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An integral representation for the Bessel form

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Abstract

We deal with integral representation problems of the Bessel form. Suitable formulations are obtained, but they are not proved for all values of the parameter. Generalizations to modified classical forms are possible.

0. Introduction

The history of the Bessel form is more tortured than that of other classical forms. The reason is certainly the fact that the Bessel form is not positive definite for any value of the parameter. The problem was to find a weight function, defined on the real axis, with respect to which the Bessel polynomials would be orthogonal. Several authors have given an integral representation through a distribution [7, 3] or an ultradistribution [5].

Here, we propose an integral representation through a true function as a consequence of the semi-classical character of the Bessel form. For another representation through a function, see [1].

1. An integral representation for semi-classical forms: the Bessel case

Let $\mathscr{P}$ be the vectorial space of complex polynomials and let $\mathscr{P}'$ its dual. Let us recall the definition of a semi-classical form. The form $u \in \mathscr{P}'$ will be called semi-classical if it satisfies (cf. [9])

(1) u is regular and

(2) there exist two polynomials Φ and ψ such that

$$D(\Phi u) + \psi u = 0 \quad (\Phi \text{ monic}, \quad \deg \psi \geqslant 1). \tag{1.1}$$

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Let u be a semi-classical form satisfying Eq. (1.1). We are looking for an integral representation of u, considering

$$\langle u,f\rangle = \int_{-\infty}^{+\infty} U(x)f(x)\,\mathrm{d}x, \quad f\in\mathscr{P}, \tag{1.2}$$

where we suppose the function U to be absolutely continuous on $\mathbb{R}$, with rapid decay and derivative U'. From (1.1), we obtain

$$\int_{-\infty}^{+\infty} ((\Phi U)' + \psi U)f(x)\,\mathrm{d}x - \Phi(x)U(x)f(x)]_{-\infty}^{+\infty} = 0, \quad f\in\mathscr{P}.$$

Hence, from the assumptions on U, the following conditions hold

$$\Phi(x)U(x)f(x)]_{-\infty}^{+\infty} = 0, \quad f\in\mathscr{P}, \tag{1.3}$$

$$\int_{-\infty}^{+\infty} ((\Phi U)' + \psi U)f(x)\,\mathrm{d}x = 0, \quad f\in\mathscr{P}. \tag{1.4}$$

Condition (1.4) implies

$$(\Phi U)' + \psi U = \lambda g, \tag{1.5}$$

where $\lambda \neq 0$ is arbitrary and g is a locally integrable function with rapid decay representing the null-form

$$\int_{-\infty}^{+\infty} x^n g(x)\,\mathrm{d}x = 0, \quad n \geqslant 0. \tag{1.6}$$

Reciprocally, if U is a solution of (1.5) verifying the hypotheses above, then (1.3) and (1.4) are fulfilled and (1.2) defines a form u which is a solution of (1.1). But, is this solution correct for our problem?

Now, we must show that the form so-constructed is not identical to the null-form which is always a solution of Eq. (1.1). Precisely, the problem is to prove

$$\int_{-\infty}^{+\infty} U(x)\,\mathrm{d}x \neq 0, \tag{1.7}$$

which is also a necessary (but not sufficient) condition for the regularity of u.

1.1. Examples of some functions g representing the null-form

The fundamental example is given by the Stieltjes function [13]:

$$s(x) = \begin{cases} 0, & x \leqslant 0, \\ \exp(-x^{1/4})\sin x^{1/4}, & x > 0. \end{cases} \tag{1.8}$$

More generally, consider the integrals [2]

$$\int_0^{+\infty} x^{p-1}\,\mathrm{e}^{-ax}\sin(mx)\,\mathrm{d}x = \frac{\Gamma(p)\sin(p\theta)}{(a^2+m^2)^{p/2}}, \quad p, a, m > 0, \tag{1.9}$$

$$\int_0^{+\infty} x^{p-1}\,\mathrm{e}^{-ax}\cos(mx)\,\mathrm{d}x = \frac{\Gamma(p)\cos(p\theta)}{(a^2+m^2)^{p/2}}, \quad p, a, m > 0, \tag{1.10}$$

with $\sin\theta = m/r$, $\cos\theta = a/r$, $0 < \theta < \pi/2$, $r = (a^2+m^2)^{1/2}$.

Put $a = \rho m$, then:

$$\sin\theta = \frac{1}{(1+\rho^2)^{1/2}}, \quad \cos\theta = \frac{\rho}{(1+\rho^2)^{1/2}}, \qquad \rho > 0.$$

For instance, let us take $\rho = (7-4\sqrt{3})^{1/2}$, then $\theta = 5\pi/12$. With $p = 12(n+1)$, $n \geqslant 0$, we have from (1.9)

$$\int_0^{+\infty} x^{12n+11}\,\mathrm{e}^{-ax}\sin\left(\frac{a}{\rho}x\right)\mathrm{d}x = 0, \quad n \geqslant 0. \tag{1.11}$$

Differentiating twice with respect to a

$$\int_0^{+\infty} x^{12(n+1)}\,\mathrm{e}^{-ax}\left\{\frac{1}{\rho}\cos\left(\frac{a}{\rho}x\right) - \sin\left(\frac{a}{\rho}x\right)\right\}\mathrm{d}x = 0, \quad n \geqslant 0, \tag{1.12}$$

$$\int_0^{+\infty} x^{12(n+1)+1}\,\mathrm{e}^{-ax}\left\{\left(1-\frac{1}{\rho^2}\right)\sin\left(\frac{a}{\rho}x\right) - \frac{2}{\rho}\cos\left(\frac{a}{\rho}x\right)\right\}\mathrm{d}x = 0, \quad n \geqslant 0. \tag{1.13}$$

So (1.11), (1.12) and (1.13) give respectively, with $a = 1$:

$$g(x) = \begin{cases} 0, & x \leqslant 0, \\ \exp(-x^{1/12})\sin\left(\frac{1}{\rho}x^{1/12}\right), & x > 0, \end{cases} \tag{1.14}$$

$$g(x) = \begin{cases} 0, & x \leqslant 0, \\ x^{1/12}\exp(-x^{1/12})\left\{\frac{1}{\rho}\cos\left(\frac{1}{\rho}x^{1/12}\right) - \sin\left(\frac{1}{\rho}x^{1/12}\right)\right\}, & x > 0, \end{cases} \tag{1.15}$$

$$g(x) = \begin{cases} 0, & x \leqslant 0, \\ x^{1/6}\exp(-x^{1/12})\left\{\left(1-\frac{1}{\rho^2}\right)\sin\left(\frac{1}{\rho}x^{1/12}\right) - \frac{2}{\rho}\cos\left(\frac{1}{\rho}x^{1/12}\right)\right\}, & x > 0. \end{cases} \tag{1.16}$$

When $\rho = 1$, then $\theta = \pi/4$, we have from (1.10) and (1.9):

$$g(x) = \begin{cases} 0, & x \leqslant 0, \\ x^{1/2} \exp(-x^{1/4}) \cos(x^{1/4}), & x > 0, \end{cases} \tag{1.17}$$

$$g(x) = \begin{cases} 0, & x \leqslant 0, \\ x^{-1/4} \exp(-x^{1/4}) \{\cos(x^{1/4}) + \sin(x^{1/4})\}, & x > 0, \end{cases} \tag{1.18}$$

$$g(x) = \begin{cases} 0, & x \leqslant 0, \\ x^{1/4} \exp(-x^{1/4}) \{\cos(x^{1/4}) - \sin(x^{1/4})\}, & x > 0. \end{cases} \tag{1.19}$$

Remark. If g is a representation of the null form, then so is $x \to x^m g(x)$ for each $m \in \mathbb{N}$ and also the convolution product $h * g$ where h is a locally integrable function with rapid decay.

When $\operatorname{supp} g \subset \mathbb{R}^+$, as for the Stieltjes function, the following functions

$$x \to |x| g(x^2) \quad \text{and} \quad x \to \tfrac{1}{2} g(|x|)$$

are even representations of the null form and

$$x \to x|x| g(x^2) \quad \text{and} \quad x \to \operatorname{sgn} x g(x^2)$$

are odd representations of the null form.

1.2. The Bessel case [8, 4, 9, 10]

In this case, we have $u = \mathscr{B}(\alpha)$, $\alpha \neq -n/2$, $n \geqslant 0$ with

$$\Phi(x) = x^2; \qquad \psi(x) = -2(\alpha x + 1),$$

$$\beta_n = \frac{1 - \alpha}{(n + \alpha - 1)(n + \alpha)}, \quad \gamma_{n+1} = -\frac{(n + 1)(n + 2\alpha - 1)}{(2n + 2\alpha - 1)(n + \alpha)^2 (2n + 2\alpha + 1)}, \qquad n \geqslant 0,$$

if $\{P_n\}_{n \geqslant 0}$ denotes the sequence of monic Bessel polynomials, verifying

$$P_0(x) = 1, \qquad P_1(x) = x - \beta_0,$$

$$P_{n+2}(x) = (x - \beta_{n+1}) P_{n+1}(x) - \gamma_{n+1} P_n(x), \quad n \geqslant 0.$$

Eq. (1.5) becomes

$$(x^2 U)' - 2(\alpha x + 1) U = \lambda g(x)$$

or equivalently

$$U(x) x^{2(1-\alpha)} \exp\left(\frac{2}{x}\right) - U(c) c^{2(1-\alpha)} \exp\left(\frac{2}{c}\right) = \lambda \int_c^x g(\xi) \xi^{-2\alpha} \exp\left(\frac{2}{\xi}\right) d\xi, \quad c > 0, \quad x \in \mathbb{R}.$$

A possible solution is

$$U(x)=\begin{cases}0, & x\leqslant 0,\\ \lambda x^{2(\alpha-1)}\exp\left(-\dfrac{2}{x}\right)\displaystyle\int_x^{+\infty}\xi^{-2\alpha}\exp\left(\dfrac{2}{\xi}\right)s(\xi)\,d\xi, & x>0,\end{cases} \tag{1.20}$$

where the function s is given by (1.8).

First, condition (1.3) is fulfilled, for we have:

$$|x^2U(x)|\leqslant|\lambda|x^{2\Re\alpha}\exp\left(-\frac{2}{x}\right)\int_x^{+\infty}\xi^{-2\Re\alpha}\exp\left(\frac{2}{\xi}\right)\exp(-\xi^{1/4})\,d\xi,\quad x>0,$$

$$|x^2U(x)|\leqslant|\lambda|x^{2\Re\alpha}\exp\left(-\frac{2}{x}\right)\int_x^{1}\xi^{-2\Re\alpha}\exp\left(\frac{2}{\xi}\right)d\xi+o(1),\quad x\to+0.$$

We apply l'Hospital's rule to the ratio

$$\lim_{x\to+0}\frac{\int_x^1\xi^{-2\Re\alpha}\exp(2/\xi)\,d\xi}{x^{-2\Re\alpha}\exp(2/x)}=\lim_{x\to+0}\frac{x^{-2\Re\alpha}\exp(2/x)}{\exp(2/x)\{2\Re\alpha x^{-2\Re\alpha-1}+2x^{-\Re\alpha-2}\}}=\lim_{x\to+0}\frac{x^2}{2\Re\alpha x+2}=0$$

so $x^2U(x)\to 0$ when $x\to+0$.

Further, when $x\to+\infty$,

$$|U(x)|\leqslant|\lambda|x^{2(\Re\alpha-1)}\int_z^{+\infty}\xi^{-2\Re\alpha}\exp(-\xi^{1/4})\,d\xi=o(\exp(-\tfrac{1}{2}x^{1/4}).$$

Finally, let us show $U\in L_1$. For $0<x\leqslant 1$,

$$U(x)=\theta(x)+O\left(x^{2(\Re\alpha-1)}\exp\left(-\frac{2}{x}\right)\right)$$

with

$$\theta(x)=\lambda x^{2(\alpha-1)}\exp\left(-\frac{2}{x}\right)\int_x^1\xi^{-2\alpha}\exp\left(\frac{2}{\xi}\right)s(\xi)\,d\xi.$$

Hence,

$$\int_0^1|\theta(x)\,dx\leqslant|\lambda|\int_0^1\xi^{-2\Re\alpha}\exp\left(\frac{2}{\xi}\right)|s(\xi)|\left(\int_0^\xi x^{2\Re\alpha-2}\exp\left(-\frac{2}{x}\right)dx\right)d\xi.$$

But,

$$\begin{aligned}\int_0^\xi x^{2\Re\alpha-2}\exp\left(-\frac{2}{x}\right)dx&=\frac{1}{2}\xi^{2\Re\alpha}\exp\left(-\frac{2}{\xi}\right)-\Re\alpha\int_0^\xi x^{2\Re\alpha-1}\exp\left(-\frac{2}{x}\right)dx\\&\leqslant\frac{1}{2}\xi^{2\Re\alpha}\exp\left(-\frac{2}{\xi}\right)+|\Re\alpha|\,\xi\int_0^\xi x^{2\Re\alpha-2}\exp\left(-\frac{2}{x}\right)dx\end{aligned}$$

and hence

$$\int_0^{\xi} x^{2\Re\alpha - 2} \exp\left(-\frac{2}{x}\right) \mathrm{d}x \leqslant \frac{1}{2} \frac{\xi^{2\Re\alpha} \exp(-2/\xi)}{1 - |\Re\alpha|\xi}, \quad 0 \leqslant \xi < \frac{1}{|\Re\alpha|}.$$

It results in

$$\int_0^1 |\theta(x)\,\mathrm{d}x < +\infty.$$

The function (1.20) is a possible representation whatever $\alpha \in \mathbb{C}$. Condition (1.7) now becomes

$$\int_0^{+\infty} U(x)\,\mathrm{d}x = \lambda \int_0^{+\infty} \xi^{-2\alpha} \exp\left(\frac{2}{\xi}\right) s(\xi) \left(\int_0^{\xi} x^{2\alpha - 2} \exp\left(-\frac{2}{x}\right) \mathrm{d}x \right) \mathrm{d}\xi = \lambda S_\alpha \neq 0, \quad (1.21)$$

with

$$S_\alpha = 4 \int_0^{+\infty} t^{3 - 8\alpha} \exp\left(\frac{2}{t^4}\right) h_{\alpha - 1}(t^4)\,\mathrm{e}^{-t} \sin t\,\mathrm{d}t, \quad (1.22)$$

$$h_\alpha(t) = \int_0^t x^{2\alpha} \exp\left(-\frac{2}{x}\right) \mathrm{d}x. \quad (1.23)$$

2. Some results about S_α

Lemma 2.1. *We have for* $\alpha > 0$

$$\frac{1}{2} \frac{t^{2\alpha + 2}}{1 + (\alpha + 1)t} \exp\left(-\frac{2}{t}\right) \leqslant h_\alpha(t) \leqslant \frac{1}{2} \frac{t^{2\alpha + 2}}{1 + \frac{1}{2}(\alpha + 1)t} \exp\left(-\frac{2}{t}\right), \quad t \geqslant 0, \quad (2.1)$$

$$\frac{1}{2} \frac{t^{2\alpha}}{1 + \alpha t} \exp\left(-\frac{2}{t}\right) \leqslant h_{\alpha - 1}(t) \leqslant \frac{1}{2} t^{2\alpha} \frac{1 + t - \frac{1}{2}\alpha t^2 + \mathrm{O}(t^3)}{1 + (\alpha + 1)t} \exp\left(-\frac{2}{t}\right), \quad t \geqslant 0. \quad (2.2)$$

From (1.23), we have upon integration by parts

$$h_\alpha(t) = \frac{1}{2} t^{2\alpha + 2} \exp\left(-\frac{2}{t}\right) - (\alpha + 1) \int_0^t x^{2\alpha + 1} \exp\left(-\frac{2}{x}\right) \mathrm{d}x. \quad (2.3)$$

The monotonicity gives

$$\frac{1}{2} t h_\alpha(t) \leqslant \int_0^t x^{2\alpha + 1} \exp\left(-\frac{2}{x}\right) \mathrm{d}x \leqslant t h_\alpha(t).$$

Hence (2.1) with (2.3). For (2.2), the first inequality of (2.1) is valid when $\alpha \to \alpha - 1$ and from (2.3) where $\alpha \to \alpha - 1$, after a new integration by parts

$$h_{\alpha - 1}(t) = \frac{1}{2} t^{2\alpha}(1 - \alpha t) \exp\left(-\frac{2}{t}\right) + \frac{1}{2} \alpha(2\alpha + 1) h_\alpha(t) \quad (2.4)$$

or with (2.3)

$$h_{\alpha-1}(t) = \frac{1}{2}t^{2\alpha}\left(1 - \alpha t + \frac{1}{2}\alpha(2\alpha+1)t^2\right)\exp\left(-\frac{2}{t}\right)$$
$$- \frac{1}{2}\alpha(\alpha+1)(2\alpha+1)\int_0^t x^{2\alpha+1}\exp\left(-\frac{2}{x}\right)dx.$$

Hence the second inequality from (2.2) follows from the first inequality from the line below (2.3) and from the first one from (2.1).

Corollary 2.2. *Putting* $F_\alpha(t) = f_\alpha(t)e^{-t}$ *with* $f_\alpha(t) = f_\alpha(t) = t^{3-8\alpha}\exp(2/t^4)h_{\alpha-1}(t^4)$. *Then, for each* $\alpha > 0$ *the function* F_α *is not decreasing for* $t \geqslant 0$.

Indeed, we have from (2.2)

$$\frac{1}{2}\frac{t^3}{1+\alpha t^4} \leqslant f_\alpha(t) \leqslant \frac{1}{2}t^3\frac{1+t^4-\frac{1}{2}\alpha t^8 + O(t^{12})}{1+(\alpha+1)t^4}, \quad t \geqslant 0.$$

Therefore $f_\alpha(t) > 0$ for $t > 0$ and $f_\alpha(0) = 0$.

Remark. This shows that it is not possible to employ the usual monotonicity property for the integral (2.6) below.

Proposition 2.3. *We have the following expression*

$$J_\alpha = \frac{1}{2^{2m}}\prod_{\mu=0}^{2m+1}(2\alpha+\mu)\int_0^{+\infty} t^{3-8\alpha}\exp\left(\frac{2}{t^4}\right)h_{\alpha+m}(t^4)e^{-t}\sin t\,dt, \quad m \geqslant 0; \quad \alpha \in \mathbb{C}. \tag{2.5}$$

From (2.4), and using the Stieltjes representation (1.8) of the null-form, we obtain

$$S_\alpha = 2\alpha(2\alpha+1)\int_0^{+\infty} t^{3-8\alpha}\exp\left(\frac{2}{t^4}\right)h_\alpha(t^4)e^{-t}\sin t\,dt.$$

Suppose (2.5) for $m \geqslant 0$ fixed. From (2.4) where $\alpha \to \alpha + m + 1$

$$h_{\alpha+m}(t) = \frac{1}{2}t^{2(\alpha+m+1)}(1-(\alpha+m+1)t)\exp\left(-\frac{2}{t}\right)$$
$$+ \frac{1}{2^2}(2(\alpha+m)+2)(2(\alpha+m)+3)h_{\alpha+m+1}(t),$$

hence easily (2.5) for $m \to m + 1$.

Corollary 2.4. *We have* $S_{-n/2} = 0$, $n \geqslant 0$.

This result is consistent with the fact that the Bessel form is not regular for these values of α.

Conjecture 2.5. The unique zeros of S_α are $\alpha_n = -n/2$, $n \geqslant 0$.

A partial answer is the following.

Proposition 2.6. *For* $\alpha \geqslant 6(2/\pi)^4$, *we have* $S_\alpha > 0$.

We need Lemma 2.7.

Lemma 2.7. *Consider the following integral*

$$S = \int_0^{+\infty} F(t) \sin t \, dt \tag{2.6}$$

where we suppose $F(t) \geqslant 0$, *continuous, increasing in* $0 < t \leqslant \bar{t}$ *and decreasing to zero for* $t > \bar{t}$. *Then, if*

$$0 < \bar{t} \leqslant \pi, \quad \int_0^{\pi} \sin t (F(t) - F(\pi + t)) \, dt \geqslant 0, \tag{2.7}$$

we have $S > 0$.

Or, if

$$0 < \bar{t} \leqslant \pi, \quad \int_0^{\pi} \sin t (F(t) - F(\pi + t) + F(2\pi + t)) \, dt \leqslant 0, \tag{2.8}$$

then $S < 0$.

Proof. Writing $S = S_1 + S_2$ with

$$S_1 = \int_0^{\pi} \sin t F(t) \, dt; \qquad S_2 = \sum_{n \geqslant 1} (-1)^n \int_0^{\pi} \sin t F(n\pi + t) \, dt,$$

we have $S_2 < 0$, for the function F is decreasing in $t \geqslant \pi$.

It is easily seen that

$$-\int_0^{\pi} \sin t F(\pi + t) \, dt < S_2 < -\int_0^{\pi} \sin t F(\pi + t) \, dt + \int_0^{\pi} \sin t F(2\pi + t) \, dt,$$

$$\int_0^{\pi} \sin t (F(t) - F(\pi + t)) \, dt < S < \int_0^{\pi} \sin t (F(t) - F(\pi + t) + F(2\pi + t)) \, dt.$$

Hence the result. □

Proof of Proposition 2.6. Let us prove (2.7) with $F(t) = F_\alpha(t)$ introduced in Corollary 2.2. The function F_α has a maximum for $t = \bar{t} = \bar{t}(\alpha)$ defined by $f_\alpha'(\bar{t}) = f_\alpha(\bar{t})$, hence

$$f_\alpha(\bar{t}) = \frac{4\bar{t}^3}{8 + (8\alpha - 3)\bar{t}^4 + \bar{t}^5}, \tag{2.9}$$

since

$$f_\alpha'(t) = \frac{4}{t^2} - \left(\frac{8\alpha - 3}{t} + \frac{8}{t^5}\right) f_\alpha(t).$$

But, from the first inequality of (2.2), we have

$$\frac{1}{2}\frac{t^3}{1 + \alpha t^4} \leqslant f_\alpha(t), \quad t \geqslant 0, \quad \alpha > 0.$$

With (2.9) necessarily: $\bar{t} \leqslant 3$.

On the other hand, the inequalities (2.1) are valid for $\alpha \to \alpha - 1$ and $\alpha \geqslant 1 - (1/2\pi)$, $0 \leqslant t \leqslant 2\pi$

$$\frac{1}{2}\frac{t^{2\alpha}}{1 + \alpha t}\exp\left(-\frac{2}{t}\right) \leqslant h_{\alpha-1}(t) \leqslant \frac{1}{2}\frac{t^{2\alpha}}{1 + \frac{1}{2}\alpha t}\exp\left(-\frac{2}{t}\right), \quad 0 \leqslant t \leqslant 2\pi$$

because the function $x \to x^{2(\alpha-1)}\exp(-2/x)$ is increasing in $0 \leqslant x \leqslant 2\pi$ for each $\alpha \geqslant 1 - 1/2\pi$. Hence,

$$\frac{1}{2}\frac{t^3}{1 + \alpha t^4} \leqslant f_\alpha(t) \leqslant \frac{1}{2}\frac{t^3}{1 + \frac{1}{2}\alpha t^4}, \quad 0 \leqslant t \leqslant 2\pi; \quad \alpha \geqslant 1 - \frac{1}{2\pi}.$$

Now, the inequality (2.7) is fulfilled if the following is verified

$$\int_0^\pi \sin t \frac{(\pi + t)^3}{1 + \frac{1}{2}\alpha(\pi + t)^4} e^{-(\pi + 1)}\,dt \leqslant \int_0^\pi \sin t \frac{t^3}{1 + \alpha t^4} e^{-t}\,dt. \tag{2.10}$$

The function $t \to t^3/(1 + \frac{1}{2}\alpha\, t^4)$ is decreasing for $t \geqslant t_1 = (6/\alpha)^{1/4}$ and $t_1 \leqslant \pi/2$ if and only if $\alpha \geqslant 6(2/\pi)^4$. We have successively

$$\int_0^\pi \sin t \frac{(\pi + t)^3}{1 + \frac{1}{2}\alpha(\pi + t)^4} e^{-(\pi + t)}\,dt \leqslant e^{-\pi}\frac{\pi^3}{1 + \frac{1}{2}\alpha\,\pi^4}\tfrac{1}{2}(1 + e^{-\pi}).$$

The integral 0 to π can be split into two parts 0 to t_1, and t_1 to π.

$$\int_{t_1}^\pi \sin t \frac{t^3}{1 + \alpha t^4} e^{-t}\,dt = \int_{t_1}^\pi \sin t \frac{t^3}{1 + \frac{1}{2}\alpha\, t^4}\frac{1 + \frac{1}{2}\alpha\, t^4}{1 + \alpha t^4} e^{-t}\,dt$$

$$\geqslant \frac{1}{2}\frac{\pi^3}{1 + \frac{1}{2}\alpha\,\pi^4}\int_{\pi/2}^\pi \sin t\, e^{-t}\,dt = \frac{1}{4}e^{-\pi}\frac{\pi^3}{1 + \frac{1}{2}\alpha\,\pi^4}(e^{\pi/2} + 1).$$

Thus, (2.10) is fulfilled if

$$\frac{1}{2}(1 + e^{-\pi})e^{-\pi}\frac{\pi^3}{1 + \frac{1}{2}\alpha\,\pi^4} \leqslant \int_0^{t_1} \sin t \frac{t^3}{1 + \alpha t^4} e^{-t}\,dt + \frac{1}{4}(e^{\pi/2} + 1)e^{-\pi}\frac{\pi^3}{1 + \frac{1}{2}\alpha\,\pi^4}. \tag{2.11}$$

But, we have

$$1 + e^{-\pi} < \tfrac{1}{2}(1 + e^{\pi/2}).$$

Therefore, the inequality (2.11) is satisfied.

Uniqueness of $\bar{t}$. For any abscissa $\tilde{t}$ of an extremum, we have easily

$$f_\alpha''(\tilde{t}) - f_\alpha'(\tilde{t}) = -\frac{4}{\tilde{t}^3} k_\alpha(\tilde{t}),$$

where

$$k_\alpha(t) = t + 1 - \frac{32 - t^5}{8 + (8\alpha - 3)t^4 + t^5}.$$

For $8\alpha - 3 \geqslant 0$, the function k_α is increasing and has a unique positive zero t^*. If $\bar{t} > 0$ is the first abscissa such that $F_\alpha'(\bar{t}) = 0$, then $F_\alpha(\bar{t})$ is a maximum and therefore $\bar{t} > t^*$. So $\tilde{t} > \bar{t}$ is not possible, since $F_\alpha''(\tilde{t}) < 0$, consequently $\tilde{t} \leqslant \bar{t}$, hence $\tilde{t} = \bar{t}$. □

Remarks. (1) The representation (1.20) is given and the condition (1.7) is proved for $\alpha = 1$ in [6].

(2) The method is applicable to any semi-classical form; in particular, to the various modified classical forms obtained by a shifting [11, 12].

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