The Most-Energetic Traveltime of Seismic Waves

SEONGJAI KIM
Department of Mathematics, University of Kentucky
Lexington, Kentucky 40506-0027 U.S.A.
skim@ms.uky.edu
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Abstract—In tomographic image processing of seismic data, the first-arrival traveltime (FATT) is often different from those of more energetic wavefronts in realistic media. Since the traveltime of most-energetic wavefront (METT) dominates the data, computing the METT is recognized as an essential element in modern seismic imaging techniques. Solving the full wave equation is extremely expensive to be impractical even for large-size computers to carry out; the solution of the eikonal equation for which the corresponding amplitude is continuous is conjectured to be the METT. © 2001 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Seismic techniques based on high frequency asymptotic representation of the acoustic/elastic Green's function require efficient and accurate methods for the computation of traveltimes [1-3]. The ray tracing is a popular and robust method for computing diffraction trajectories for most of small or moderate velocity contrasts. An alternative to the method is to compute traveltimes by solving directly the eikonal equation on a regular grid by finite difference (FD) schemes; see [4-13].

However, most of the above methods have been developed to compute the first-arrival traveltime (FATT), while more energetic traveltimes are often different from the FATT and dominate the data acquired from realistic media. In cases, seismic imaging techniques utilizing the FATT perform poorly. No doubt that more energetic traveltimes can better imprint physical characteristics [14,15]. Here the difficulty is that the energetic traveltimes are discontinuous and multivalued. Furthermore, a computation method for energetic traveltimes should be able to compute the energy amplitudes of each of wavefronts, because otherwise there is no way to choose more energetic wavefronts among many of those coming to a place.

In many areas of seismic applications such as oil exploration (velocity inversion and migration) and earthquake analysis (imaging earthquake fault rupture and simulating seismic ground
motion), it has been a long time for researchers and engineers to consider the most-energetic traveltime (METT) as an essential element for geophysical image processing with data from highly heterogeneous media. The METT is single-valued, but is discontinuous in general and requires the computation of the amplitudes of wavefronts.

Currently in the oil industry, a popular technique is the method of the minimum raypath traveltime (MRTT). The MRTT technique utilizes a simple postulate that the wave loses more energy while traveling a longer distance. It chooses the traveltime whose corresponding raypath has the shortest length among all the rays coming to the target position. Note that the MRTT is not the METT, but has been considered as an approximation. It is not difficult to find a velocity model in which the MRTT approximates the METT poorly. Furthermore, the MRTT techniques often fail to compute the amplitudes, which is the main reason that oil exploration geophysicists have chosen the minimum raypath as their weapon rather than the highest energy. Then, the question is: how can we compute the METT and the corresponding energy amplitude with an acceptable accuracy and computation cost? It is not easy to answer the question. However, in this letter, we will try some positive aspects to help answer the question.

2. PRELIMINARIES

We first review principles in wave theory that are utilized in this letter: Fermat’s Principle, Snell’s Law, and Huygens’s Principle. Concepts of refraction and diffraction are also reviewed.

FERMAT’S PRINCIPLE. The seismic raypath between two points is that for which traveltime is stationary, usually minimum, compared with those for neighboring paths. (Named for Pierre Fermat (1601–1665), French mathematician.) If the intervening media have different velocities, the raypath is not straight, but will be such that the overall traveltime is minimized. It can be observed through the refraction, the change in direction of a ray upon passing into a medium with different velocity.

SNELL’S LAW. Snell’s Law follows from Fermat’s Principle. When a wave crosses a boundary of two different media, the wave changes the direction such that

\[
\frac{\sin \theta_i}{v_i} = \frac{\sin \theta_r}{v_r},
\]

where \( \theta_i \) is the angle of the incident wave with the velocity \( v_i \) (medium 1) and \( \theta_r \) is the angle of refraction in medium 2 which has the velocity \( v_r \). It is named for Willebrord Snell (1591–1626), Dutch mathematician. Snell’s Law is also called Descartes’s Law, named for René Descartes (1596–1650), French philosopher and scientist.

HUYGENS’S PRINCIPLE. The concept that each point on the advancing wavefront can be regarded as the source of a secondary wave and that a later wavefront is the envelope tangent to all the secondary waves. Such a wave phenomenon is easily observed as diffraction, the bending of wave energy around obstacles without obeying Snell’s Law. Named for Christian Huygens (1629–1695), Dutch mathematician. The amplitude of a diffracted wavefront decreases exponentially as the diffraction angle increases.

3. THE HIGH FREQUENCY ASYMPTOTICS

In this section, we consider the differential equations for traveltimes and amplitudes. The well-known eikonal equation and the transport equation read

(a) \[ \nabla \tau \cdot \nabla \tau = \frac{1}{v^2(x)}, \]
(b) \[ 2\nabla \tau \cdot \nabla a + a \nabla^2 \tau = 0, \]

where \( \tau = \tau(x_s, x) \) is the traveltime function from the source \( x_s \) to the location \( x \), \( a = a(x) \) is the amplitude field, and \( v(x) \) denotes the velocity of propagating wavefront at \( x \).
The above equations can be obtained from the acoustic model in the frequency domain

$$\mathcal{L}_{\omega} p \equiv -\left(\frac{\omega^2}{v^2} + \nabla^2\right) p = \delta(x - x_s),$$  \hspace{1cm} (3)

with the solution substituted by the test function of the form

$$p(x_s, x, \omega) = \exp\{-i\omega\tau(x_s, x)\} \sum_{j=0}^{\infty} \frac{a_j(x_s, x)}{(i\omega)^j}. \hspace{1cm} (4)$$

Let us consider a little bit more details. When (4) is substituted into (3), the equation reads

$$\mathcal{L}_{\omega} p(x_s, x, \omega) = \exp\{-i\omega\tau(x_s, x)\} \sum_{j=0}^{\infty} b_j(x_s, x) \omega^{2-j} = \delta(x - x_s), \hspace{1cm} (5)$$

where

$$b_0 = \nabla \tau \cdot \nabla \tau - \frac{1}{v^2(x)},$$
$$b_1 = i \left(2\nabla \tau \cdot \nabla a_0 + a_0 \nabla^2 \tau - b_0 a_1\right),$$
$$\ldots$$

To satisfy (5), the coefficients $b_j$, $j = 0, 1, \ldots$, should be all zero except $x = x_s$. Equations (2a) and (2b) come from $b_0 = 0$ and $b_1 = 0$, respectively. Equivalently, the equations in (2) can be obtained by approximating the solution of (3) as

$$p(x_s, x, \omega) \approx \exp\{-i\omega\tau(x_s, x)\} a(x_s, x). \hspace{1cm} (6)$$

The approximation technique is known as the high frequency asymptotics or the WKBJ approximation, which is popular in the computation of traveltimes and amplitudes [16, pp. 416-418]. Here “WKBJ” stands for Wentzel, Kramers, Brillouin, and Jeffreys who studied approximate solutions of the equation

$$\frac{d^2 \phi}{dx^2} + \frac{\omega^2}{v^2} \phi = 0.$$  

Let $u = -\log a$. Then, for, e.g., $(z+)$-directional (down-going) wavefronts, we can rewrite (2) as the following Hamilton-Jacobi differential equations:

(a) \hspace{1cm} \tau_z = H(\tau_x, \tau_y, x),$$
(b) \hspace{1cm} u_z = \Psi(u_x, u_y, x), \hspace{1cm} \hspace{1cm} (7)$$

where

$$H(\tau_x, \tau_y, x) = \sqrt{s^2 - \tau_x^2 - \tau_y^2}, \hspace{1cm} s = \frac{1}{v},$$
$$\Psi(u_x, u_y, x) = \tau_z^{-1} \left(\frac{1}{2} \nabla^2 \tau - \tau_x u_x - \tau_y u_y\right).$$

Here $s$, the reciprocal of the velocity, is called the slowness.

REMARK. For wavefronts marching in other directions, one can analogously reformulate (2) as done in (7) for the wavefronts in the $(z+)$-direction. Numerical computation is often carried out with the reformulations. For a stable, second-order finite difference scheme for the FATT, see [7].

REMARK. The traveltime $\tau$ often develops discontinuities in heterogeneous media. An upwind difference formula is requisite to sharply resolve the discontinuities. So the computed solution reveals the first-order accuracy near shocks. Note that the transport equation incorporates the traveltime Laplacian; it is difficult to expect an accurate simulation for amplitudes.
4. FATT VERSUS METT

In this section, we consider differences between wavefronts of the FATT and the METT, and some important aspects in their numerical solutions. See Figure 1, where the contours of the FATT are superposed on a vertical slice of a real velocity model provided from Shell E&P Technology Co., Houston, Texas. The headwave is defined as a wave which enters and leaves a high velocity medium at a critical angle. Here the critical angle is defined as the angle of incidence for which the refracted ray grazes the surface of contact between two media. (See (8) below for the critical angle in a two-layer velocity model.) Headwaves are clearly observed in two regions, near (x, z) = (13, 1.5) and (x, z) = (8, 5), due to the presence of the high velocities.

![Figure 1. A real velocity model and the FATT. A real velocity model obtained from a survey of the Gulf of Mexico is provided from Shell E&P Technology Co., Houston, Texas. The model utilized in the computation consists of 150 x 150 x 136 cells (= 3,123,737 grid points) of the edge length 45.7m each. The minimum velocity is 1,502.7m/s in the top portion of the model, sea water, and the maximum is 4,419.6m/s in the salt domes. (The high velocities greater than 4,000m/s correspond to salt domes.) A point source is located at the top center of the model. Traveltime contours are superposed on a vertical slice (in x-direction) of the model (y = 12116m).](image_url)

The METT may differ from the FATT in the regions where headwaves appear, in particular, near high velocity contrasts. If a source is located at (x, z) = (6, 0) in the slice of the model shown in Figure 1, headwaves will appear both above and below the salt dome in the right side. Headwaves can be considered as the main source of poor performances of the seismic imaging techniques utilizing the FATT. Getting rid of headwaves is an important issue in computational seismology. Some seismologists have tried to remove them by keeping only the downward wavefronts [17], which clearly does not work for downward headwaves.

Consider the two-layer model depicted in Figure 2. There synthetic contours of the FATT (solid curves $\gamma_i$, $i = 1, 2, 3$) and the METT (the dash curve) are superposed. Compared with $\gamma_2$, the dash curve is quite different in the region where headwaves appear.

Let $\theta_c$ be the critical angle. Then, it can be found by setting $\sin \theta_c = 1$ in (1)

$$\theta_c = \sin^{-1} \frac{v_1}{v_2},$$

which is 30° for the case in Figure 2. The ray departing from the source in the critical angle reaches at $(x, z) = (2/\sqrt{3}, 2)$ on the interface in $4/\sqrt{3}$ seconds. Then it travels through the interface with the speed 2Km/s, diffracting headwaves upward.
Figure 2. A two-layer velocity model and the traveltime contours. The velocities are chosen as $v_1 = 1$ Km/s and $v_2 = 2$ Km/s for the top layer of 2 Km thick and the bottom layer, respectively. The point $S$ denotes the source position and the solid curves $\gamma_i$, $i = 1, 2, 3$, are contours for the FATT. The dash curve denotes a contour for the METT.

In the numerical simulation of advancing interfaces such as the FATT, the METT, and $u$ of (7b), one of fundamental requirements is to find the correct upwind direction. The upwind direction for the FATT can be easily selected using the causality principle: wavefronts of the FATT travel from regions of smaller traveltimes to regions of larger values. So the upwind direction of the FATT is up-going near the velocity interface in Figure 2, when $r > 4/\sqrt{3}$. For the METT, the criterion for determining the upwind direction should be the amplitude or equivalently its negative logarithm, $u = -\log a$. It can be shown from (7b) that the function $u$ increases as the wavefront changes its propagation direction. Therefore in Figure 2, the function $u$ corresponding to the METT should have larger values below the velocity interface. Now, it is clear to say that the upwind direction of the METT near the velocity interface is downward. So the contour of the METT has been obtained as such in Figure 2.

5. THE MOST-ENERGETIC TRAVELTIME

The most-energetic traveltime (METT) is, by definition, the traveltime of the wavefront whose energy level (amplitude) is the highest among all those coming to the point under consideration. The METT can be computed numerically, with a desirable accuracy, by solving the acoustic model

$$\frac{1}{v^2(x)} \frac{\partial^2 u}{\partial t^2} - \nabla^2 u = \delta(x - x_s)\delta(t), \quad t > 0,$$

$$u(x, t) \equiv 0, \quad t \leq 0,$$

and monitoring the amplitude of the solution at each grid point at every time level. Since solving (9) is quite expansive, we may try to to find an approximation of the METT by solving (2a) with some other physical conditions.

Consider two wavefronts of different amplitudes coming to a point. Then, due to the Huygens's Principle, the traveltime function of the stronger wavefront will appear to push or overlap that of the weaker one up to the point where the amplitudes become the same. The physical observation can be extended for arbitrarily many wavefronts coming to a point. We can summarize the physical observation as follows.

**Theorem 5.1.** The amplitude corresponding to the METT is continuous over the whole domain.

The theorem seems to be new in such an explicit form; as a matter of fact, the METT is not yet studied systematically.

If the METT is computed by solving (9a), Theorem 5.1 is hardly beneficial in numerical computation. The theorem just implies that the amplitude of the METT forms a continuous surface at each time level. Here we want to extend the theorem to the WKBJ approximation of the solution of (9). We formulate it as follows.
CONJECTURE 5.2. Let the traveltime $\tau$ in (2a) be computed in a way that the amplitude $a$ of (2b) is continuous over the whole domain. Then $\tau$ is the METT.

We call the traveltime $\tau$ considered in the conjecture the \textit{continuous amplitude traveltime} (CATT). So the conjecture says, \textit{"The CATT is the METT"}. The conjecture may be verified numerically by comparing the METT of (9) with the CATT. Computing the CATT requires lots of fancy numerical strategies and careful implementations. Note that the traveltime $\tau$ should be computed accurately enough to deliver a positive order of accuracy to the numerical approximation for the traveltime Laplacian $\nabla^2 \tau$, and therefore, for the amplitude, while the amplitude must be accurate in order for the CATT to capture the physical characteristics. The difficulty is that any high-order numerical schemes turn out to become first-order accurate near the discontinuities of the CATT; naive applications of numerical techniques fail to simulate the CATT. Effective finite difference schemes for the CATT and the corresponding continuous amplitude will appear elsewhere [18].

6. CONCLUSIONS AND DISCUSSION

The most-energetic traveltime (METT) can be different from the first-arrival traveltime (FATT) and dominate the data obtained from realistic media. The numerical computation of the METT becomes more important than ever for modern applications of seismic waves. Solving the wave equation is extremely expensive to be impractical even for large-size computers to carry out; the continuous-amplitude traveltime (CATT) of the high frequency asymptotics has been conjectured to be the METT.

The conjecture seems very hard to solve mathematically, while its numerical verification requires developing sophisticated numerical techniques for both the eikonal equation and the transport equation to advance discontinuous wavefronts accurately. Once the conjecture is confirmed to be true, at least, numerically, one can easily expect lots of applications to tomography in oil exploration, e.g., velocity inversion and true amplitude migration; earthquake analysis, in particular, simulation of seismic ground motion; and shallow seismic reflection problems such as waste-disposal site characterization, archaeological search, and ground security analysis.

Up to this point, we have considered isotropic seismic waves, i.e., acoustics. The simplest anisotropy of broad geophysical applicability is \textit{transverse isotropy} (TI) for which materials have the same property value when measured in a plane, but a different value when measured perpendicular to the plane. The symmetry axis of the TI medium is by definition the line normal to the symmetry plane. Most earth media are TI of the vertical symmetry axis (VTI), due to the layered deposition of rocks and gravity effects. Elastic waves have three different modes: quasi-P, quasi-SV, and quasi-SH waves. The conjecture can be applied to each mode of elastic waves in TI media, when the anisotropic eikonal and transport equations are considered correspondingly.

REFERENCES


