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On varieties of almost minimal degree I: Secant loci of rational normal scrolls

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ABSTRACT

To provide a geometrical description of the classification theory and the structure theory of varieties of almost minimal degree, that is of non-degenerate irreducible projective varieties whose degree exceeds the codimension by precisely 2, a natural approach is to investigate simple projections of varieties of minimal degree. Let $\tilde{X} \subset \mathbb{P}_K^{r+1}$ be a variety of minimal degree and of codimension at least 2, and consider $X_p = \pi_p(\tilde{X}) \subset \mathbb{P}_K^r$ where $p \in \mathbb{P}_K^{r+1} \setminus \tilde{X}$. By Brodmann and Schenzel (2007) [1], it turns out that the cohomological and local properties of X_p are governed by the secant locus $\Sigma_p(\tilde{X})$ of $\tilde{X}$ with respect to p.

Along these lines, the present paper is devoted to giving a geometric description of the secant stratification of $\tilde{X}$, that is of the decomposition of $\mathbb{P}_K^{r+1}$ via the types of secant loci. We show that there are at most six possibilities for the secant locus $\Sigma_p(\tilde{X})$, and we precisely describe each stratum of the secant stratification of $\tilde{X}$, each of which turns out to be a quasi-projective variety.

As an application, we obtain a different geometrical description of non-normal del Pezzo varieties $X \subset \mathbb{P}_K^r$, first classified by Fujita (1985) [3, Theorem 2.1(a)] by providing a complete list of pairs $(\tilde{X}, p)$, where $\tilde{X} \subset \mathbb{P}_K^{r+1}$ is a variety of minimal degree, $p \in \mathbb{P}_K^{r+1} \setminus \tilde{X}$ and $X_p = X \subset \mathbb{P}_K^r$.

1. Introduction

Throughout this paper, we work over an algebraically closed field K of arbitrary characteristic. We denote by $\mathbb{P}_K^r$ the projective r-space over K.

Let $X \subset \mathbb{P}_K^r$ be a non-degenerate irreducible projective variety. It is well known that $\deg(X) \geq \operatorname{codim}(X) + 1$. In case equality holds, X is called a variety of minimal degree. Varieties of minimal degree were completely classified more than hundred years ago by P. del Pezzo and E. Bertini, and now they are very well understood from several points of view. A variety $X \subset \mathbb{P}_K^r$ is of minimal degree if and only if it is either $\mathbb{P}_K^r$ or a quadric hypersurface or (a cone over) the Veronese surface in $\mathbb{P}_K^5$ or a rational normal scroll.

In the next case, that is if $\deg(X) = \operatorname{codim}(X) + 2$, one calls X a variety of almost minimal degree. The results of [3,4] imply that these varieties can be divided into two classes:

1. $X \subset \mathbb{P}_K^r$ is linearly normal and X is normal;
2. $X \subset \mathbb{P}_K^r$ is non-linearly normal or X is non-normal.

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By [3, Theorem 2.1(b)], see also (6.4.6) and (9.2) in [5], X is of the first type if and only if it is a normal del Pezzo variety. By [3, Theorem 2.1(a)], X is of the second type if and only if $X = \pi_p(\widetilde{X})$ where $\widetilde{X} \subset \mathbb{P}_K^{r+1}$ is a variety of minimal degree and of codimension at least two and $\pi_p : \widetilde{X} \to \mathbb{P}_K^r$ is the linear projection of $\widetilde{X}$ from a closed point p in $\mathbb{P}_K^{r+1} \setminus \widetilde{X}$. For details, we refer the reader to Notation and Remarks 2.3. Smooth del Pezzo varieties are completely classified by Fujita (cf. (8.11) and (8.12) in [5]).

Now, a natural approach to understand varieties of almost minimal degree which are not normal del Pezzo is to investigate simple projections of varieties of minimal degree. In this situation, i.e. in the case where $X = \pi_p(\widetilde{X})$, one can naturally expect that all the properties of X may be precisely described in terms of the relative location of p with respect to $\widetilde{X}$. For the cohomological and local properties of X, this expectation turns out to be true in [1]. Indeed those properties are governed by the secant locus $\Sigma_p(\widetilde{X})$ of $\widetilde{X}$ with respect to p, which is the scheme-theoretic intersection of $\widetilde{X}$ and the union of all secant lines to X passing through p.

Along these lines, the main purpose of the paper is to classify the pairs $(\widetilde{X}, p)$ with $\widetilde{X} \subset \mathbb{P}_K^{r+1}$ a variety of minimal degree via the analysis of the secant locus $\Sigma_p(\widetilde{X})$.

The classification theory of varieties of minimal degree says that $\widetilde{X}$ is either (a cone over) the Veronese surface in $\mathbb{P}_K^5$ or else a rational normal scroll of degree ≥ 3. When $\widetilde{X}$ is the Veronese surface in $\mathbb{P}_K^5$, it is well known that $\Sigma_p(\widetilde{X})$ is either empty or a smooth plane conic. The case where $\widetilde{X}$ is a cone over the Veronese surface can be easily dealt with from this fact. For details, see Remark 6.3.

When $\widetilde{X}$ is a rational normal scroll of degree ≥ 3, Proposition 3.2 and Corollary 3.4 show that there are at most six different possibilities for the secant locus $\Sigma_p(\widetilde{X})$.

According to this, the secant locus $\Sigma_p(\widetilde{X})$ can be at most of six different types, giving a decomposition of $\mathbb{P}_K^{r+1} \setminus \widetilde{X}$ which will be called *the secant stratification of* $\widetilde{X}$, see Theorem 4.2.

Finally, Theorem 6.2 gives a different proof of the classification of non-normal del Pezzo varieties $X = \pi_p(\widetilde{X}) \subset \mathbb{P}_K^r$. If X is not a cone, then either $\widetilde{X}$ is $S(a) \subset \mathbb{P}_K^a$ with $a \geq 3$; or $S(1, b) \subset \mathbb{P}_K^{b+3}$ with $b \geq 2$; or $S(2, b) \subset \mathbb{P}_K^{b+4}$ with $b \geq 2$; or $S(1, 1, c) \subset \mathbb{P}_K^{c+4}$ with $c \geq 1$. This provides a geometric picture of non-normal del Pezzo varieties from the view point of linear projections and normalizations, reproving some of the results of Fujita contained in [3,4]. In particular, as shown in [3,4], we provide a different proof of the fact that the dimension of $X \subset \mathbb{P}_K^r$, not a cone, does not exceed three although there is no upper bound on the degree of X.

2. Preliminaries

Notation and Remark 2.1. Let $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ be a variety of minimal degree. That is, $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ is a non-degenerate irreducible projective subvariety of $\mathbb{P}_K^{r+1}$ such that $\deg(\tilde{X}) = \operatorname{codim}(\tilde{X}) + 1$. According to the well-known classification of varieties of minimal degree (cf. [2]), $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ is either

(i) $\mathbb{P}_K^{r+1}$;
(ii) a quadric hypersurface;
(iii) (a cone over) the Veronese surface in $\mathbb{P}_K^5$; or
(iv) a rational normal scroll.

In particular $\tilde{X}$ is always arithmetically Cohen–Macaulay.

Notation and Remark 2.2. Let $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ be as above and let $p \in \mathbb{P}_K^{r+1} \setminus \tilde{X}$ be a fixed closed point. The union of all secant lines to X passing through p is called the *secant cone of* $\tilde{X}$ with respect to p and denoted by $\mathrm{Sec}_p(\tilde{X})$, i.e.

$$\mathrm{Sec}_p(\tilde{X}) := \bigcup_{q \in \tilde{X}, \operatorname{length}(\tilde{X} \cap \langle p, q \rangle) \geq 2} \langle p, q \rangle.$$

We use the convention that $\mathrm{Sec}_p(\tilde{X}) = \{p\}$ when there is no secant line to X passing through p. Observe that $\mathrm{Sec}_p(\tilde{X})$ is a cone with vertex p. We define the *secant locus* $\Sigma_p(\tilde{X})$ of $\tilde{X}$ with respect to p as the scheme-theoretic intersection of $\tilde{X}$ and $\mathrm{Sec}_p(\tilde{X})$. Therefore

$$\Sigma_p(\tilde{X})_{\mathrm{red}} = \{q \in \tilde{X} \mid \operatorname{length}(\widetilde{X} \cap \langle p, q \rangle) \geq 2\}.$$

Thus $\Sigma_p(\tilde{X})$ is the *entry locus* of $\tilde{X}$ with respect to p in the sense of [6].

Notation and Remarks 2.3. (A) Let $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ and $p \in \mathbb{P}_K^{r+1} \setminus \tilde{X}$ be as above. We fix a projective space $\mathbb{P}_K^r$ and consider the linear projection

$$\pi_p : \widetilde{X} \to X_p := \pi_p(\tilde{X}) \subseteq \mathbb{P}_K^r$$

of $\widetilde{X}$ from p. As the morphism $\pi_p : \widetilde{X} \to X_p$ is finite, we have

$$\operatorname{codim}(\tilde{X}) = \operatorname{codim}(X_p) + 1 \leq \deg(X_p) \mid \deg(\tilde{X}) = \operatorname{codim}(\tilde{X}) + 1.$$

If $\operatorname{codim}(\tilde{X}) = 1$, then $\tilde{X}$ is a quadric and π_p is a double covering of $\mathbb{P}^r_K$. On the other hand, if $\operatorname{codim}(\tilde{X}) > 1$ then π_p is birational and $X_p \subseteq \mathbb{P}^r_K$ is of almost minimal degree (that is $\deg(X_p) = \operatorname{codim}(X_p) + 2$).

(B) Suppose that $\tilde{X}$ is smooth. Then $\operatorname{Sec}(\tilde{X})$ will denote the secant variety of $\tilde{X}$, i.e. the closure of the union of chords joining pairs of distinct points of $\tilde{X}$. Also $\operatorname{Tan}(\tilde{X})$ will denote the tangent variety of $\tilde{X}$, i.e. the closure of the union of the tangent spaces of $\tilde{X}$. Thus X_p is smooth if and only if $p \notin \operatorname{Sec}(\tilde{X})$.

3. Possible secant loci

Throughout this section we keep the previously introduced notation. Let $\tilde{X} \subseteq \mathbb{P}^{r+1}_K$ be a rational normal scroll. The aim of this section is to show which secant loci $\Sigma_p(\tilde{X}) \subseteq \operatorname{Sec}_p(\tilde{X})$ may occur at all.

We first treat the case in which $\tilde{X}$ is a smooth rational normal scroll.

Notation and Remark 3.1. Let $\widetilde{X} \subset \mathbb{P}^{r+1}_K$ be a smooth rational normal scroll. Thus there is a projection morphism $\varphi : \widetilde{X} \to \mathbb{P}^1_K$. For each $x \in \mathbb{P}^1_K$, let $\mathbb{L}(x) = \varphi^{-1}(x)$ denote the ruling of $\widetilde{X}$ over x.

Proposition 3.2. *Let $\widetilde{X} \subset \mathbb{P}^{r+1}_K$ be a smooth rational normal scroll and of codimension at least 2 and let $p \in \operatorname{Sec}(\widetilde{X}) \setminus \widetilde{X}$. Then*

(1) $\operatorname{Sec}_p(\widetilde{X}) = \mathbb{P}^{t-1}_K \subset \mathbb{P}^{r+1}_K$ *for some* $t \geq 2$, $\Sigma_p(\widetilde{X}) \subset \mathbb{P}^{t-1}_K$ *is a quadric hypersurface,* $\operatorname{Sing}(X_p) = \pi_p(\Sigma_p(\widetilde{X})) = \mathbb{P}^{t-2}_K$ *is the non-normal locus of* X_p *and* $\operatorname{depth}(X_p) = t$, *where* depth *is the arithmetic depth of* X_p.

(2) *One of the following holds:*
   (a) $\operatorname{Sec}_p(\widetilde{X}) = \mathbb{P}^1_K$ *and* $\Sigma_p(\widetilde{X}) \subset \mathbb{P}^1_K$ *is either a double point or the union of two simple points.*
   (b) $\operatorname{Sec}_p(\widetilde{X}) = \mathbb{P}^2_K$ *and* $\Sigma_p(\widetilde{X}) \subset \mathbb{P}^2_K$ *is either a smooth conic or the union of a line L which is contained in a ruling $\mathbb{L}(x) \subset \widetilde{X}$ and a line section L' of $\widetilde{X}$.*
   (c) $\operatorname{Sec}_p(\widetilde{X}) = \mathbb{P}^3_K$ *and* $\Sigma_p(\widetilde{X}) \subset \mathbb{P}^3_K$ *is a smooth quadric surface.*

Proof. (1) For the geometric description of the secant cone and the secant locus, let us recall that $\widetilde{X}$ satisfies Vermeire's condition K_2 which means that the homogeneous ideal of $\widetilde{X}$ is generated by quadrics and the Koszul relations among them are generated by linear syzygies (cf. [2, Lemma 2.1]). Thus the simple argument of [7, Proposition 2.8] (see also [8, Theorem 2.2]) enables us to deduce that $\operatorname{Sec}_p(\widetilde{X}) = \mathbb{P}^{t-1}_K \subset \mathbb{P}^{r+1}_K$ for some $t \geq 2$ and $\Sigma_p(\widetilde{X}) \subset \mathbb{P}^{t-1}_K$ is a quadratic hypersurface. This implies that $\pi_p(\Sigma_p(\widetilde{X})) = \mathbb{P}^{t-2}_K$ is precisely the singular and also the non-normal locus of X_p since $\pi_p : \widetilde{X} \to X_p$ is the normalization map of X_p. For the fact that $\operatorname{depth}(X_p) = t$, we refer the reader to [1, Theorem 1.1].

(2) Let the notation be as in (3.1) and let $f : \Sigma_p(\widetilde{X}) \to \mathbb{P}^1_K$ be the restriction of φ to $\Sigma_p(\widetilde{X})$. If $t = 2$, then $\Sigma_p(\widetilde{X}) \subset \mathbb{P}^1_K$ is a hyperquadric and so we get statement (a). Suppose that $t \geq 3$. Since $p \notin \widetilde{X}$ and since $p \in \langle \Sigma_p(\widetilde{X}) \rangle$, the morphism f is surjective. If $t = 3$, then $\Sigma_p(\widetilde{X})$ is a smooth conic or the union of two distinct lines L and L' or a double line supported on a section L of φ. In the second case, it is easily shown that one of the two lines L or L' is contained in a ruling and the other one is a section of φ. To rule out the third case it suffices to remark that for two distinct points $q_1, q_2 \in L$, letting $x_i = \varphi(q_i)$, we have $T_{q_i}\Sigma_p(\widetilde{X}) \subseteq T_{q_i}\widetilde{X} = \langle \mathbb{L}(x_i), q_j \rangle$ with $i \neq j$, and $T_{q_1}\widetilde{X} \cap T_{q_2}\widetilde{X} = L$ so that $L \supseteq T_{q_1}\Sigma_p(\widetilde{X}) \cap T_{q_2}\Sigma_p(\widetilde{X}) \supseteq \operatorname{Sec}_p(\widetilde{X}) = \mathbb{P}^2_K$, a contradiction. If $t \geq 4$, then $\Sigma_p(\widetilde{X})$ is irreducible. In this case, if $\Sigma_p(\widetilde{X})$ is not a smooth quadric surface then $\operatorname{Pic}(\Sigma_p(\widetilde{X})) = \langle \mathcal{O}_{\Sigma_p(\widetilde{X})}(1) \rangle$. This contradiction yields that $t = 4$ and that $\Sigma_p(\widetilde{X})$ is a smooth quadric surface. □

We now consider the case in which the scroll $\tilde{X}$ is not necessarily smooth. First we introduce some notation.

Notation and Remark 3.3. Let $\tilde{X} \subseteq \mathbb{P}^{r+1}_K$ be a rational normal scroll of codimension at least 2 and with the vertex $\operatorname{Vert}(\tilde{X}) = \mathbb{P}^h_K$ for some $h \geq -1$. Let $\dim \widetilde{X} = n + h + 1$ and note that $\widetilde{X}$ is a cone over an n-fold rational normal scroll $\tilde{X}_0$ in $\langle \tilde{X}_0 \rangle = \mathbb{P}^{r-h}_K$. Consider the projection map

$$\psi : \mathbb{P}^{r+1}_K \setminus \operatorname{Vert}(\tilde{X}) \twoheadrightarrow \langle \tilde{X}_0 \rangle = \mathbb{P}^{r-h}_K.$$

For a closed point $p \in \mathbb{P}^{r+1}_K \setminus \operatorname{Vert}(\tilde{X})$, we denote $\psi(p)$ by $\overline{p}$. □

The following result is an immediate consequence of Proposition 3.2 and of the previous definitions.

Corollary 3.4. *Assume that the rational normal scroll $\tilde{X} \subseteq \mathbb{P}^{r+1}_K$ has codimension at least 2 and let $p \in \mathbb{P}^{r+1}_K \setminus \tilde{X}$. Then we have either*

(a) $\operatorname{Sec}_p(\tilde{X}) = \langle \operatorname{Vert}(\tilde{X}), p \rangle = \mathbb{P}^{h+1}$ *and* $\Sigma_p(\tilde{X}) = 2\operatorname{Vert}(\tilde{X}) \subseteq \mathbb{P}^{h+1}_K$.

(b) $\operatorname{Sec}_p(\tilde{X}) = \langle \operatorname{Vert}(\tilde{X}), L \rangle = \mathbb{P}^{h+2}_K$ *for some line* $L \subseteq \langle \tilde{X}_0 \rangle$ *and either*
   (i) $\Sigma_p(\tilde{X}) = \operatorname{Join}(\operatorname{Vert}(\tilde{X}), Z) \subseteq \mathbb{P}^{h+2}_K$, *where* $Z \subseteq L$ *consists of two simple points; or*
   (ii) $\Sigma_p(\tilde{X}) = 2\langle \operatorname{Vert}(\tilde{X}), Z \rangle \subseteq \mathbb{P}^{h+2}_K$, *where* $Z \subseteq L$ *consists of one simple point.*

(c) $\operatorname{Sec}_p(\tilde{X}) = \langle \operatorname{Vert}(\tilde{X}), P \rangle = \mathbb{P}^{h+3}_K$ *for some plane* $P \subseteq \langle \tilde{X}_0 \rangle$ *and* $\Sigma_p(\tilde{X}) = \operatorname{Join}(\operatorname{Vert}(\tilde{X}), W)$, *where* $W \subseteq P$ *is either a smooth conic or the union of two lines* $L, L' \subseteq P$ *such that* $L \subseteq \mathbb{L}(x) \cap \tilde{X}_0$ *for some* $x \in \mathbb{P}^1_K$ *and* L' *is a line section of* $\tilde{X}_0$.

(d) $\operatorname{Sec}_p(\tilde{X}) = \langle \operatorname{Vert}(\tilde{X}), D \rangle = \mathbb{P}^{h+4}_K$ *for some 3-space* $D \subseteq \langle \tilde{X}_0 \rangle$ *and* $\Sigma_p(\tilde{X}) = \operatorname{Join}(\operatorname{Vert}(\tilde{X}), V)$, *where* $V \subseteq D$ *is a smooth quadric surface.*

4. The secant stratification in the smooth case

According to Proposition 3.2 there are at most six different possibilities for the secant locus $\Sigma_p(\tilde{X})$ of smooth rational normal scrolls $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ with respect to the point $p \in \mathbb{P}_K^{r+1} \setminus \tilde{X}$. This gives a decomposition of $\mathbb{P}_K^{r+1} \setminus \tilde{X}$ into at most six disjoint strata. The aim of this section is to describe this stratification in geometric terms.

Definition and Remark 4.1. (A) Let

$$\tilde{X} = S(a_1, \ldots, a_n) \subseteq \mathbb{P}_K^{r+1}$$

be an n-dimensional smooth rational normal scroll of type $(a_1, \ldots, a_n)$ with $1 \leq a_1 \leq \cdots \leq a_n$. We assume that $\operatorname{codim}(\widetilde{X}) \geq 2$, or equivalently, $a_1 + \cdots + a_n \geq 3$. We write

$$k := \begin{cases} \max\{i \in \{1, \ldots, n\} | a_i = 1\} & \text{if } a_1 = 1, \text{ and} \\ 0 & \text{if } a_1 > 1 \end{cases}$$

and

$$m := \begin{cases} \max\{i \in \{k+1, \ldots, n\} | a_i = 2\} & \text{if } a_{k+1} = 2, \text{ and} \\ k & \text{if } a_{k+1} \neq 2. \end{cases}$$

So, we have $k \in \{0, \ldots, n\}$ and $m \in \{k, \ldots, n\}$ and may write

$$\tilde{X} = S(\underbrace{1, \ldots, 1}_{k}, \underbrace{2, \ldots, 2}_{m-k}, a_{m+1}, \ldots, a_n) = S(\underline{1}, \underline{2}, \underline{a})$$

with $3 \leq a_{m+1} \leq \cdots \leq a_n$.
(B) Consider the (possibly empty) scrolls

$$S(\underline{1}) = S(\underbrace{1, \ldots, 1}_{k}) \subset \mathbb{P}_K^{2k-1} \quad \text{and} \quad S(\underline{2}) = S(\underbrace{2, \ldots, 2}_{m-k}) \subset \mathbb{P}_K^{3m-3k-1}$$

which are contained in $\widetilde{X}$. Obviously $S(\underline{1}) \cong \mathbb{P}_K^1 \times \mathbb{P}_K^{k-1}$ and $S(\underline{2}) \cong C \times \mathbb{P}_K^{m-k-1}$ for a smooth plane conic $C \subset \mathbb{P}_K^2$. For each $\alpha \in \mathbb{P}_K^{k-1}$, we denote the line $\mathbb{P}_K^1 \times \{\alpha\}$ in $S(\underline{1})$ by L_α. Similarly, for each $\beta \in \mathbb{P}_K^{m-k-1}$, we denote the smooth plane conic $C \times \{\beta\}$ in $S(\underline{2})$ by C_β. Also $S(\underline{2})$ can be regarded as a subvariety of the Segre variety

$$\Delta := \dot{\bigcup_{\beta \in \mathbb{P}_K^{m-k-1}}} \langle C_\beta \rangle = \mathbb{P}_K^2 \times \mathbb{P}_K^{m-k-1} \subset \mathbb{P}_K^{3m-3k-1}.$$

(C) We define the following sets in $\mathbb{P}_K^{r+1} \setminus \tilde{X}$:

$$SL^{\emptyset}(\tilde{X}) := \{p \in \mathbb{P}_K^{r+1} \setminus \tilde{X} | \Sigma_p(\tilde{X}) = \emptyset\}; \tag{4.1}$$

$$SL^{q_1,q_2}(\tilde{X}) := \{p \in \mathbb{P}_K^{r+1} \setminus \tilde{X} | \Sigma_p(\tilde{X}) \text{ consists of two simple points}\}; \tag{4.2}$$

$$SL^{2q}(\tilde{X}) := \{p \in \mathbb{P}_K^{r+1} \setminus \tilde{X} | \Sigma_p(\tilde{X}) \text{ is a double point in some straight line } \mathbb{P}_K^1 \subseteq \mathbb{P}_K^{r+1}\}; \tag{4.3}$$

$$SL^{L_1 \cup L_2}(\tilde{X}) := \{p \in \mathbb{P}_K^{r+1} \setminus \tilde{X} | \Sigma_p(\tilde{X}) \text{ is the union of two distinct coplanar simple lines}\}; \tag{4.4}$$

$$SL^{C}(\tilde{X}) := \{p \in \mathbb{P}_K^{r+1} \setminus \tilde{X} | \Sigma_p(\tilde{X}) \text{ is a smooth plane conic}\}; \tag{4.5}$$

$$SL^{Q}(\tilde{X}) := \{p \in \mathbb{P}_K^{r+1} \setminus \tilde{X} | \Sigma_p(\tilde{X}) \text{ is a smooth quadric in a 3-space}\}. \tag{4.6}$$

Note that some of the above sets can be empty. According to Proposition 3.2 we have

$$\mathbb{P}_K^{r+1} = \tilde{X} \,\dot{\cup}\, SL^{\emptyset}(\tilde{X}) \,\dot{\cup}\, SL^{q_1,q_2}(\tilde{X}) \,\dot{\cup}\, SL^{2q}(\tilde{X}) \,\dot{\cup}\, SL^{L_1 \cup L_2}(\tilde{X}) \,\dot{\cup}\, SL^{C}(\tilde{X}) \,\dot{\cup}\, SL^{Q}(\tilde{X}). \tag{4.7}$$

This decomposition will be called *the secant stratification of* $\widetilde{X} \subset \mathbb{P}_K^{r+1}$.

We now describe the strata (4.1)–(4.6).

Theorem 4.2. *Assume that the rational normal scroll $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ is of codimension at least 2 and smooth. Let (cf. Definition and Remark 4.1(B))*

$$A := \langle S(\underline{1}) \rangle;$$
$$B := \operatorname{Join}(S(\underline{1}), \widetilde{X});$$
$$U := \operatorname{Join}(A, \Delta).$$

Then $\tilde{X} \subseteq B,\ A \subseteq B \cap U,\ B \cup U \subseteq \operatorname{Tan}(\tilde{X})$ and

(a) $SL^{\emptyset}(\tilde{X}) = \mathbb{P}_K^{r+1} \setminus \operatorname{Sec}(\tilde{X})$;

(b) $SL^{q_1,q_2}(\tilde{X}) = \mathrm{Sec}(\tilde{X})\backslash\mathrm{Tan}(\tilde{X})$;
(c) $SL^{2q}(\tilde{X}) = \mathrm{Tan}(\tilde{X})\backslash(B \cup U)$;
(d) $SL^{L_1\cup L_2}(\tilde{X}) = B\backslash(A \cup \tilde{X})$;
(e) $SL^{C}(\tilde{X}) = U\backslash B$;
(f) $SL^{Q}(\tilde{X}) = A\backslash\tilde{X}$.

In order to prove this theorem we will need a preliminary result we mention now:

Lemma 4.3. *Let $\widetilde{X} \subset \mathbb{P}_K^{r+1}$ be a smooth rational normal scroll and let p be a closed point contained in $\mathbb{P}_K^{r+1} \setminus \widetilde{X}$.*

(1) *If* $\dim \Sigma_p(\widetilde{X}) > 0$, *then* $p \in \mathrm{Tan}(\widetilde{X})$.
(2) *Let* $\widetilde{X} = S(\underbrace{1, \ldots, 1}_{n}) \subset \mathbb{P}_K^{2n-1}$ *with* $n \geq 2$. *Then* $\Sigma_p(\widetilde{X})$ *is a smooth quadric surface.*
(3) *Let* $\widetilde{X} = S(1, 2) \subset \mathbb{P}_K^{4}$. *Then* $\dim \Sigma_p(\widetilde{X}) = 1$.
(4) *Let* $\widetilde{X} = S(\underbrace{1, \ldots, 1}_{n-1}, 2) \subset \mathbb{P}_K^{2n}$ *with* $n \geq 2$. *Then* $\dim \Sigma_p(\widetilde{X}) \geq 1$.

Proof. (1): The claim is obvious by Proposition 3.2.
(2), (3), (4): The statements follow by keeping in mind that for every $p \in \mathrm{Sec}(\widetilde{X}) \setminus \widetilde{X}$ we have

$$(2n + 1) - \dim \mathrm{Sec}(\widetilde{X}) \leq \dim \Sigma_p(\widetilde{X}) \tag{4.8}$$

with equality holding for general $p \in \mathrm{Sec}(\widetilde{X}) \setminus \widetilde{X}$.

For $\widetilde{X} = S(\underbrace{1, \ldots, 1}_{n}) \subset \mathbb{P}_K^{2n-1}$ we have $\dim \Sigma_p(\widetilde{X}) \geq 2n + 1 - (2n - 1) = 2$ since $\mathrm{Sec}(\widetilde{X}) \subseteq \mathbb{P}_K^{2n-1}$. Therefore by Proposition 3.2, $\Sigma_p(\widetilde{X})$ is a smooth quadric surface for every $p \in \mathrm{Sec}(\widetilde{X}) \setminus \widetilde{X}$. For $\widetilde{X} = S(1, 2) \subset \mathbb{P}_K^{4}$ we have $\dim \Sigma_p(\widetilde{X}) \geq 1$ by (4.8) so that equality holds. Moreover, since two general rulings of $\widetilde{X} = S(\underbrace{1, \ldots, 1}_{n-1}, 2) \subset \mathbb{P}_K^{2n}$ with $n \geq 2$ span a linear space of dimension $2n - 1$ contained in $\mathrm{Sec}(\widetilde{X})$, we have $\mathrm{Sec}(\widetilde{X}) = \mathbb{P}_K^{2n}$ and hence $\dim \Sigma_p(\widetilde{X}) \geq 1$ with equality for general $p \in \mathrm{Sec}(\widetilde{X}) \setminus \widetilde{X}$. $\square$

Now we give the

Proof of Theorem 4.2. The inclusions $\tilde{X} \subseteq B$ and $A \subseteq U$ are obvious. As $A = \mathrm{Sec}(S(\underline{1}))$ we get $A \subseteq \mathrm{Join}(S(\underline{1}), \tilde{X}) = B$, hence $A \subseteq B \cap U$.

To prove the inclusions $B \subseteq \mathrm{Tan}(\tilde{X})$ and $U \subseteq \mathrm{Tan}(\tilde{X})$ we write

$$S(\underline{1}) = \bigcup_{\alpha \in \mathbb{P}_K^{k-1}} L_\alpha \quad \text{and} \quad \Delta = \bigcup_{\beta \in \mathbb{P}_K^{m-k-1}} \langle C_\beta \rangle$$

(cf. Definition and Remark 4.1(B)). Now, the first inclusion follows from the equalities

$$\begin{aligned} B &= \mathrm{Join}(S(\underline{1}), \widetilde{X}) \\ &= \bigcup_{\alpha \in \mathbb{P}_K^{k-1}} \mathrm{Join}(L_\alpha, \widetilde{X}) \\ &= \bigcup_{\alpha \in \mathbb{P}_K^{k-1}} \mathrm{Join}\left(L_\alpha, \bigcup_{x \in \mathbb{P}_K^{1}} \mathbb{L}(x)\right) \\ &= \bigcup_{\alpha \in \mathbb{P}_K^{k-1}, x \in \mathbb{P}_K^{1}} \langle L_\alpha, \mathbb{L}(x) \rangle. \end{aligned} \tag{4.9}$$

To prove the second inclusion we first observe that

$$\begin{aligned} U &= \mathrm{Join}(A, \Delta) \\ &= \mathrm{Join}\left(A, \bigcup_{\beta \in \mathbb{P}_K^{m-k-1}} \langle C_\beta \rangle\right) \\ &= \bigcup_{\beta \in \mathbb{P}_K^{m-k-1}} \langle A, \langle C_\beta \rangle \rangle. \end{aligned} \tag{4.10}$$

For each $\beta \in \mathbb{P}_K^{m-k-1}$ consider the smooth rational normal scroll

$$S(\underline{1},2)_\beta := \langle A, \langle C_\beta \rangle\rangle \cap \widetilde{X} \subset \langle A, \langle C_\beta\rangle\rangle.$$

Since $S(\underline{1},2)_\beta$ spans the linear space $\langle A, \langle C_\beta\rangle\rangle$, Lemma 4.3(4) implies that

$$\langle A, \langle C_\beta\rangle\rangle = \operatorname{Tan} S(\underline{1},2)_\beta \subset \operatorname{Tan}(\widetilde{X}).$$

This gives the desired inclusion $U \subseteq \operatorname{Tan}(\tilde{X})$.

We now prove statements (a), (b), (f), (d), (e) and (c).

(a): This follows by Notation and Remarks 2.3(B).

(b): "$\subseteq$": Let $p \in SL^{q_1,q_2}(\widetilde{X})$. Then $\Sigma_p(\widetilde{X}) = \{q_1, q_2\}$ and $L := \langle q_1, q_2\rangle$ is a secant line to $\widetilde{X}$ and hence $p \in \operatorname{Sec}(\widetilde{X})$. Now assume that $p \in \operatorname{Tan}(\widetilde{X})$. Then there exists $q \in \widetilde{X}$ such that $p \in T_q\widetilde{X}$. Therefore $q \in \Sigma_p(\widetilde{X})$. This implies that $q = q_1$ or $q = q_2$. In particular, L is a tri-secant line to $\widetilde{X}$ and so we get the contradiction that $p \in L \subseteq \widetilde{X}$. Therefore $p \in \operatorname{Sec}(\widetilde{X}) \setminus \operatorname{Tan}(\widetilde{X})$.

"$\supseteq$": Let $p \in \operatorname{Sec}(\widetilde{X}) \setminus \operatorname{Tan}(\widetilde{X})$. By statement (a) we have $\Sigma_p(\widetilde{X}) \neq \emptyset$. By Lemma 4.3(1) we have $\dim \Sigma_p(\widetilde{X}) \leq 0$. Thus $\Sigma_p(\widetilde{X})$ has dimension zero. Now Proposition 3.2 and $p \notin \operatorname{Tan}(\widetilde{X})$ guarantee that $\Sigma_p(\widetilde{X})$ is the union of two simple points.

(f): "$\subseteq$": Let $p \in SL^{Q}(\widetilde{X})$ so that $\operatorname{Sec}_p(\widetilde{X}) = \mathbb{P}_K^3$ and $\Sigma_p(\widetilde{X}) \subset \mathbb{P}_K^3$ is a smooth quadric surface. Remember that $\Sigma_p(\widetilde{X})$ contains two disjoint families of lines $\{L_\lambda\}$, $\{M_\lambda\}$, each parameterized by $\lambda \in \mathbb{P}_K^1$. Also since $p \notin \widetilde{X}$, the restriction map $\varphi|_{\Sigma_p(\widetilde{X})} : \Sigma_p(\widetilde{X}) \to \mathbb{P}_K^1$ is surjective. Therefore one of the two families, say $\{L_\lambda\}$, consists of line sections of $\widetilde{X}$. Thus $\Sigma_p(\widetilde{X}) \subset S(\underline{1})$ and hence $p \in \operatorname{Sec}_p(\widetilde{X}) = \langle \Sigma_p(\widetilde{X})\rangle \subset A = \langle S(\underline{1})\rangle$.

"$\supseteq$": Let $p \in A \setminus \widetilde{X}$. Then $k > 1$ (s. Definition and Remark 4.1(B)). Therefore Lemma 4.3(2) shows that $\Sigma_p(S(\underline{1}))$ is a smooth quadric surface. Since $\Sigma_p(S(\underline{1})) \subset \Sigma_p(\widetilde{X})$ this implies that $\dim\, \Sigma_p(\widetilde{X}) \geq 2$. Now by Proposition 3.2 we see that $\Sigma_p(\widetilde{X}) = \Sigma_p(S(\underline{1}))$ is a smooth quadric surface and hence $p \in SL^{Q}(\widetilde{X})$.

(d): "$\subseteq$": Let $p \in SL^{L_1\cup L_2}(\widetilde{X})$ so that $\operatorname{Sec}_p(\widetilde{X}) = \mathbb{P}_K^2$ and $\Sigma_p(\widetilde{X}) = L_1 \cup L_2$ where L_i are lines such that L_1 is a line section of $\widetilde{X}$. Thus $L_1 \subset S(\underline{1})$ and $L_2 \subset \mathbb{L}(x)$ for some $x \in \mathbb{P}_K^1$. This shows that $p \in \langle L_1, L_2\rangle = \operatorname{Sec}_p(\widetilde{X}) \subset \operatorname{Join}(S(\underline{1}), \widetilde{X}) = B$. On the other hand statement (f) implies $p \notin A$. Therefore, $p \in B \setminus (A \cup \widetilde{X})$.

"$\supseteq$": Let $p \in B \setminus (A \cup \widetilde{X})$. By (4.8), there exists a line $L_\alpha \subset S(\underline{1})$ and a point $x \in \mathbb{P}_K^1$ such that $p \in \langle L_\alpha, \mathbb{L}(x)\rangle = \mathbb{P}_K^n$. Let L denote the line $\langle p, L_\alpha\rangle \cap \mathbb{L}(x)$. Then clearly $L_\alpha \cup L \subset \Sigma_p(\widetilde{X})$. On the other hand, Proposition 3.2 and statement (f) show that $\Sigma_p(\widetilde{X})$ has dimension at most one so that $\Sigma_p(\widetilde{X}) = L_\alpha \cup L$ and hence $p \in SL^{L_1\cup L_2}(\widetilde{X})$.

(e): "$\subseteq$": Let $p \in SL^{C}(\widetilde{X})$. This means that $\operatorname{Sec}_p(\widetilde{X}) = \mathbb{P}_K^2$ and $\Sigma_p(\widetilde{X}) \subset \mathbb{P}_K^2$ is a smooth plane conic curve. Clearly $\varphi|_{\Sigma_p(\widetilde{X})} : \Sigma_p(\widetilde{X}) \to \mathbb{P}_K^1$ is a surjective map. We will show that indeed $\Sigma_p(\widetilde{X})$ is a conic section of $\tilde{X}$, or equivalently, that $\varphi|_{\Sigma_p(\widetilde{X})}$ is bijective. Suppose to the contrary that there is a point $x \in \mathbb{P}_K^1$ such that $\mathbb{L}(x)$ meets $\Sigma_p(\widetilde{X})$ in two distinct points q_1 and q_2. Then the two lines $T_{q_1}\Sigma_p(\widetilde{X})$ and $T_{q_2}\Sigma_p(\widetilde{X})$ meet at a point $z \in \operatorname{Sec}_p(\widetilde{X})\setminus\tilde{X}$. Thus we have

$$T_{q_1}(\widetilde{X}) = \langle \mathbb{L}(x), z\rangle = T_{q_2}(\widetilde{X}),$$

which is impossible. Therefore, $\Sigma_p(\widetilde{X})$ is a conic section of $\tilde{X}$. In particular $\Sigma_p(\widetilde{X}) \subset S(\underline{1},\underline{2})$.

Next we claim that $\operatorname{Sec}_p(\widetilde{X}) \cap A = \emptyset$. Suppose to the contrary that there is a closed point z in $\operatorname{Sec}_p(\widetilde{X}) \cap A$. If $z \notin S(\underline{1})$, then $k > 1$ and so $\operatorname{Sec}_z(\widetilde{X})$ contains the 3-dimensional space $\operatorname{Sec}_z(S(\underline{1}))$ (s. Lemma 4.3(2)) and the point p. This implies that $\operatorname{Sec}_z(\widetilde{X})$ has dimension at least 4, a contradiction to Proposition 3.2. If $z \in S(\underline{1})$, then let L_z be the unique line section of $\widetilde{X}$ which passes through z. Note that $z \in L_z \cap \Sigma_p(\widetilde{X})$ since otherwise $\langle p, z\rangle$ is a proper tri-secant line to $\widetilde{X}$, which is not possible since $\widetilde{X}$ is cut out by quadrics. As $\Sigma_p(\widetilde{X})$ is a smooth plane conic curve, we have $L_z \nsubseteq \operatorname{Sec}_p(\tilde{X}) = \langle \Sigma_p(\tilde{X})\rangle$, since otherwise $\tilde{X}$ would have tri-secant lines. Observe that the 3-dimensional linear space $\langle \operatorname{Sec}_p(\widetilde{X}), L_z\rangle$ contains both p and the surface

$$S := \overline{\bigcup_{x\in\mathbb{P}_K^1\setminus\{\varphi(z)\}} \langle \mathbb{L}(x) \cap \Sigma_p(\widetilde{X}), \mathbb{L}(x) \cap L_z\rangle} \subset \tilde{X}.$$

As the conic $\Sigma_p(\tilde{X})$ is contained in S, S cannot be a plane. So the generic line in $\langle \operatorname{Sec}_p(\tilde{X}), L_z\rangle$ passing through p is a secant line to S, whence to $\tilde{X}$. It follows that $\Sigma_p(\widetilde{X})$ contains S, which contradicts the fact that $\dim\ \Sigma_p(\tilde{X}) = 1$. This completes the proof that $\operatorname{Sec}_p(\widetilde{X}) \cap A = \emptyset$.

Now consider the canonical projection map

$$\pi : \langle S(\underline{1},\underline{2})\rangle \setminus A \to \langle S(\underline{2})\rangle$$

which fixes $\langle S(\underline{2})\rangle$. The image $\pi(\Sigma_p(\tilde{X}))$ is contained in $S(\underline{2})$. Also it is again a smooth plane conic as $\operatorname{Sec}_p(\tilde{X}) \cap A = \emptyset$. Moreover, for all $x \in \mathbb{P}_K^1$ we have $\pi(\mathbb{L}(x) \cap (\langle S(\underline{1},\underline{2})\rangle\setminus A)) \subseteq \mathbb{L}(x)$ so that $\sharp(\pi(\Sigma_p(\widetilde{X})) \cap \mathbb{L}(x)) \leq 1$ for all such x. Therefore $\pi(\Sigma_p(\tilde{X}))$ is a conic section of $\tilde{X}$ and hence is equal to C_β for some $\beta \in \mathbb{P}_K^{m-k-1}$. Therefore $\pi(\operatorname{Sec}_p(\widetilde{X})) = \langle C_\beta\rangle$ which guarantees that

$$p \in \operatorname{Sec}_p(\widetilde{X}) \subset \operatorname{Join}(A, \langle C_\beta\rangle) \subset U.$$

On the other hand, $p \notin (B \setminus (A \cup \widetilde{X})) \cup (A \setminus \widetilde{X}) \cup \widetilde{X} = B$ by statements (d) and (f). This completes the proof that $p \in U \setminus B$.

"$\supseteq$": Let $p \in U \setminus B$. By statements (a), (b), (d), (f) and Proposition 3.2 we have

$$p \in SL^{2q}(\widetilde{X}) \,\dot{\cup}\, SL^{C}(\widetilde{X}).$$

Thus it suffices to show that $\dim \Sigma_p(\widetilde{X}) \geq 1$. By (4.9), there exists a point $\beta \in \mathbb{P}_K^{m-k-1}$ such that $p \in \langle A, \langle C_\beta\rangle\rangle \setminus A$. If $p \in \langle C_\beta\rangle$, then $C_\beta \subset \Sigma_p(\widetilde{X})$ and hence $\dim \Sigma_p(\widetilde{X}) \geq 1$. Now assume that $p \notin \langle C_\beta\rangle$, so that $A \neq \emptyset$, and consider the canonical projection map (which fixes A)

$$\varrho : \langle A, \langle C_\beta\rangle\rangle \setminus \langle C_\beta\rangle \to A.$$

Let $q = \varrho(p)$. If $q \in S(\underline{1})$, let L be the unique line section of $\widetilde{X}$ which passes through q. Then

$$p \in \mathrm{Join}(L, \langle C_\beta\rangle) = \mathbb{P}_K^4.$$

Moreover, $\mathbb{P}_K^4$ contains the smooth rational normal surface scroll

$$S(1,2) = \bigcup_{v\in L, w\in C_\beta, \varphi(v)=\varphi(w)} \langle v, w\rangle \subset \widetilde{X}.$$

Since $\Sigma_p(S(1,2)) \subset \Sigma_p(\widetilde{X})$, Lemma 4.3(3) shows that $\dim \Sigma_p(\widetilde{X}) \geq 1$. If $q \in A \setminus S(\underline{1})$, we have $k > 1$ and Lemma 4.3(2) implies that $\Sigma_q(S(\underline{1}))$ is a smooth quadric surface. Moreover

$$p \in \mathrm{Join}(\langle \Sigma_q(S(\underline{1}))\rangle, \langle C_\beta\rangle) = \mathbb{P}_K^6.$$

For each point $x \in \mathbb{P}_K^1$ consider the line $L_x = \Sigma_q(S(\underline{1})) \cap \mathbb{L}(x)$. Now $\mathbb{P}_K^6$ contains the threefold rational normal scroll

$$S(1,1,2) = \bigcup_{x\in \mathbb{P}_K^1, w\in C_\beta, \varphi(w)=x} \langle L_x, w\rangle \subset \widetilde{X}.$$

As $\Sigma_p(S(1,1,2)) \subset \Sigma_p(\widetilde{X})$, it remains to show that $\dim \Sigma_p(S(1,1,2)) \geq 1$. This follows by Lemma 4.3(4).

(c): It is easy to see that $\mathbb{P}_K^{r+1}\setminus\tilde{X}$ is the disjoint union of the sets $\mathbb{P}_K^{r+1}\setminus\mathrm{Sec}(\tilde{X})$, $\mathrm{Sec}(\tilde{X})\setminus\mathrm{Tan}(\tilde{X})$, $\mathrm{Tan}(\tilde{X})\setminus(B\cup U)$, $B\setminus(A\cup\tilde{X})$, $U\setminus B$ and $A\setminus\tilde{X}$. Now, statements (a), (b), (d), (e), (f) and the equality (4.7) imply that $SL^{2q}(\tilde{X}) = \mathrm{Tan}(\tilde{X})\setminus(B\cup U)$. $\square$

5. The secant stratification in the general case

We now treat the secant stratification in the general case, that is in the case where the scroll $\tilde{X}$ is not necessarily smooth. We keep all the previous hypotheses and notations.

We first appropriately generalize the concepts defined in Definition and Remark 4.1.

Definition and Remark 5.1. (A) Let $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ be a rational normal scroll of codimension at least 2 and with vertex $\mathrm{Vert}(\tilde{X}) = \mathbb{P}_K^h$ for some $h \geq 0$. As in Notation and Remark 3.3, let $\dim \tilde{X} = n+h+1$ and note that $\tilde{X}$ is a cone over an n-fold rational normal scroll $\tilde{X}_0$ in $\langle \tilde{X}_0\rangle = \mathbb{P}_K^{r-h}$. Let $(a_1, \ldots, a_n)$ be the type of $\tilde{X}_0$. Let the integers k and m and the subvarieties $S(\underline{1})$, $S(\underline{2})$ and Δ of $\mathbb{P}_K^{r-h}$ be as in Definition and Remark 4.1. Again we consider the projection map (cf. Notation and Remark 3.3)

$$\psi : \mathbb{P}_K^{r+1} \setminus \mathrm{Vert}(\tilde{X}) \twoheadrightarrow \langle \tilde{X}_0\rangle = \mathbb{P}_K^{r-h}$$

and write $\overline{p} := \psi(p)$ for a closed point $p \in \mathbb{P}_K^{r+1}\setminus\mathrm{Vert}(\tilde{X})$.

(B) We define the following sets in $\mathbb{P}_K^{r+1}\setminus\tilde{X}$:

$$SL^{\emptyset}(\tilde{X}) := \{p \in \mathbb{P}_K^{r+1}\setminus\tilde{X} \mid \Sigma_p(\tilde{X}) = 2\mathrm{Vert}(\tilde{X}) \subseteq \langle \mathrm{Vert}(\tilde{X}), p\rangle\}; \tag{5.1}$$

$$SL^{q_1,q_2}(\tilde{X}) := \{p \in \mathbb{P}_K^{r+1}\setminus\tilde{X} \mid \Sigma_p(\tilde{X}) = \mathrm{Join}(\mathrm{Vert}(\tilde{X}), \{x,y\}) \text{ where } x, y \text{ are two distinct points in some line } \mathbb{P}_K^1 \subseteq \langle \tilde{X}_0\rangle\}; \tag{5.2}$$

$$SL^{2q}(\tilde{X}) := \{p \in \mathbb{P}_K^{r+1}\setminus\tilde{X} \mid \Sigma_p(\tilde{X}) = 2\mathrm{Join}(\mathrm{Vert}(\tilde{X}), x) \subseteq \mathrm{Join}(\mathrm{Vert}(\tilde{X}), L), \text{ where } L \subseteq \langle \tilde{X}_0\rangle \text{ is a line and } x \in L\}; \tag{5.3}$$

$$SL^{L_1\cup L_2}(\tilde{X}) := \{p \in \mathbb{P}_K^{r+1}\setminus\tilde{X} \mid \Sigma_p(\tilde{X}) = \mathrm{Join}(\mathrm{Vert}(\tilde{X}), L\cup L') \text{ where } L, L' \subseteq \langle \tilde{X}_0\rangle \text{ are two distinct coplanar simple lines}\}; \tag{5.4}$$

$$SL^{C}(\tilde{X}) := \{p \in \mathbb{P}_K^{r+1}\setminus\tilde{X} \mid \Sigma_p(\tilde{X}) = \mathrm{Join}(\mathrm{Vert}(\tilde{X}), V) \text{ where } V \subseteq \langle \tilde{X}_0\rangle \text{ is a smooth plane conic}\}; \tag{5.5}$$

$$SL^{Q}(\tilde{X}) := \{p \in \mathbb{P}_K^{r+1}\setminus\tilde{X} \mid \Sigma_p(\tilde{X}) = \mathrm{Join}(\mathrm{Vert}(\tilde{X}), W) \text{ where } W \subseteq \langle \tilde{X}_0\rangle \text{ is a smooth quadric surface in a 3-space}\}. \tag{5.6}$$

Clearly, if $\tilde{X}$ is smooth, the strata defined in (5.1)–(5.6) respectively coincide with the corresponding strata defined in (4.1)–(4.6). According to Corollary 3.4 we have

$$\mathbb{P}_K^{r+1} = \tilde{X} \,\dot{\cup}\, SL^{\emptyset}(\tilde{X}) \,\dot{\cup}\, SL^{q_1,q_2}(\tilde{X}) \,\dot{\cup}\, SL^{2q}(\tilde{X}) \,\dot{\cup}\, SL^{L_1\cup L_2}(\tilde{X}) \,\dot{\cup}\, SL^{C}(\tilde{X}) \,\dot{\cup}\, SL^{Q}(\tilde{X}). \tag{5.7}$$

This decomposition will be called *the secant stratification of* $\widetilde{X} \subset \mathbb{P}_K^{r+1}$.

Table 1
The secant stratification.

$p \in$	$\Sigma_{\overline{p}}(\tilde{X}_0) \subseteq \tilde{X}_0$	$\Sigma_p(\tilde{X}) \subseteq \tilde{X}$	$\dim(\Sigma_p(\tilde{X}))$	$\operatorname{depth}(X_p)$
$\mathbb{P}_K^{r+1} \backslash W$	$\emptyset$	$2Z \subseteq \langle Z, p\rangle$	h	$h+2$
$W \backslash V$	$\{q_1, q_2\}$ $q_1, q_2 \in \tilde{X}_0, q_1 \neq q_2$	$\langle Z, q_1\rangle \cup \langle Z, q_2\rangle$	$h+1$	$h+3$
$V \backslash (B \cup U)$	$2q \in \langle \overline{p}, q\rangle$ $q \in \tilde{X}_0$	$2\langle Z, q\rangle \subseteq \langle Z, p, q\rangle$	$h+1$	$h+3$
$B \backslash (A \cup \tilde{X})$	$L_1 \cup L_2 \subseteq \tilde{X}_0$ two coplanar lines	$\langle Z, L_1\rangle \cup \langle Z, L_2\rangle$	$h+2$	$h+4$
$U \backslash B$	$C \subseteq \tilde{X}_0$ a smooth plane conic	$\operatorname{Join}(Z, C)$	$h+2$	$h+4$
$A \backslash \tilde{X}$	$Q \subseteq \tilde{X}_0$, a smooth quadric surface	$\operatorname{Join}(Z, Q)$	$h+3$	$h+5$

Lemma 5.2. *Let $p \in \mathbb{P}_K^{r+1} \backslash \tilde{X}$ and let $\overline{p} \in \langle \tilde{X}_0\rangle$ be defined according to Definition and Remark 5.1(A). Then*

$$p \in SL^*(\tilde{X}) \Longleftrightarrow \overline{p} \in SL^*(\tilde{X}_0)$$

where $*$ runs through the set of symbols $\{\emptyset, (q_1, q_2), 2q, L_1 \cup L_2, C, Q\}$.

Proof. Clear from Corollary 3.4. □

Theorem 5.3. *Let $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ be a rational normal scroll of codimension ≥ 2. In the notations of Definition and Remark 5.1, let*

$$V := \operatorname{Join}(\operatorname{Vert}(\widetilde{X}), \operatorname{Tan}(\tilde{X}_0));$$
$$W := \operatorname{Join}(\operatorname{Vert}(\widetilde{X}), \operatorname{Sec}(\tilde{X}_0));$$
$$A := \langle \operatorname{Vert}(\widetilde{X}), \langle S(\underline{1})\rangle\rangle;$$
$$B := \operatorname{Join}(\operatorname{Vert}(\widetilde{X}), \operatorname{Join}(S(\underline{1}), \tilde{X}_0));$$
$$U := \operatorname{Join}(A, \Delta).$$

Then $\tilde{X} \subseteq B$, $A \subseteq B$, $B \cup U \subseteq V \subseteq W$ and

(a) $SL^{\emptyset}(\tilde{X}) = \mathbb{P}_K^{r+1} \backslash W$;
(b) $SL^{q_1,q_2}(\tilde{X}) = W \backslash V$;
(c) $SL^{2q}(\tilde{X}) = V \backslash (B \cup U)$;
(d) $SL^{L_1 \cup L_2}(\tilde{X}) = B \backslash (A \cup \tilde{X})$;
(e) $SL^{C}(\tilde{X}) = U \backslash B$;
(f) $SL^{Q}(\tilde{X}) = A \backslash \tilde{X}$.

Proof. For any closed subset $T \subseteq \langle \tilde{X}_0\rangle$, we have $\psi^{-1}(T) = \operatorname{Join}(\operatorname{Vert}(\widetilde{X}), T) \backslash Z$. Therefore we get the relations

$$\psi^{-1}(\tilde{X}_0) = \tilde{X} \backslash \operatorname{Vert}(\widetilde{X});$$
$$\psi^{-1}(\langle S(\underline{1})\rangle) = A \backslash \operatorname{Vert}(\widetilde{X});$$
$$\psi^{-1}(\operatorname{Join}(S(\underline{1}), \tilde{X}_0)) = B \backslash \operatorname{Vert}(\widetilde{X});$$
$$\psi^{-1}(\operatorname{Join}(\langle S(\underline{1}), \Delta)) = U \backslash \operatorname{Vert}(\widetilde{X});$$
$$\psi^{-1}(\operatorname{Tan}(\tilde{X}_0)) = V \backslash \operatorname{Vert}(\widetilde{X}) \quad \text{and}$$
$$\psi^{-1}(\operatorname{Sec}(\tilde{X}_0)) = W \backslash \operatorname{Vert}(\widetilde{X}).$$

Thus we get our claim by combining Theorem 4.2 and Lemma 5.2. □

Remark 5.4. Let $Z = \operatorname{Vert}(\widetilde{X})$ and let $p \in \mathbb{P}_K^{r+1} \backslash \tilde{X}$. By Proposition 3.2(1) and Theorem 5.3, the secant loci $\Sigma_p(\tilde{X})$, $\Sigma_{\overline{p}}(\tilde{X}_0)$ and the arithmetic depth of the projection $X_p \subset \mathbb{P}_K^r$ depend on the position of p as shown in Table 1.

6. Non-normal del Pezzo varieties

Remark 6.1. (A) A non-degenerate closed integral subscheme $X \subseteq \mathbb{P}_K^r$ is called a *maximal del Pezzo variety* if it is arithmetically Cohen–Macaulay and satisfies $\deg(X) = \operatorname{codim}(X) + 2$ (cf. [1, Definition 6.3]). A *del Pezzo variety* is a projective variety $X \subseteq \mathbb{P}^r_K$ which is an isomorphic projection of a maximal del Pezzo variety. It is equivalent to say that the polarized pair $(X, \mathcal{O}_X(1))$ is del Pezzo in the sense of Fujita [5], (cf. [1, Theorem 6.8]). So in particular we can say:

$$\text{A del Pezzo variety } X \subseteq \mathbb{P}^r_K \text{ is maximally del Pezzo if and only if it is linearly normal.} \tag{6.1}$$

(B) Let $X \subseteq \mathbb{P}^r_K$ be a non-degenerate closed integral subscheme. According to [3, Theorem 2.1(a)] and Proposition 3.2,

$$\begin{aligned} &X \subseteq \mathbb{P}^r_K \text{ is a non-normal maximal del Pezzo variety if and only if } X = \pi_p(\widetilde{X}) \text{ where } \tilde{X} \subseteq \mathbb{P}^{r+1}_K \\ &\quad \text{is of minimal degree with } \operatorname{codim}(\widetilde{X}) \geq 2,\ p \text{ is a closed point in } \mathbb{P}^{r+1}_K \backslash \tilde{X} \text{ with } \dim \Sigma_p(\widetilde{X}) = \dim(\widetilde{X}) - 1. \end{aligned} \tag{6.2}$$

Obviously $\widetilde{X}$ is either (a cone over) the Veronese surface in $\mathbb{P}^5_K$ or a rational normal scroll (cf. Notation and Remarks 2.1). We say that the del Pezzo variety X is *exceptional* (resp. *non-exceptional*) if $\tilde{X}$ is (a cone over) the Veronese surface (resp. a rational normal scroll).

In view of (6.2) it suffices to classify the pairs $(\widetilde{X}, p)$ where $X_p = \pi_p(\tilde{X}) \subseteq \mathbb{P}^r_K$ is arithmetically Cohen–Macaulay in order to classify the non-normal maximal del Pezzo varieties in $\mathbb{P}^r_K$, obtaining a different geometric description of the classification of singular del Pezzo varieties obtained by Fujita in [3, Theorem 2.1], see also [4,5]. This we do now for the case of non-exceptional non-normal maximal del Pezzo varieties. For the exceptional case, see Remark 6.3.

Theorem 6.2. *Let $\tilde{X} \subseteq \mathbb{P}^{r+1}_K$ be a rational normal scroll with $\operatorname{codim}(\widetilde{X}) \geq 2$ and with vertex $Z := \operatorname{Vert}(\widetilde{X}) = \mathbb{P}^h_K$ for some $h \geq -1$. Let $\tilde{X}_0 \subset \mathbb{P}^{r-h}_K$ be a smooth rational normal scroll such that $\tilde{X} = \operatorname{Join}(Z, \tilde{X}_0)$. Then for a closed point p in $\mathbb{P}^{r+1}_K \backslash \tilde{X}$, the variety*

$$X_p := \pi_p(\widetilde{X}) \subseteq \mathbb{P}^r_K$$

is arithmetically Cohen–Macaulay, and hence non-normal maximally del Pezzo precisely in the following cases:

(a) $\tilde{X}_0 = S(a)$ *for some integer* $a > 2$ *and* $p \in \operatorname{Join}(Z, \operatorname{Sec}(\tilde{X}_0))\backslash\tilde{X}$.
(b) (i) $\tilde{X}_0 = S(1, 2)$ *and* $p \in \mathbb{P}^{r+1}_K \backslash \tilde{X}$;
(ii) $\tilde{X}_0 = S(1, b)$ *with* $b > 2$ *and* $p \in \operatorname{Join}(Z, \operatorname{Join}(S(1), \tilde{X}_0))\backslash\tilde{X}$;
(iii) $\tilde{X}_0 = S(2, 2)$ *and* $p \in \operatorname{Join}(Z, \Delta)\backslash\tilde{X}$;
(iv) $\tilde{X}_0 = S(2, b)$ *for some integer* $b > 2$ *and* $p \in \operatorname{Join}(Z, \langle S(2)\rangle)\backslash\tilde{X}$.
(c) (i) $\tilde{X}_0 = S(1, 1, 1)$ *and* $p \in \mathbb{P}^{r+1}_K \backslash \tilde{X}$;
(ii) $\tilde{X}_0 = S(1, 1, c)$ *for some integer* $c > 1$ *and* $p \in \operatorname{Join}(Z, \langle S(1, 1)\rangle)\backslash\tilde{X}$.

Proof. Let $n + h + 1$ denote the dimension of $\tilde{X}$. Thus $\tilde{X}_0$ has dimension n. According to (6.2), X_p is arithmetically Cohen–Macaulay if and only if $\Sigma_p(\widetilde{X})$ has dimension equal to $n + h$. Since $\dim\ \Sigma_p(\widetilde{X}) = h + j$ $(0 \leq j \leq 3)$ (cf. Corollary 3.4 and Remark 5.4), it follows that X_p is arithmetically Cohen–Macaulay if and only if $n = j$ with $j \in \{0, 1, 2, 3\}$. Obviously the case $j = 0$ cannot occur. Therefore $n \in \{1, 2, 3\}$.

"(a)": Assume that $n = 1$, so that $\tilde{X}_0 = S(a)$ for some integer $a \geq 3$. Then Remark 5.4 yields that $\operatorname{depth}(X_p) = h + 3$ if and only if

$$p \in (W\backslash V) \cup (V\backslash B \cup U) = W\backslash B \cup U.$$

Since $B = \tilde{X}$, $U = Z$ and $W = \operatorname{Join}(Z, \operatorname{Sec}(\tilde{X}_0))$, it follows that X_p is arithmetically Cohen–Macaulay if and only if $p \in \operatorname{Join}(Z, \operatorname{Sec}(\tilde{X}_0))\backslash\tilde{X}$.

"(b)": Assume that $n = 2$, so that $\tilde{X}_0 = S(a, b)$ for some integers a, b with $1 \leq a \leq b$ and $b \geq 2$. Then Remark 5.4 yields that $\operatorname{depth}(X_p) = h + 4$ if and only if

$$p \in (B\backslash A \cup \tilde{X}) \cup (U\backslash B) = (B \cup U)\backslash(A \cup \tilde{X}). \tag{6.3}$$

If $a \geq 3$, then we have $\Delta = S(\underline{1}) = \emptyset$ and hence $B = \tilde{X}$ and $U = Z \subseteq \tilde{X}$, which leaves no possibility for p. Therefore $a \leq 2$.

Observe that for all $b \geq 2$ we have

$$A = \begin{cases} \operatorname{Join}(Z, \langle S(1)\rangle), & \text{if } a = 1; \\ Z, & \text{if } a = 2. \end{cases} \tag{6.4}$$

As $\langle S(1)\rangle = S(1)$ it follows $A \subseteq \tilde{X}$ for all $b \geq 2$. So, the condition (6.3) imposed on p now simply becomes

$$p \in (B \cup U)\backslash\tilde{X}. \tag{6.5}$$

Also we get for all $b \geq 2$

$$B = \begin{cases} \operatorname{Join}(Z, \operatorname{Join}(S(1), \tilde{X}_0)), & \text{if } a = 1; \\ \tilde{X}, & \text{if } a = 2. \end{cases} \tag{6.6}$$

Observe that Δ is given by

$$\Delta = \begin{cases} \langle S(2)\rangle, & \text{if } a = 1 \text{ and } b = 2; \\ \emptyset, & \text{if } a = 1 \text{ and } b > 2; \\ \mathbb{P}^2_K \times \mathbb{P}^1_K, & \text{if } a = b = 2; \\ \langle S(2)\rangle, & \text{if } a = 2 < b. \end{cases}$$

So, we get the following possibilities for $U = \text{Join}(Z, \text{Join}(\langle S(\underline{1})\rangle, \Delta)$

$$U = \begin{cases} \mathbb{P}^{r+1}_K, & \text{if } a = 1 \text{ and } b = 2; \\ \text{Join}(Z, S(1)) \subseteq \tilde{X}, & \text{if } a = 1 \text{ and } b > 2; \\ \text{Join}(Z, \mathbb{P}^2_K \times \mathbb{P}^1_K), & \text{if } a = b = 2; \\ \text{Join}(Z, \langle S(2)\rangle), & \text{if } a = 2 < b. \end{cases} \tag{6.7}$$

Combining (6.6) and (6.7), we now get for $T := (B \cup U)\backslash\tilde{X}$ the following values

$$T = \begin{cases} \mathbb{P}^{r+1}_K\backslash\tilde{X}, & \text{if } a = 1 \text{ and } b = 2; \\ \text{Join}(Z, \text{Join}(S(1), \tilde{X}_0))\backslash\tilde{X} & \text{if } a = 1 \text{ and } b > 2; \\ \text{Join}(Z, \mathbb{P}^2_K \times \mathbb{P}^1_K)\backslash\tilde{X}, & \text{if } a = b = 2; \\ \text{Join}(Z, \langle S(2)\rangle)\backslash\tilde{X}, & \text{if } a = 2 < b. \end{cases}$$

This proves statement (b).

"(c)": Assume that $n = 3$, so that $\tilde{X}_0 = S(a, b, c)$ for some integers a, b, c with $1 \le a \le b \le c$. According to Remark 5.4 we have $\text{depth}(X_p) = h + 5$ if and only if $p \in A\backslash\tilde{X}$. If $a > 1$, we have $A = Z$, so that no choice for p is left. Therefore $a = 1$. Now, for A we get the following possibilities:

$$A = \begin{cases} \langle Z, \langle\tilde{X}_0\rangle\rangle = \mathbb{P}^{r+1}_K, & \text{if } a = b = c = 1; \\ \text{Join}(Z, \langle S(1, 1)\rangle), & \text{if } a = b = 1 < c; \\ \text{Join}(Z, \langle S(1)\rangle), & \text{if } a = 1 < b. \end{cases}$$

If $a = 1 < b$, we have $A = \text{Join}(Z, S(1)) \subseteq \tilde{X}$, so that no possibility is left for p. Therefore $b = 1$. This proves claim (c). $\square$

Remark 6.3. In order to understand all non-normal del Pezzo varieties it suffices now to know the exceptional cases in which $X = X_p \subseteq \mathbb{P}^r_K$ where $\tilde{X} \subseteq \mathbb{P}^{r+1}_K$ is a cone over the Veronese surface $S \subset \mathbb{P}^5_K$.
(A) Recall that $\text{Sec}(S)$ is a cubic hypersurface. Let p be a closed point in $\mathbb{P}^5_K \setminus S$. It belongs to folklore that the following statements are equivalent:

(a) $p \in \text{Sec}(S) \setminus S$.
(b) $\Sigma_p(S)$ is a smooth plane conic curve.
(c) $\pi_p(S) \subset \mathbb{P}^4_K$ is a complete intersection of two quadrics and so it is arithmetically Cohen–Macaulay.

Therefore the secant stratification of $S \subset \mathbb{P}^5_K$ is

$$\mathbb{P}^5_K = S \,\dot{\cup}\, SL^C(S) \,\dot{\cup}\, SL^{\emptyset}(S) \tag{6.8}$$

where $SL^C(S)$ is equal to $\text{Sec}(S) \setminus S$.
(B) In the same way as in Corollary 3.4 and Theorem 5.3, one can get the secant stratification of $\tilde{X} \subset \mathbb{P}^{r+1}_K$ from (6.8). In particular, $X_p \subset \mathbb{P}^r_K$ is a maximal del Pezzo variety if and only if $p \in \text{Join}(\text{Vert}(\tilde{X}), \text{Sec}(S))\backslash\tilde{X}$.

Remark 6.4. (A) Let $\tilde{X} \subseteq \mathbb{P}^{r+1}_K$ be a variety of minimal degree with $\text{codim}(\tilde{X}) \ge 2$ and let p be a closed point in $\mathbb{P}^{r+1}_K \setminus \tilde{X}$. As a consequence of Theorem 6.2 and Remark 6.3, we finally have a complete list of pairs $(\tilde{X}, p)$ for which X_p is a non-normal del Pezzo variety. This provides a complete picture of non-normal del Pezzo varieties from the point of view of linear projections and normalizations.
(B) Let the notations and hypotheses be as in Theorem 6.2 and Remark 6.3. Since $X_p = \pi_p(\tilde{X})$ is a cone with vertex $\pi_p(\text{Vert}(\tilde{X}))$, the non-normal del Pezzo varieties which are not cones are the varieties described in Theorem 6.2 and Remark 6.3 for which $\text{Vert}(\tilde{X}) \ne \emptyset$. More precisely, the non-normal maximal del Pezzo varieties which are not cones are precisely the following ones:

(i) Projections of a rational normal curve $S(a) \subseteq \mathbb{P}^a_K$ with $a > 2$ from a point $p \in \text{Sec}(S(a))\backslash S(a)$.
(ii) Projections of the Veronese surface $S \subseteq \mathbb{P}^5_K$ from a point $p \in \text{Sec}(S)\backslash S$.
(iii) Projections of a smooth cubic surface scroll $S(1, 2) \subseteq \mathbb{P}^4_K$ from a point $p \in \mathbb{P}^4_K\backslash S(1, 2)$.
(iv) Projections of a smooth rational normal scroll $S(1, b) \subseteq \mathbb{P}^{b+2}_K$ with $b > 2$ from a point $p \in \text{Join}(S(1), S(1, b))\backslash S(1, b)$.
(v) Projections of a smooth quartic surface scroll $S(2, 2) \subseteq \mathbb{P}^5_K$ from a point $p \in \mathbb{P}^2_K \times \mathbb{P}^1_K\backslash S(2, 2)$.
(vi) Projections of a smooth surface scroll $S(2, b) \subseteq \mathbb{P}^{b+3}_K$ with $b > 2$ from a point $p \in \langle S(2)\rangle\backslash S(2, b)$.

(vii) Projections of a smooth 3-fold scroll $S(1, 1, 1) \subseteq \mathbb{P}^5_K$ from a point $p \in \mathbb{P}^5_K \backslash S(1, 1, 1)$.
(viii) Projections of a smooth 3-fold scroll $S(1, 1, c) \subseteq \mathbb{P}^{c+4}_K$ with $c > 1$ from a point $p \in \langle S(1, 1)\rangle \backslash S(1, 1, c)$.

Therefore if $X \subset \mathbb{P}^r_K$ is a non-normal del Pezzo and is not a cone, then the dimension of X is ≤ 3 while there is no upper bound of the degree of X. This fact was first shown by Fujita (cf. (2.9) in [4] and (9.10) in [5]).

Remark 6.5. Using the same method as above, one can indeed classify all varieties $X \subset \mathbb{P}^r_K$ of almost minimal degree and codimension ≥ 2, via their arithmetic depth, as projections from rational normal scrolls of given numerical type and the position of the center of the projection. We shall give a detailed exposition of this in a later paper.

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