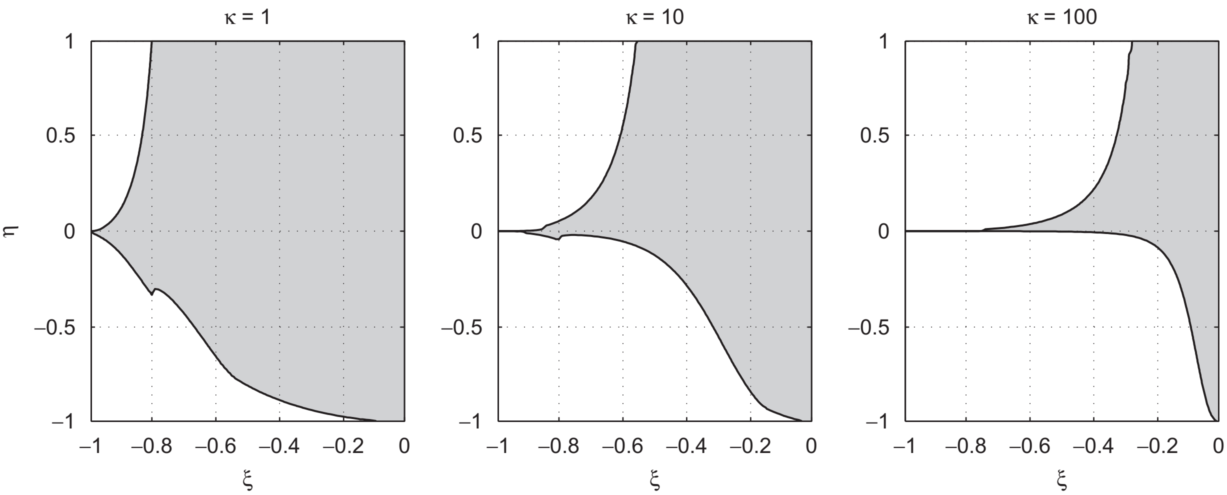

Fig. 4. Compound step, ROS1 with forward quadratic interpolation. Asymptotic stability domains (gray areas) for $\kappa = 1, 10, 100$.

types of interpolation. We present these domains in the (ξ, η)-plane for three values of $\kappa = 10^j$, $j = 0, 1, 2$. We observe that for these multirate schemes the stability region decreases with the increasing of κ.

From Figs. 1 and 2 it is seen that the combination of ROS1 and linear interpolation is unconditionally stable for both multirate strategies if the coupling parameter $\eta \geqslant 0$. For the $\eta < 0$ case, both strategies have instability regions which increase when κ becomes large. In this case stability regions for the recursive refinement strategy are somehow larger than for the compound step strategy.

For the ROS1 with forward quadratic interpolation (Figs. 3 and 4), both multirate schemes become unstable for large κ, except the trivial case $\eta = 0$. Both strategies have almost the same stability regions. The recursive refinement strategy has slightly larger stability area for $\eta > 0$. For $\eta < 0$ there exist a small set of points (close to $\xi = -0.8$) where the compound step strategy is asymptotically stable but the recursive refinement strategy is unstable. However, in general the recursive refinement strategy in the experiments in this section is slightly more stable.

The case $\eta \geqslant 0$ is relevant to the semi-discrete systems which are obtained by the central spatial discretization of the heat equation. The results obtained here suggest that the both strategies, based on ROS1 and linear interpolation, are stable for these semi-discrete systems. The results also show that for both strategies it is not possible to have an unconditionally stable second-order multirate scheme based on ROS1. Using linear interpolation/extrapolation we get better stability properties, however, we may lose one order due to stiffness (see the analysis in [4]).

5. Asymptotic stability for multirate ROS2

5.1. Recursive refinement strategy

In our recursive strategy, first we take the global step

$$\overline{w}_n = w_{n-1} + \tfrac{3}{2}\bar{k}_1 + \tfrac{1}{2}\bar{k}_2,$$
$$(I - \gamma Z)\bar{k}_1 = Z w_{n-1},$$
$$(I - \gamma Z)\bar{k}_2 = Z(w_{n-1} + \bar{k}_1) - 2\bar{k}_1, \tag{5.1}$$

from which we also obtain an approximation $v_{\mathrm{I}}(t_{n-1/2})$ for the second component at the intermediate time level $t_{n-1/2}$ by interpolation.

We continue with the first update step for the first component

$$u_{n-1/2} = u_{n-1} + \tfrac{3}{2}\tilde{k}_1 + \tfrac{1}{2}\tilde{k}_2,$$
$$(1 - \tfrac{1}{2}\gamma z_{11})\tilde{k}_1 = \tfrac{1}{2}(z_{11}u_{n-1} + z_{12}v_{n-1}) + \tfrac{1}{4}\gamma z_{12}\tau v_{\mathrm{I}}'(t_{n-1}),$$
$$(1 - \tfrac{1}{2}\gamma z_{11})\tilde{k}_2 = \tfrac{1}{2}(z_{11}(u_{n-1} + \tilde{k}_1) + z_{12}v_{\mathrm{I}}(t_{n-1/2})) - \tfrac{1}{4}\gamma z_{12}\tau v_{\mathrm{I}}'(t_{n-1}) - 2\tilde{k}_1, \tag{5.2}$$

where the time derivative term is approximated with

$$\tau v_{\mathrm{I}}'(t_{n-1}) = \overline{v}_n - v_{n-1}. \tag{5.3}$$

Since this approximation is used for the $\tau^2 F_t$ term in (2.2), it follows that the order of the method does not change by (5.3).

At this point we get the numerical approximation of the solution at time $t_{n-1/2}$

$$w_{n-1/2} = \begin{pmatrix} u_{n-1/2} \\ v_{\mathrm{I}}(t_{n-1/2}) \end{pmatrix}. \tag{5.4}$$

We proceed further with the second update step

$$u_n = u_{n-1/2} + \tfrac{3}{2}\hat{k}_1 + \tfrac{1}{2}\hat{k}_2,$$
$$(1 - \tfrac{1}{2}\gamma z_{11})\hat{k}_1 = \tfrac{1}{2}(z_{11}u_{n-1/2} + z_{12}v_{\mathrm{I}}(t_{n-1/2})) + \tfrac{1}{4}\gamma z_{12}\tau v_{\mathrm{I}}'(t_{n-1/2}),$$
$$(1 - \tfrac{1}{2}\gamma z_{11})\hat{k}_2 = \tfrac{1}{2}(z_{11}(u_{n-1/2} + \hat{k}_1) + z_{12}\overline{v}_n) - \tfrac{1}{4}\gamma z_{12}\tau v_{\mathrm{I}}'(t_{n-1/2}) - 2\hat{k}_1, \tag{5.5}$$

where, again, we approximate

$$\tau v_{\mathrm{I}}'(t_{n-1/2}) = \overline{v}_n - v_{n-1}. \tag{5.6}$$

The final numerical value of the solution at time t_n is given by

$$w_n = \begin{pmatrix} u_n \\ \overline{v}_n \end{pmatrix}. \tag{5.7}$$

5.2. Compound step strategy

In the compound step strategy, the first micro-step for the first component

$$\begin{aligned} u_{n-1/2} &= u_{n-1} + \tfrac{3}{2}\bar{k}_1 + \tfrac{1}{2}\bar{k}_2, \\ (1 - \tfrac{1}{2}\gamma z_{11})\bar{k}_1 &= \tfrac{1}{2}(z_{11}u_{n-1} + z_{12}v_{n-1}) + \tfrac{1}{4}\gamma z_{12}\tau v_{\mathrm{I}}'(t_{n-1}), \\ (1 - \tfrac{1}{2}\gamma z_{11})\bar{k}_2 &= \tfrac{1}{2}(z_{11}(u_{n-1} + \bar{k}_1) + z_{12}v_{\mathrm{I}}(t_{n-1/2})) - \tfrac{1}{4}\gamma z_{12}\tau v_{\mathrm{I}}'(t_{n-1}) - 2\bar{k}_1 \end{aligned} \tag{5.8}$$

and the time step for the second component

$$\begin{aligned} v_n &= v_{n-1} + \tfrac{3}{2}\hat{k}_1 + \tfrac{1}{2}\hat{k}_2, \\ (1 - \gamma z_{22})\hat{k}_1 &= (z_{21}u_{n-1} + z_{22}v_{n-1}) + \gamma z_{21}\tau u_{\mathrm{I}}'(t_{n-1}), \\ (1 - \gamma z_{22})\hat{k}_2 &= (z_{21}u_{\mathrm{E}}(t_n) + z_{22}(v_{n-1} + \hat{k}_1)) - \gamma z_{21}\tau u_{\mathrm{I}}'(t_{n-1}) - 2\hat{k}_1 \end{aligned} \tag{5.9}$$

are computed at the same time. Then we continue with the second micro-step

$$\begin{aligned} u_n &= u_{n-1/2} + \tfrac{3}{2}\tilde{k}_1 + \tfrac{1}{2}\tilde{k}_2, \\ (1 - \tfrac{1}{2}\gamma z_{11})\tilde{k}_1 &= \tfrac{1}{2}(z_{11}u_{n-1/2} + z_{12}v_{\mathrm{I}}(t_{n-1/2})) + \tfrac{1}{4}\gamma z_{12}\tau v_{\mathrm{I}}'(t_{n-1/2}), \\ (1 - \tfrac{1}{2}\gamma z_{11}\tilde{k}_2 &= \tfrac{1}{2}(z_{11}(u_{n-1/2} + \tilde{k}_1) + z_{12}v_n) - \tfrac{1}{4}\gamma z_{12}\tau v_{\mathrm{I}}'(t_{n-1/2}) - 2\tilde{k}_1. \end{aligned} \tag{5.10}$$

The time derivative terms are approximated by

$$\tau u_{\mathrm{I}}'(t_{n-1}) = 2(u_{n-1/2} - u_{n-1}), \tag{5.11}$$

$$\tau v_{\mathrm{I}}'(t_{n-1}) = v_n - v_{n-1}, \tag{5.12}$$

$$\tau v_{\mathrm{I}}'(t_{n-1/2}) = v_n - v_{n-1}. \tag{5.13}$$

Again, these approximations will not affect the order of the method.

A multirate scheme based on a third-order Rosenbrock method and compound step strategy was considered in [2]. Due to stability constraints, instead of the third-order method the embedded second-order method was used for time stepping. Extra and interpolations were done via internal stages.

5.3. Results

Again, both considered strategies can be written in the form of a partitioned Rosenbrock methods (for example by adding some artificial extra stages to the original method). Therefore the eigenvalues of the amplification matrix of the multirate schemes will depend on three parameters κ, η and ξ (see Section 7).

The domains of asymptotic stability are shown in Figs. 5–10 for both strategies and all considered types of interpolation/extrapolation. We present these domains in the (ξ, η)-plane for three values of $\kappa = 10^j$, $j = 0, 1, 2$. From Figs. 5 and 6 it is seen that the combination of ROS2 and linear interpolation is unconditionally stable for both multirate strategies if $\eta \geqslant 0$. An instability region appears at η close to -1. The instability region for the recursive refinement strategy is smaller than for the compound step strategy.

For ROS2 with forward quadratic interpolation (Figs. 7 and 8), both multirate schemes become unstable for large κ, unless $\eta = 0$. In this case the recursive refinement strategy has larger stability regions than the compound step strategy. A curious fact is that for $\kappa = 1$ and 10 the recursive refinement strategy is stable almost for all the values of η when $\xi = \xi^*$, where ξ^* is a number close to -0.9. For $\kappa = 100$ this property is not valid anymore.

Fig. 9 shows that the combination of ROS2 and backward quadratic interpolation is almost unconditionally stable for the recursive refinement strategy. There is a small set of points in the bottom-right corner of the domain where this strategy is unstable. In this case the stability domain is getting larger with the increase of the stiffness parameter κ,

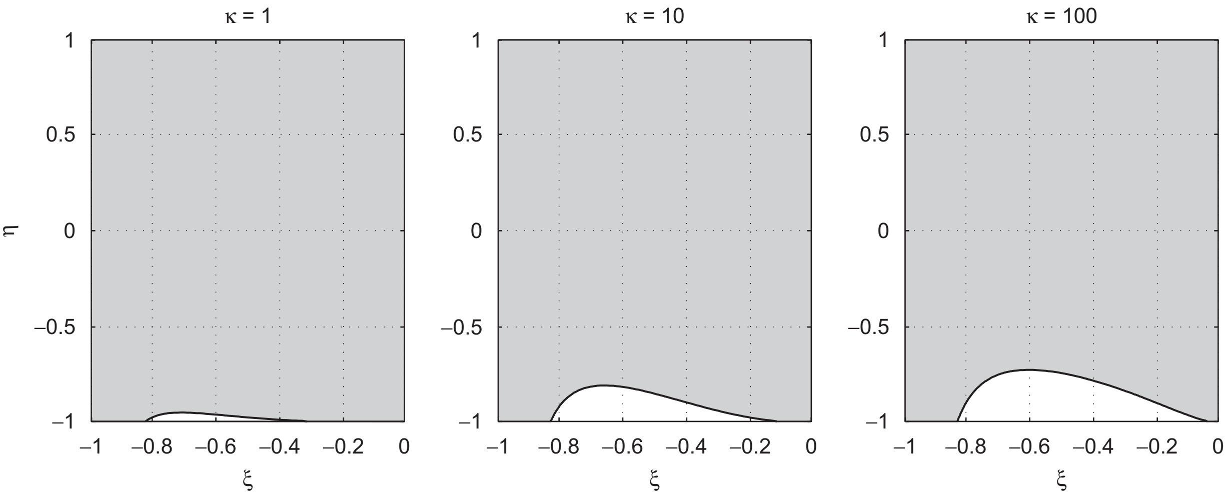

Fig. 5. Recursive refinement, ROS2 with linear interpolation. Asymptotic stability domains (gray areas) for $\kappa = 1, 10, 100$.

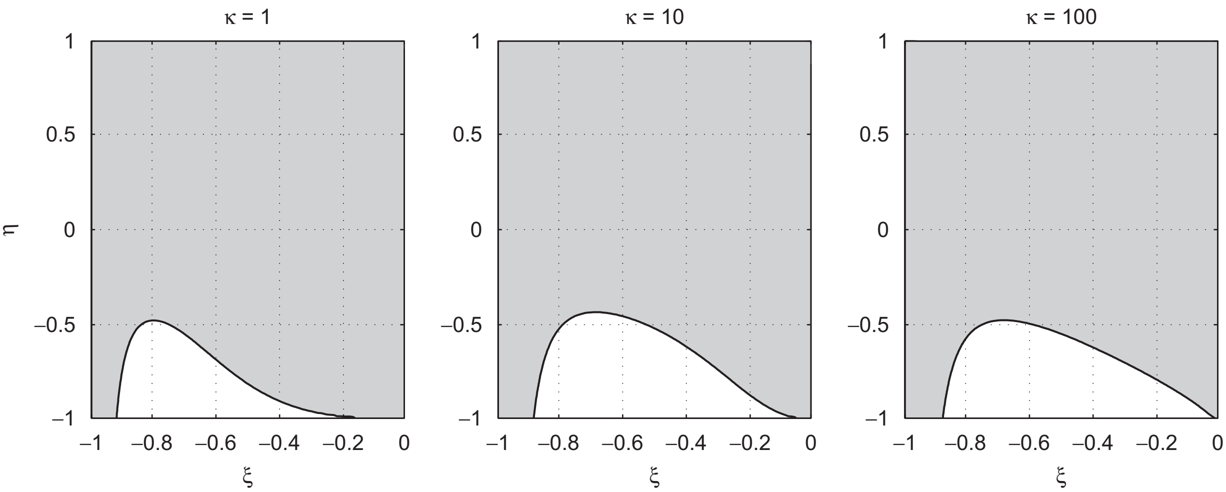

Fig. 6. Compound step, ROS2 with linear interpolation. Asymptotic stability domains (gray areas) for $\kappa = 1, 10, 100$.

probably due to the L-stability of the ROS2 scheme. As shown in Fig. 10, the compound step strategy used with ROS2 and backward quadratic interpolation has large instability regions, which in this case is a disadvantage of this strategy in comparison with the recursive refinement strategy.

In the case of linear and forward quadratic interpolation, for both strategies stability regions decrease with the increase of κ. However, in the case of backward quadratic interpolation, the stability region of the recursive refinement strategy increases with the increase of κ. The compound step strategy, used with backward quadratic interpolation, has irregular large stability regions, which shows that it can lead to unpredictable stability problems.

In this section we showed some results for ROS2 with the choice $\gamma = 1 - \frac{1}{2}\sqrt{2}$. We also performed some tests for $\gamma = 1 + \frac{1}{2}\sqrt{2}$ and $\gamma = \frac{1}{2}$. The results we obtained are very similar to the ones with $\gamma = 1 - \frac{1}{2}\sqrt{2}$. The asymptotic instability regions were a bit larger for $\gamma = 1 + \frac{1}{2}\sqrt{2}$ than for $\gamma = 1 - \frac{1}{2}\sqrt{2}$. The only significant difference was that ROS2 with $\gamma = \frac{1}{2}$ and backward quadratic interpolation was as unstable as ROS2 with $\gamma = \frac{1}{2}$ and forward quadratic interpolation.

The main result of this section is that for the recursive refinement strategy there exists a second-order multirate scheme, based on ROS2 and backward quadratic interpolation, which is unconditionally asymptotically stable (except

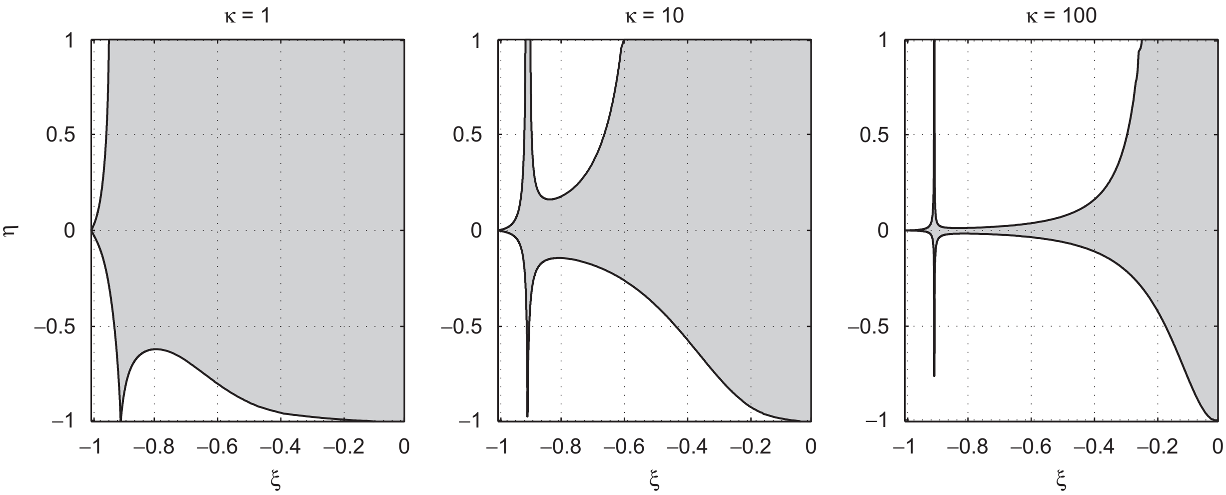

Fig. 7. Recursive refinement, ROS2 with forward quadratic interpolation. Asymptotic stability domains (gray areas) for $\kappa = 1, 10, 100$.

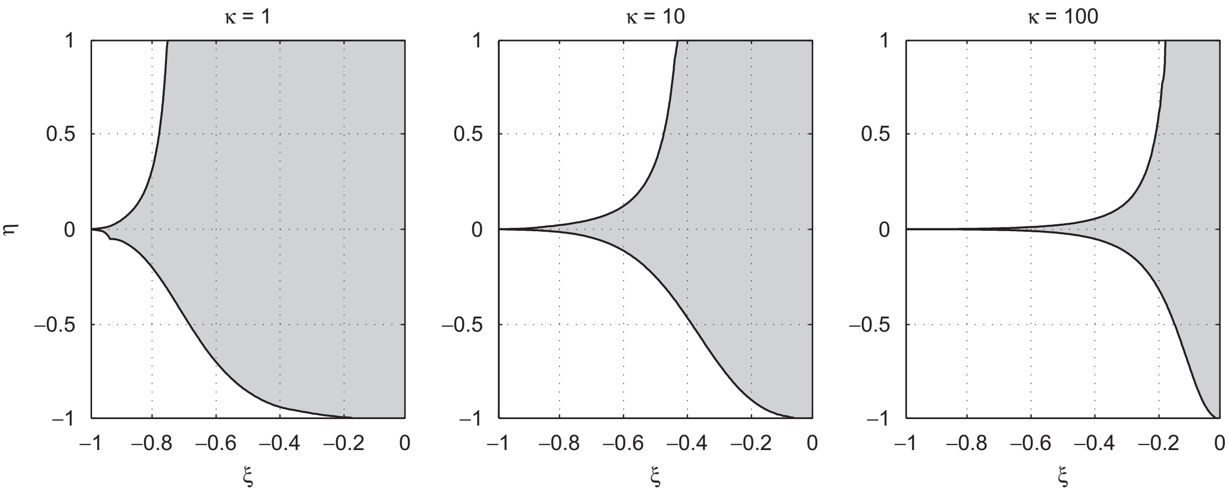

Fig. 8. Compound step, ROS2 with forward quadratic interpolation. Asymptotic stability domains (gray areas) for $\kappa = 1, 10, 100$.

for a very small region). For the compound step strategy it is not possible to have a second-order multirate scheme with this stability property.

6. Relevance of the linear 2 × 2 test problem

Asymptotic stability guarantees $\|S^n\| \to 0$ as $n \to \infty$. This also implies boundedness of

$$M = \sup_{n \geqslant 0} \|S^n\|, \tag{6.1}$$

but this bound M may depend on τ and A, and in particular on the stiffness of the problem. There is also lack of theory which would extend the results of stability analysis for multirate schemes for the linear 2×2 test equation to general systems of ODEs. Therefore, in order to see how relevant the asymptotic stability results for the linear 2×2 test problem are we did some stability tests in $\mathbb{R}^m$ to determine M for some interesting matrices A. In this section we consider $m = 50$ and we assume that the first 25 components of the system are fast and the last 25 components are slow.

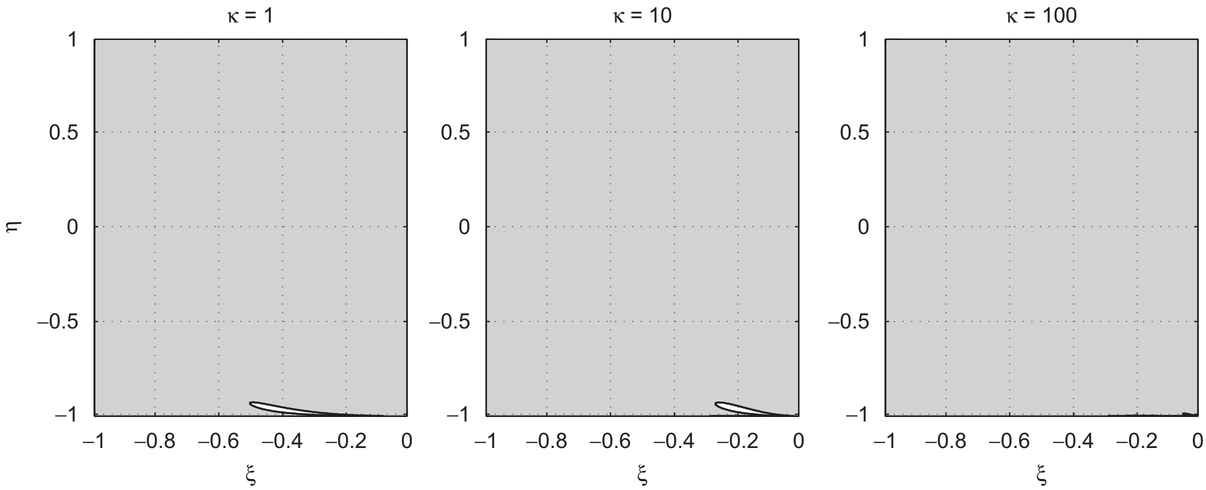

Fig. 9. Recursive refinement, ROS2 with backward quadratic interpolation. Asymptotic stability domains (gray areas) for $\kappa = 1, 10, 100$.

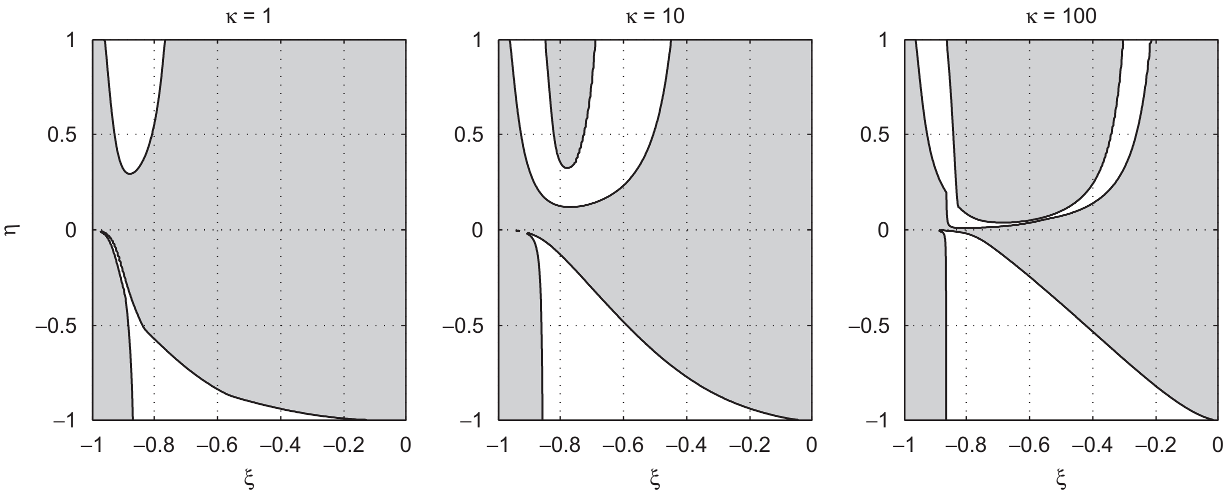

Fig. 10. Compound step, ROS2 with backward quadratic interpolation. Asymptotic stability domains (gray areas) for $\kappa = 1, 10, 100$.

We use ROS2 as our main time integration method. Forward quadratic interpolation showed bad asymptotic stability properties in the 2×2 tests and therefore we do not consider it anymore in the following numerical tests.

6.1. The heat equation

Let us consider the heat equation

$$u_t = \mathrm{d}u_{xx}. \tag{6.2}$$

Applying the second-order central discretization on a uniform spatial grid leads to a semi-discrete system

$$w'(t) = Aw(t), \tag{6.3}$$

where A is a $m \times m$ matrix

$$A = \mu\,\mathrm{tridiag}(1, -2, 1) \tag{6.4}$$

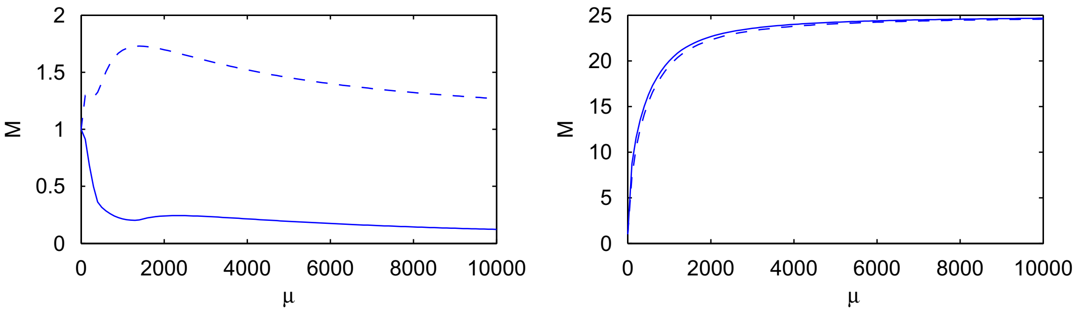

Fig. 11. Problem (6.2). Plot of the bound value M for ROS2 with recursive refinement (left) and compound step (right) strategies, used with linear (solid line) and backward quadratic interpolation (dashed line).

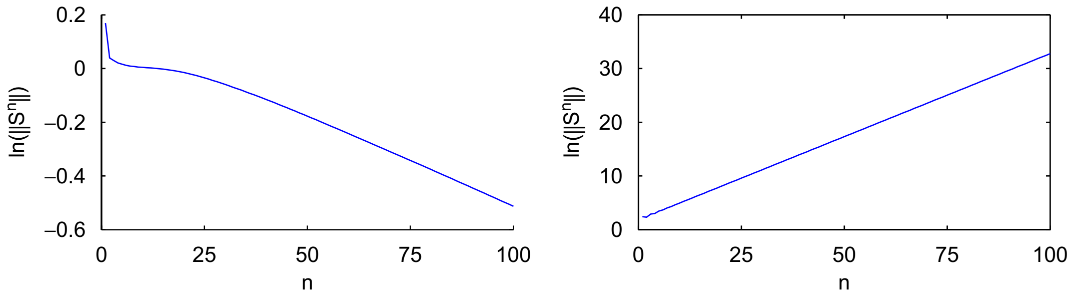

Fig. 12. Problem (6.5). Plot of the $\ln(||S^n||)$ for ROS2 with recursive refinement (left) and compound step (right) strategies, used with backward quadratic interpolation.

and $\mu > 0$ will depend on m and d. For matrices A of type (6.4), with $m = 50$, numerical tests for the recursive refinement and compound step strategies based on ROS2 and backward quadratic interpolation showed boundedness for the powers of the amplification matrix of the scheme in the maximum norm. From Fig. 11 it is seen that in this case $\|S^n\|_\infty$ is bounded by 2 and 25, for any choice of n and μ, for the recursive refinement and the compound step strategy, respectively. The bound value $M = 25$ for the compound step is much larger than $M = 2$ for the recursive refinement strategy. For the compound step strategy M becomes larger with the increase of m; numerical experiments suggest that for this strategy $M = \frac{1}{2}m$, which can be viewed as a weak instability.

However, if we consider the heat equation with a non-constant diffusion coefficient

$$u_t = \mathrm{d}(x)u_{xx} \tag{6.5}$$

then with the same spatial discretization we obtain a semi-discrete system (6.3) with

$$A = \mathrm{diag}(\mu_1, \ldots, \mu_m)\mathrm{tridiag}(1, -2, 1). \tag{6.6}$$

If, for this type of systems, we take $\mu_i = \frac{7}{6}$ for $i \leqslant 25$ and $\mu_i = \frac{35}{3}$ for $i > 25$ then the compound step strategy based on ROS2 and backward quadratic interpolation becomes unstable. Fig. 12 shows that for this choice of the coefficients μ_i, $\|S^n\|_\infty$ is bounded by 2 for any n for the recursive refinement strategy, whereas for the compound step strategy an exponential growth in n is observed.

These numerical results are in accordance with the results obtained for the linear 2×2 test problem. The 2×2 version of the matrix (6.4) would correspond to $\kappa = 1$ and $\eta = \frac{1}{7}$. Figs. 5,6,9, and 10 show that for these values of κ and η both multirate strategies are asymptotically stable. The 2×2 version of the matrix (6.6) corresponds to $\kappa = 10$, $\eta = \frac{1}{7}$ and $\xi = -0.7$. For these values the compound step strategy is asymptotically unstable (Fig. 10), but the recursive refinement strategy is stable (Fig. 9).

The numerical tests presented in this subsection suggest that the conclusions obtained in Section 5 are also valid for more general systems. The following conjecture can be formulated: the recursive refinement strategy, based on ROS2

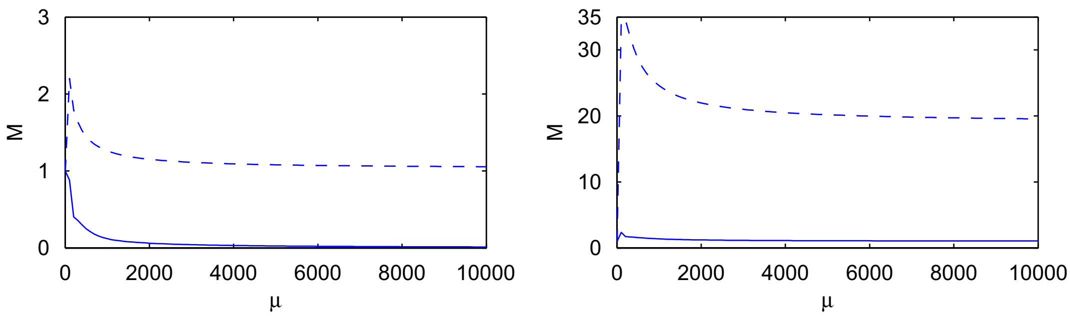

Fig. 13. Problem (6.7), first-order upwind spatial discretization. Plot of the bound value M for ROS2 with recursive refinement (left) and compound step (right) strategies, used with linear (solid line) and backward quadratic interpolation (dashed line).

and linear or backward quadratic interpolation, is stable if it is applied to the discrete system obtained by second-order spatial discretization of the heat equation. In the same context, the compound step strategy is stable if is used with linear interpolation, but it can lead to instabilities when is used with backward quadratic interpolation.

6.2. The advection equation

As a second test problem we consider the advection equation

$$u_t + au_x = 0. \tag{6.7}$$

Applying the first-order upwind discretization on a uniform spatial grid leads to a semi-discrete system

$$w'(t) = Aw(t), \tag{6.8}$$

where A is a $m \times m$ matrix

$$A = \mu\,\mathrm{tridiag}(1, -1, 0). \tag{6.9}$$

For the matrices A of type (6.9), numerical tests for the recursive refinement and compound step strategies based on ROS2 and backward quadratic interpolation showed uniform boundedness for the powers of the amplification matrix of the scheme. From Fig. 13 it is seen that in this case $||S^n||_\infty$ is bounded by 3 and 35, for any choice of n and μ, for the recursive refinement and the compound step strategy, respectively. The bound $M = 35$ for the compound step strategy is larger than the bound $M = 3$ for the recursive refinement strategy. However, for this case (6.9) it was observed in further numerical tests that both these bounds do not change significantly, with increasing m, in contrast to (6.4).

We also consider the case of the second-order central spatial discretization of the advection term for the problem (6.7). With this discretization we obtain a semi-discrete system (6.8) with

$$A = \mu\ \mathrm{tridiag}(1, 0, -1). \tag{6.10}$$

Numerical tests showed that both multirate strategies used with ROS2 are unstable for the system (6.8) with matrices A of type (6.10). Fig. 14 shows that the infinity norm of the powers of the amplification matrix S for the case $\mu = 100$ is not bounded.

Again, the results from this subsection agree with those obtained for the linear 2×2 test problem. The 2×2 version of the matrix (6.9) would correspond to $\kappa = 1$ and $\eta = 0$. Figs. 5–10 show that for these values of κ and η both multirate strategies are asymptotically stable. The 2×2 version of the matrix (6.10) corresponds to $\eta = -1$ and $\xi = 0$. The same figures show that these values of κ and ξ can lead to asymptotic instabilities of both strategies. All this suggests that both strategies, based on ROS2 and linear or backward quadratic interpolation, are stable when applied to the semi-discrete system obtained by first-order upwind spatial discretization of the advection equation. They are unstable if, instead, the second-order central spatial discretization is used.

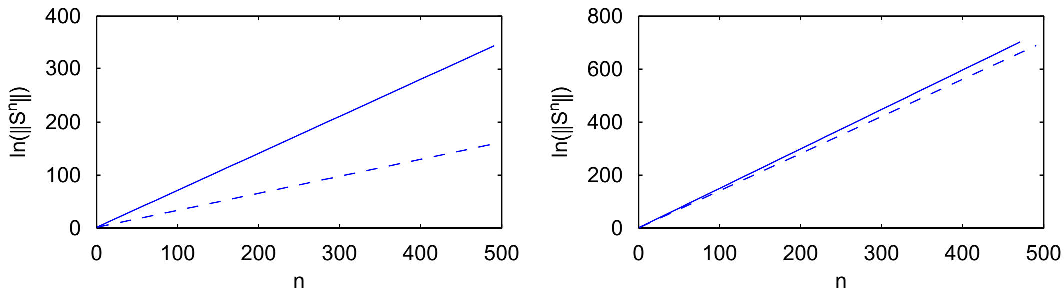

Fig. 14. Problem (6.7), second order central spatial discretization. Plot of the $\ln(||S^n||)$ for ROS2 with recursive refinement (left) and compound step (right) strategies, used with linear (solid line) and backward quadratic interpolation (dashed line), ROS2.

7. A property of the eigenvalues of the amplification matrix for partitioned Rosenbrock methods

All multirate schemes considered in this paper can be transformed into a partitioned Rosenbrock method, for example by adding some artificial extra stages; see [2], for example.

For a system

$$\begin{aligned} u' &= F_1(u, v), \\ v' &= F_2(u, v), \end{aligned} \tag{7.1}$$

a partitioned Rosenbrock method is given by

$$u_n = u_{n-1} + \sum_{i=1}^{s_1} \bar{b}_i \bar{k}_i, \tag{7.2}$$

$$v_n = v_{n-1} + \sum_{i=1}^{s_2} \hat{b}_i \hat{k}_i, \tag{7.3}$$

$$\begin{aligned} \bar{k}_i = \tau F_1 &\left(u_{n-1} + \sum_{j=1}^{i-1} \bar{\alpha}_{ij} \bar{k}_j,\, v_{n-1} + \sum_{j=1}^{\bar{p}_i} \bar{\beta}_{ij} \hat{k}_j \right) \\ &+ \tau F_{1u} \sum_{j=1}^{i} \bar{\gamma}_{ij} \bar{k}_j + \tau F_{1v} \sum_{j=1}^{s_2} \bar{\delta}_{ij} \hat{k}_j, \quad i = 1, \ldots, s_1, \end{aligned} \tag{7.4}$$

$$\begin{aligned} \hat{k}_i = \tau F_2 &\left(u_{n-1} + \sum_{j=1}^{\hat{p}_i} \hat{\alpha}_{ij} \bar{k}_j,\, v_{n-1} + \sum_{j=1}^{i-1} \hat{\beta}_{ij} \hat{k}_j \right) \\ &+ \tau F_{2u} \sum_{j=1}^{s_1} \hat{\gamma}_{ij} \bar{k}_j + \tau F_{2v} \sum_{j=1}^{i} \hat{\delta}_{ij} \hat{k}_j, \quad i = 1, \ldots, s_2, \end{aligned} \tag{7.5}$$

where $F_{iu} = \partial F_i / \partial u$ and $F_{iv} = \partial F_i / \partial v$.

We mention that if

$$\bar{p}_i \leqslant i \quad \text{and} \quad \hat{p}_i \leqslant i \tag{7.6}$$

then the system Eqs. (7.4)–(7.5) can be solved by sequentially computing the values of the pairs $(\bar{k}_i, \hat{k}_i)$. The recursive refinement strategy leads to a multirate scheme which can be written as a partitioned Rosenbrock method with property (7.6). In the compound step strategy the macro-step and the first micro-step are computed simultaneously. The micro-step uses the information obtained from the interpolation of the results from the macro-step. The macro-step uses the information obtained by the extrapolation of the results from the micro-step. The partitioned Rosenbrock method derived from the multirate scheme obtained with the compound step strategy may not satisfy (7.6). This happens if backward quadratic interpolation is used. In this case all micro-steps are computed using interpolation which depends on the value of the solution calculated at the last micro-step. Therefore for the compound step strategy, Eqs. (7.4)–(7.5) can result in large implicit systems. In practice, backward quadratic interpolation is not used.

In the case of our 2×2 linear test problem the system (7.1) can be written as

$$\begin{aligned} u' &= a_{11}u + a_{12}v, \\ v' &= a_{21}u + a_{22}v. \end{aligned} \tag{7.7}$$

If we write the method Eqs. (7.2)–(7.5) in a short form

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = S \begin{pmatrix} u_{n-1} \\ v_{n-1} \end{pmatrix}, \tag{7.8}$$

with $S = (S_{ij})$, $i, j = 1, 2$, then we can prove the following theorem.

Theorem 1. *The eigenvalues of the amplification matrix S can be written as functions of the three variables* z_{11}, z_{22} *and* $\det(Z)$.

Proof. For the problem (7.7) the formulas Eqs. (7.4)–(7.5) reduce to

$$\bar{k}_i = z_{11}\left(u_{n-1} + \sum_{j=1}^{i} \bar{\alpha}^*_{ij}\bar{k}_j\right) + z_{12}\left(v_{n-1} + \sum_{j=1}^{s_2} \bar{\beta}^*_{ij}\hat{k}_j\right), \tag{7.9}$$

$$\hat{k}_i = z_{21}\left(u_{n-1} + \sum_{j=1}^{s_1} \hat{\alpha}^*_{ij}\bar{k}_j\right) + z_{22}\left(v_{n-1} + \sum_{j=1}^{i} \hat{\beta}^*_{ij}\hat{k}_j\right). \tag{7.10}$$

If we set $(u_{n-1}, v_{n-1})^{\mathrm{T}} = (1, 0)^{\mathrm{T}}$ then we get $(S_{11}, S_{21})^{\mathrm{T}} = (u_n, v_n)^{\mathrm{T}}$. By defining $\hat{k}_i = z_{21}\hat{k}^*_i$ from Eqs. (7.9)–(7.10) we obtain

$$\bar{k}_i = z_{11}\left(1 + \sum_{j=1}^{i} \bar{\alpha}^*_{ij}\bar{k}_j\right) + z_{12}z_{21}\sum_{j=1}^{s_2} \bar{\beta}^*_{ij}\hat{k}^*_j, \quad i = 1, \ldots, s_1, \tag{7.11}$$

$$\hat{k}^*_i = 1 + \sum_{j=1}^{s_1} \hat{\alpha}^*_{ij}\bar{k}_j + z_{22}\sum_{j=1}^{i} \hat{\beta}^*_{ij}\hat{k}^*_j, \quad i = 1, \ldots, s_2. \tag{7.12}$$

The solution of system Eqs. (7.11)–(7.12) depends only on z_{11}, z_{22} and $\det(Z)$. Therefore we have

$$S_{11} = u_n = 1 + \sum_{i=1}^{s_1} \bar{b}_i\bar{k}_i = f_{11}(z_{11}, z_{22}, \det(Z)), \tag{7.13}$$

$$S_{21} = v_n = z_{21}\sum_{i=1}^{s_2} \hat{b}_i\hat{k}^*_i = z_{21}f_{21}(z_{11}, z_{22}, \det(Z)). \tag{7.14}$$

In a similar way, by setting $(u_{n-1}, v_{n-1})^{\mathrm{T}} = (0, 1)^{\mathrm{T}}$ one can show that

$$S_{12} = z_{12}f_{21}(z_{11}, z_{22}, \det(Z)) \quad \text{and} \quad S_{22} = f_{22}(z_{11}, z_{22}, \det(Z)). \tag{7.15}$$

Finally from

$$S=\begin{pmatrix} f_{11}(z_{11},z_{22},\det(Z)) & z_{12}f_{21}(z_{11},z_{22},\det(Z)) \\ z_{21}f_{21}(z_{11},z_{22},\det(Z)) & f_{22}(z_{11},z_{22},\det(Z)) \end{pmatrix} \tag{7.16}$$

the proof of the theorem directly follows. □

This property was already observed for some special methods in [4,6,8].

8. Conclusions

In this paper we presented a comparison of asymptotic stability properties for the multirate recursive refinement and the compound step strategies. We also discussed how the obtained results can be used in the context of stability of the more general schemes. For most of the tests in the paper the recursive refinement strategy does have the asymptotic stability regions somewhat larger than the compound step strategy. Sometimes the difference is very small (ROS1 and quadratic interpolation), in other cases the difference is significant (ROS2 and backward quadratic interpolation).

The scheme based on the recursive refinement strategy used with ROS2 and backward quadratic interpolation is clearly the favorite among the considered second-order schemes. It has a very small instability region. There are no multirate schemes based on the compound step strategy, which are of second-order for stiff problems and have good stability properties.

The numerical tests for more general systems presented in the paper gave results which are in accordance with those obtained for the 2×2 linear test problem. Therefore, the simple 2×2 case already gives a good indication for stability properties for more general systems, such as the semi-discrete systems obtained from the spatial discretization of the heat equation and the advection equation.

Finally we mention that the compound step strategy, by avoiding the extra work of doing the macro-step for all the components, looses some stability properties compared to the recursive refinement strategy, and it can also lead to more complex implicit systems which are difficult to solve. The recursive refinement strategy is very simple and it has better stability properties.

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