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Sphericity test in a GMANOVA–MANOVA model with normal error

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ABSTRACT

For the GMANOVA–MANOVA model with normal error: $Y = XB_1Z_1' + B_2Z_2' + \mathcal{E}$, $\mathcal{E} \sim N_{q\times n}(0, I_n \otimes \Sigma)$, we study in this paper the sphericity hypothesis test problem with respect to covariance matrix: $\Sigma = \lambda I_q$ (λ is unknown). It is shown that, as a function of the likelihood ratio statistic Λ, the null distribution of $\Lambda^{2/n}$ can be expressed by Meijer's $G^{q,0}_{q,q}$ function, and the asymptotic null distribution of $-2\log\Lambda$ is $\chi^2_{q(q+1)/2-1}$ (as $n \to \infty$). In addition, the Bartlett type correction $-2\rho\log\Lambda$ for $\log\Lambda$ is indicated to be asymptotically distributed as $\chi^2_{q(q+1)/2-1}$ with order n^{-2} for an appropriate Bartlett adjustment factor -2ρ under null hypothesis.

1. Introduction

The model considered here is a GMANOVA–MANOVA model which can be defined as

$$Y = XB_1Z_1' + B_2Z_2' + \mathcal{E}, \tag{1}$$

where Y is a $q \times n$ observable random response matrix, X is a $q \times p$ known constant matrix, Z_1 and Z_2 are the $n \times m$ and $n \times s$ known design matrices, respectively, B_1 and B_2 are the $p \times m$ and $q \times s$ unknown regression coefficient matrices, respectively, $\mathcal{E}$ is a $q \times n$ unobservable random error matrix, and A' denotes the transpose of matrix A. The model (1) was first proposed by Chinchilli and Elswick [1], and was extensively applied to various fields including biology, medicine and economics. The error matrix $\mathcal{E}$ is often assumed to be normal:

$$\mathcal{E} \sim N_{q\times n}(0, I_n \otimes \Sigma), \tag{2}$$

i.e. $\mathcal{E}_1, \ldots, \mathcal{E}_n \overset{i.i.d}{\sim} N_q(0, \Sigma)(\mathcal{E} \hat{=} (\mathcal{E}_1, \ldots, \mathcal{E}_n))$, where $\Sigma(> 0)$ is a $q \times q$ unknown covariance matrix. Under the assumption (2), a variety of investigations have been made to handle the statistical inferences with respect to the parameter matrices B_1, B_2 and Σ, a good summary for the related results can be found in Kollo and von Rosen [2], and the excessive published papers will not be listed here for being irrelative to our subject. The available materials clearly show that most of the published works relating to the model (1) and (2) focused their attention on the statistical inferences for B_1 and B_2, and few took Σ into account. In this paper, we study an inference with respect to Σ, which is referred to as sphericity hypothesis and can be described as

$$H: \ \Sigma = \lambda I_q, \quad \lambda(> 0) \text{ is unknown}. \tag{3}$$

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To the best of our knowledge, the likelihood ratio test for the above hypothesis in the model (1) under the assumption (2) has not been done before. The remainder of this article is arranged as follows: Section 2 gives the likelihood ratio statistic Λ for sphericity hypothesis (3). In Section 3, the exact null density function of $\Lambda^{2/n}$ is expressed by Meijer's $G^{q,0}_{q,q}$ function, the asymptotic null distribution of $-2\log\Lambda$ is shown to be $\chi^2_{q(q+1)/2-1}$(as $n \to \infty$), and $-2\rho\log\Lambda$ is indicated to be asymptotically distributed as $\chi^2_{q(q+1)/2-1}$ with order n^{-2} for an appropriate Bartlett adjustment factor -2ρ for $\log\Lambda$ under null hypothesis.

2. Likelihood ratio statistic

In order to obtain the likelihood ratio test statistic for sphericity hypothesis (3), we need the following results. We follow the symbols and notations in Muirhead [3] without specification.

Lemma 1 (*Bai [4]*). *For the GMANOVA–MANOVA model (1) with normal error (2), the maximum likelihood estimates of B_1, B_2 and Σ are given by (with probability one)*

$$\begin{cases} \hat{B}_1 = (X'S^{-1}X)^{-}X'S^{-1}YQ_{Z_2}Z_1(Z_1'Q_{Z_2}Z_1)^{-}, \\ \hat{B}_2 = (Y - X\hat{B}_1Z_1')Z_2(Z_2'Z_2)^{-}, \\ \hat{\Sigma} = \dfrac{1}{n}(Y - X\hat{B}_1Z_1')Q_{Z_2}(Y - X\hat{B}_1Z_1')', \end{cases}$$

respectively, where $S = YQ_ZY'$, $Z \mathrel{\hat{=}} (Z_1, Z_2)$, $P_A \mathrel{\hat{=}} A(A'A)^{-}A'$, $Q_A \mathrel{\hat{=}} I_p - P_A$ (A is a $p \times q$ matrix) and A^{-} denotes an arbitrary g-inverse of A such that $AA^{-}A = A$.

Remark 1. The sufficient and necessary conditions for the random matrix S in Lemma 1 being positive definite with probability one are $n \geq rk(Z) + q$ (Okamato [5]), where $rk(A)$ denotes the rank of matrix A. In addition, although the expressions of both $\hat{B}_1$ and $\hat{B}_2$ contain the g-inverses, we have

$$\hat{\Sigma} = \frac{1}{n}\{S + [I_q - X(X'S^{-1}X)^{-}X'S^{-1}](S_2 - S)[I_q - X(X'S^{-1}X)^{-}X'S^{-1}]'\}, \tag{4}$$

which and $R(X') = R(X'S^{-1})$ show that $\hat{\Sigma}$ is unique, where $S_2 = YQ_{Z_2}Y'$ and $R(A)$ denotes the linear subspace spanned by the columns of matrix A.

Lemma 2. *Let $K(B_1, B_2, \Sigma|Y)$ denote the likelihood function of (B_1, B_2, Σ) based on Y in the model (1) and (2), i.e.*

$$K(B_1, B_2, \Sigma; Y) = (2\pi)^{-qn/2}|\Sigma|^{-n/2}\mathrm{etr}\left\{-\frac{1}{2}(Y - XB_1Z_1' - B_2Z_2')' \times \Sigma^{-1}(Y - XB_1Z_1' - B_2Z_2')\right\}, \quad B_1 \in R^{p\times m}, B_2 \in R^{q\times s}, \Sigma > 0, \tag{5}$$

then

$$\sup_{B_1\in R^{p\times m}, B_2\in R^{q\times s}, \lambda>0} K(B_1, B_2, \lambda I_q; Y) = (2\pi e\hat{\lambda})^{-qn/2}, \tag{6}$$

where $\hat{\lambda} = \frac{1}{qn}tr(P_XS + Q_XS_2)$.

Proof. It follows from (5) that

$$\begin{aligned} L(B_1, B_2, \lambda; Y) &\mathrel{\hat{=}} \log K(B_1, B_2, \lambda I_q; Y) \\ &= -\frac{qn}{2}\log(2\pi\lambda) - \frac{1}{2\lambda}tr\left\{(Y - XB_1Z_1' - B_2Z_2')(Y - XB_1Z_1' - B_2Z_2')'\right\}, \\ &\quad B_1 \in R^{p\times m}, B_2 \in R^{q\times s}, \lambda > 0, \end{aligned} \tag{7}$$

which implies that

$$\begin{aligned} \tilde{L}(B_1, B_2; Y) &\mathrel{\hat{=}} \sup_{\lambda>0} L(B_1, B_2, \lambda; Y) \\ &= -\frac{qn}{2}\log\{2\pi e\tilde{\lambda}(B_1, B_2)\}, \quad B_1 \in R^{p\times m}, B_2 \in R^{q\times s}, \end{aligned} \tag{8}$$

where $\tilde{\lambda}(B_1, B_2) = \frac{1}{qn} tr\{(Y - XB_1Z_1' - B_2Z_2')(Y - XB_1Z_1' - B_2Z_2')'\}$ and $tr(A)$ denotes the trace of matrix A. Note that

$$\begin{aligned}&(Y - XB_1Z_1' - B_2Z_2')(Y - XB_1Z_1' - B_2Z_2')'\\ &\quad = (Y - XB_1Z_1')Q_{Z_2}(Y - XB_1Z_1')' + (B_2 - \tilde{B}_2(B_1))Z_2'Z_2(B_2 - \tilde{B}_2(B_1))', \quad B_2 \in R^{q\times s}, B_1 \in R^{p\times m},\end{aligned}$$

where $\tilde{B}_2(B_1) = (Y - XB_1Z_1')Z_2(Z_2'Z_2)^-$, hence

$$tr\{(Y - XB_1Z_1' - B_2Z_2')(Y - XB_1Z_1' - B_2Z_2')'\} \geq tr\{(Y - XB_1Z_1')Q_{Z_2}(Y - XB_1Z_1')'\}, \quad B_1 \in R^{p\times m}, \tag{9}$$

where the equality holds if $B_2 = \tilde{B}_2(B_1)$, $B_1 \in R^{p\times m}$. Again note that

$$\begin{aligned}(Y - XB_1Z_1')Q_{Z_2}(Y - XB_1Z_1')' &= [Y - X\hat{B}_{10}Z_1' - X(B_1 - \hat{B}_{10})Z_1']Q_{Z_2}[Y - X\hat{B}_{10}Z_1' - X(B_1 - \hat{B}_{10})Z_1']'\\ &= [Q_XYP_{Q_{Z_2}Z_1} - X(B_1 - \hat{B}_{10})Z_1'Q_{Z_2}][Q_XYP_{Q_{Z_2}Z_1} - X(B_1 - \hat{B}_{10})Z_1'Q_{Z_2}]' + S,\\ &\quad B_1 \in R^{p\times m},\end{aligned}$$

where $\hat{B}_{10} = (X'X)^-X'YQ_{Z_2}Z_1(Z_1'Q_{Z_2}Z_1)^-$, thus

$$\begin{aligned}tr\{(Y - XB_1Z_1')Q_{Z_2}(Y - XB_1Z_1')'\} &= tr(P_XS + Q_XS_2) + tr\{X(B_1 - \hat{B}_{10})Z_1'Q_{Z_2}Z_1(B_1 - \hat{B}_{10})X'\}\\ &\geq tr(P_XS + Q_XS_2),\end{aligned} \tag{10}$$

where the equality holds if $B_1 = \hat{B}_{10}$. From the definition of $\tilde{\lambda}(B_1, B_2)$, (9) and (10), we have

$$\tilde{\lambda}(B_1, B_2) \geq \hat{\lambda} \triangleq \frac{1}{qn} tr(P_XS + Q_XS_2), \quad B_1 \in R^{p\times m}, B_2 \in R^{q\times s},$$

where the equality holds if $B_2 = \tilde{B}_2(B_1)$, $B_1 = \hat{B}_{10}$. This and (8) show that

$$\begin{aligned}\sup_{B_1\in R^{p\times m}, B_2\in R^{q\times s}, \lambda>0} L(B_1, B_2, \lambda; Y) &= \sup_{B_1\in R^{p\times m}, B_2\in R^{q\times s}} \tilde{L}(B_1, B_2; Y)\\ &\leq -\frac{qn}{2}\log(2\pi e\hat{\lambda}),\end{aligned}$$

where the equality holds if $B_2 = \tilde{B}_2(B_1)$, $B_1 = \hat{B}_{10}$. Therefore, from (7), we obtain (6). □

From Lemmas 1 and 2, we immediately have

Corollary 1. *For the model (1) and (2), the likelihood ratio statistic for testing the sphericity (3) is*

$$\Lambda = \left(\frac{\hat{\Sigma}}{\hat{\lambda}^q}\right)^{n/2}, \tag{11}$$

where $\hat{\Sigma}$ and $\hat{\lambda}$ are given by Remark 1 and Lemma 2, respectively.

Proof. It follows from Lemma 1 and (5) that

$$\sup_{B_1\in R^{p\times m}, B_2\in R^{q\times s}, \Sigma>0} K(B_1, B_2, \Sigma; Y) = (2\pi e)^{-qn/2}|\hat{\Sigma}|^{-n/2},$$

which and (6) mean that the likelihood ratio statistic for testing the sphericity (3) is given by (11). □

3. Null distribution

In this section, we will establish the exact null density function of $\Lambda^{2/n}$ and the asymptotic null distributions of $-2\log\Lambda$. The following theorem plays the key role for deriving the results mentioned above.

Theorem 1. *The null distribution of likelihood ratio statistic Λ is determined by*

$$\Lambda^{2/n} \overset{d}{=} q^q \frac{|U||V|}{[tr(U) + tr(V) + W]^q}, \tag{12}$$

where $X \overset{d}{=} Y$ denotes that the random variables X and Y have the same distribution, U, V, W are mutually independent and

$$\begin{cases} U \sim W_{rk(X)}(n_1, I_{rk(X)}),\\ V \sim W_{q-rk(X)}(n_2, I_{q-rk(X)}),\\ W \sim \chi^2_{rk(X)(q-rk(X))},\end{cases} \tag{13}$$

where $n_1 = n - rk(Z) - q + rk(X)$ *and* $n_2 = n - rk(Z_2)$.

Proof. Let the singular value decomposition of X be

$$X = P\begin{pmatrix}\Delta & 0\\ 0 & 0\end{pmatrix}Q', \tag{14}$$

where P and Q are the $q \times q$ and $p \times p$ orthogonal matrices, respectively, Δ is a $rk(X) \times rk(X)$ nonsingular diagonal matrix, then

$$P_X = P\begin{pmatrix}I_{rk(X)} & 0\\ 0 & 0\end{pmatrix}P', \qquad Q_X = P\begin{pmatrix}0 & 0\\ 0 & I_{q-rk(X)}\end{pmatrix}P'. \tag{15}$$

Make the transformation

$$T \hat{=} \begin{pmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{pmatrix} = P'SP, \tag{16}$$

where T_{11} is a $rk(X) \times rk(X)$ random matrix, then

$$S^{-1} = PT^{-1}P' = P\begin{pmatrix}T_{11\cdot2}^{-1} & -T_{11\cdot2}^{-1}T_{12}T_{22}^{-1}\\ -T_{22}^{-1}T_{21}T_{11\cdot2}^{-1} & T_{22\cdot1}^{-1}\end{pmatrix}P', \tag{17}$$

where $T_{11\cdot2} = T_{11} - T_{12}T_{22}^{-1}T_{21}$, $T_{22\cdot1} = T_{22} - T_{21}T_{11}^{-1}T_{12}$, hence from (14) and (17), we have

$$X'S^{-1}X = Q\begin{pmatrix}\Delta T_{11\cdot2}^{-1}\Delta & 0\\ 0 & 0\end{pmatrix}Q',$$

which means that

$$(X'S^{-1}X)^{-} = Q\begin{pmatrix}\Delta^{-1}T_{11\cdot2}\Delta^{-1} & C_{12}\\ C_{21} & C_{22}\end{pmatrix}Q', \tag{18}$$

where C_{12}, C_{21} and C_{22} are arbitrary. Substitute (14), (16)–(18) into (4) to yield

$$\hat{\Sigma} = \frac{1}{n}P\left[T + \begin{pmatrix}0 & T_{12}T_{22}^{-1}\\ 0 & I_{q-rk(X)}\end{pmatrix}P'(\tilde{S}_2 - S)P\begin{pmatrix}0 & 0\\ T_{22}^{-1}T_{21} & I_{q-rk(X)}\end{pmatrix}\right]P', \tag{19}$$

where $\tilde{S}_2 = \mathcal{E}Q_{Z_2}\mathcal{E}'$, $S = \mathcal{E}Q_Z\mathcal{E}'$. Furthermore, make the transformation

$$\tilde{T} \hat{=} \begin{pmatrix}\tilde{T}_{11} & \tilde{T}_{12}\\ \tilde{T}_{21} & \tilde{T}_{22}\end{pmatrix} = P'(\tilde{S}_2 - S)P, \tag{20}$$

where $\tilde{T}_{11}$ is a $rk(X) \times rk(X)$ random matrix, then from (16) and (19), we obtain

$$\hat{\Sigma} = \frac{1}{n}P\begin{pmatrix}T_{11} + T_{12}T_{22}^{-1}\tilde{T}_{22}T_{22}^{-1}T_{21} & T_{12}(I_{q-rk(X)} + T_{22}^{-1}\tilde{T}_{22})\\ (I_{q-rk(X)} + \tilde{T}_{22}T_{22}^{-1})T_{21} & T_{22} + \tilde{T}_{22}\end{pmatrix}P',$$

which shows that (Theorem A5.3 in [3])

$$|\hat{\Sigma}| = \frac{1}{n^q}|T_{11\cdot2}||T_{22} + \tilde{T}_{22}|. \tag{21}$$

In addition, it follows from (15), (16) and (20) that

$$\begin{aligned}\hat{\lambda} &= \frac{1}{qn}tr\{S + Q_X(\tilde{S}_2 - S)\}\\ &= \frac{1}{qn}[tr(T_{11\cdot2}) + tr(T_{12}T_{22}^{-1}T_{21}) + tr(T_{22} + \tilde{T}_{22})].\end{aligned} \tag{22}$$

When the sphericity hypothesis (3) holds, from (2), we have

$$\mathcal{E} \sim N_{q\times n}(0, \lambda I_n \otimes I_q), \tag{23}$$

which implies that

$$\begin{cases}S \sim W_q(n - rk(Z), \lambda I_q),\\ \tilde{S}_2 - S \sim W_q(rk(Z) - rk(Z_2), \lambda I_q).\end{cases} \tag{24}$$

Note that $Q_Z(Q_{Z_2} - Q_Z) = 0$, hence from (23) and Theorem 10.24 in Schott [6], $S = \mathcal{E} Q_Z \mathcal{E}'$ and $\tilde{S}_2 - S = \mathcal{E}(Q_{Z_2} - Q_Z)\mathcal{E}'$ are mutually independent. Therefore, from (16), (20) and (24), we know that T and $\tilde{T}$ are mutually independent and

$$\begin{cases} T \sim W_q(n - rk(Z), \lambda I_q), \\ \tilde{T} \sim W_q(rk(Z) - rk(Z_2), \lambda I_q), \end{cases}$$

which and Theorem 3.2.10 in Muirhead [3] indicate that

$$\begin{cases} T_{11\cdot 2} \sim W_{rk(X)}(n_1, \lambda I_{rk(X)}), \\ T_{12}|T_{22} \sim N_{rk(X)\times(q-rk(X))}(0, \lambda T_{22} \otimes I_{rk(X)}), \\ T_{22} \sim W_{q-rk(X)}(n - rk(Z), \lambda I_{q-rk(X)}), \\ \tilde{T}_{22} \sim W_{q-rk(X)}(rk(Z) - rk(Z_2), \lambda I_{q-rk(X)}), \end{cases} \tag{25}$$

and $T_{11\cdot 2}$ is independent of (T_{12}, T_{22}). It follows from the independence between T and $\tilde{T}$ that $(T_{11\cdot 2}, T_{12}, T_{22})$ and $\tilde{T}_{22}$ are independent, hence from the independence between $T_{11\cdot 2}$ and (T_{12}, T_{22}), we know that $T_{11\cdot 2}$, (T_{12}, T_{22}), $\tilde{T}_{22}$ are mutually independent. Note that from the second equality in (25), we have

$$T_{12}T_{22}^{-1}T_{21}|T_{22} \sim W_{rk(X)}(q - rk(X), \lambda I_{rk(X)}),$$

which means that $T_{12}T_{22}^{-1}T_{21}$ and T_{22} are independent and

$$T_{12}T_{22}^{-1}T_{21} \sim W_{rk(X)}(q - rk(X), \lambda I_{rk(X)}). \tag{26}$$

Thus from the independence among $T_{11\cdot 2}$, (T_{12}, T_{22}) and $\tilde{T}_{22}$, we know that $T_{11\cdot 2}$, $T_{12}T_{22}^{-1}T_{21}$, T_{22}, $\tilde{T}_{22}$ are mutually independent. Let

$$\begin{cases} U = \dfrac{1}{\lambda} T_{11\cdot 2}, \\ V = \dfrac{1}{\lambda}(T_{22} + \tilde{T}_{22}), \\ W = \dfrac{1}{\lambda} tr(T_{12}T_{22}^{-1}T_{21}), \end{cases} \tag{27}$$

then U, V, W are mutually independent, and from (25) and (26), we obtain (13). Finally, it follows from (21), (22) and (27) that

$$|\hat{\Sigma}| = \left(\frac{\lambda}{n}\right)^q |U||V|, \qquad \hat{\lambda} = \frac{\lambda}{qn}[tr(U) + tr(V) + W],$$

which and Corollary 1 shows that (12) holds. □

In order to obtain the exact null density function of $\Lambda^{2/n}$ based on Theorem 1, we need the following definition and lemma.

Definition 1. If the $m \times m$ nonnegative definite random matrix X has the density function

$$\frac{1}{2^{ma}\Gamma_m(a)|\Sigma|^a}|x|^{a-(m+1)/2} \mathrm{e}\, tr\left(-\frac{1}{2}\Sigma^{-1}x\right)(\mathrm{d}x), \quad x > 0, \tag{28}$$

where $\mathrm{Re}(a) > \frac{1}{2}(m-1)$, Σ is a $m \times m$ symmetric matrix such that $\mathrm{Re}(\Sigma) > 0$, then X is said to have an m-variate gamma distribution with parameter (a, Σ) and is denoted by $X \sim \Gamma_m(a, \Sigma)$. When $a = \frac{n}{2}$, n is an integer, $\Gamma_m(a, \Sigma)$ is the Wishart distribution $W_m(n, \Sigma)$ (Muirhead [3]).

Lemma 3. *If $X \sim \Gamma_m(a, \Sigma)$, then $tr(\Sigma^{-1}X) \sim \Gamma_1(ma, 1)$.*

Proof. Make the transformation

$$\Sigma^{-1/2}X\Sigma^{-1/2} = T'T, \tag{29}$$

where $T = (T_{ij})_{m\times m}$ is upper-triangular with positive diagonal elements, then (Theorem 2.1.9 in Muirhead [3])

$$(\mathrm{d}X) = |\Sigma|^{(m+1)/2} 2^m \prod_{i=1}^m T_{ii}^{m+1-i} \bigwedge_{i\le j}^m dT_{ij},$$

which and (28) means that the joint density function of T_{ij}, $1 \le i \le j \le m$ can be written as

$$\prod_{i<j}^m \left[\frac{1}{(2\pi)^{1/2}} \exp\left(-\frac{t_{ij}^2}{2}\right)\mathrm{d}t_{ij}\right] \cdot \prod_{i=1}^m \frac{1}{2^{a-(i-1)/2}\Gamma[a-(i-1)/2]}(t_{ii}^2)^{a-(i+1)/2} \exp\left(-\frac{t_{ii}^2}{2}\right)\mathrm{d}t_{ii}^2,$$

which shows that $T_{ij} \sim N(0,1)$, $1 \le i < j \le m$, $T_{ii}^2 \sim \Gamma_1(a - \frac{1}{2}(i-1), 1)$, $i = 1, \ldots, m$ and T_{ij}, $1 \le i \le j \le m$ are mutually independent. Therefore, from (29) and additivity of gamma distribution [7], we have

$$tr(\Sigma^{-1}X) = tr(T'T) = \sum_{i \le j}^{m} T_{ij}^2 \sim \Gamma_1(ma, 1). \quad \square$$

Theorem 2. *Let $\tilde{\Lambda} = \Lambda^{2/n}$, when the sphericity hypothesis (3) holds, we have*

$$E(\tilde{\Lambda}^z) = q^{qz} \frac{\Gamma_{rk(X)}(n_1/2+z)}{\Gamma_{rk(X)}(n_1/2)} \frac{\Gamma_{q-rk(X)}(n_2/2+z)}{\Gamma_{q-rk(X)}(n_2/2)} \frac{\Gamma(n_3/2)}{\Gamma(n_3/2+qz)}, \quad \mathrm{Re}(z) \ge 0, \tag{30}$$

where $n_3 = rk(X)(n - rk(Z)) + (q - rk(X))(n - rk(Z_2))$.

Proof. When the sphericity hypothesis (3) holds, from Theorem 1, we know that

$$\begin{aligned} E(\tilde{\Lambda}^z) &= q^{qz} E\left\{ \frac{|U|^z |V|^z}{[tr(U) + tr(V) + W]^{qz}} \right\} \\ &= (2q)^{qz} \frac{\Gamma_{rk(X)}(n_1/2+z)}{\Gamma_{rk(X)}(n_1/2)} \frac{\Gamma_{q-rk(X)}(n_2/2+z)}{\Gamma_{q-rk(X)}(n_2/2)} E\left\{ \frac{1}{[tr(\tilde{U}) + tr(\tilde{V}) + W]^{qz}} \right\}, \quad \mathrm{Re}(z) \ge 0, \end{aligned} \tag{31}$$

where $\tilde{U}, \tilde{V}, W$ are mutually independent and

$$\begin{cases} \tilde{U} \sim \Gamma_{rk(X)}\left(\frac{1}{2}n_1 + z, I_{rk(X)}\right), \\ \tilde{V} \sim \Gamma_{q-rk(X)}\left(\frac{1}{2}n_2 + z, I_{q-rk(X)}\right). \end{cases} \tag{32}$$

It follows from the additivity of gamma distribution [7], Lemma 3, (13) and (32) that

$$tr(\tilde{U}) + tr(\tilde{V}) + W \sim \Gamma_1\left(\frac{1}{2}n_3 + qz, 1\right),$$

which indicates that

$$E\left\{ \frac{1}{[tr(\tilde{U}) + tr(\tilde{V}) + W]^{qz}} \right\} = 2^{-qz} \frac{\Gamma(n_3/2)}{\Gamma(n_3/2+qz)}, \quad \mathrm{Re}(z) \ge 0.$$

Substitute the above equality into (31) to get (30). $\square$

Theorem 3. *As the function of likelihood ratio statistic, the null density function of $\tilde{\Lambda} = \Lambda^{2/n}$ can be expressed as*

$$f_{\tilde{\Lambda}}(\tilde{\lambda}) = \frac{(2\pi)^{(q-1)/2}}{q^{(n_3-1)/2}} \frac{\pi^{[rk(X)(rk(X)-1)+(q-rk(X))(q-rk(X)-1)]/4}\Gamma(n_3/2)}{\Gamma_{rk(X)}(n_1/2)\Gamma_{q-rk(X)}(n_2/2)} G_{q,q}^{q,0}\left(\tilde{\lambda}\,\middle|\,{}^{a_1,\ldots,a_q}_{b_1,\ldots,b_q}\right), \quad 0 < \tilde{\lambda} < 1, \tag{33}$$

where $G_{m,n}^{p,q}\left(z\,\middle|\,{}^{a_1,\ldots,a_p}_{b_1,\ldots,b_q}\right)$ denotes the Meijer's G-function, $a_i = \frac{1}{2}(n_1 - i - 1)$, $1 \le i \le rk(X)$, $a_i = \frac{1}{2}(n_2 - i + rk(X) - 1)$, $rk(X) + 1 \le i \le q$, $b_i = \frac{1}{2q}[n_3 + 2(i-1)] - 1$, $1 \le i \le q$.

Proof. It follows from Theorem 2 that the Mellin transform of null density of $\tilde{\lambda}$ is

$$\begin{aligned} g_{\tilde{\Lambda}}(z) &= E(\tilde{\Lambda}^{z-1}) \\ &= q^{q(z-1)} \frac{\Gamma_{rk(X)}(n_1/2+z-1)}{\Gamma_{rk(X)}(n_1/2)} \frac{\Gamma_{q-rk(X)}(n_2/2+z-1)}{\Gamma_{q-rk(X)}(n_2/2)} \frac{\Gamma(n_3/2)}{\Gamma[n_3/2+q(z-1)]}, \quad \mathrm{Re}(z) \ge 1. \end{aligned} \tag{34}$$

From Gauss's multiplication formula for gamma function [8], we have

$$\Gamma\left[\frac{1}{2}n_3 + q(z-1)\right] = \frac{q^{(n_3-1)/2+q(z-1)}}{(2\pi)^{(q-1)/2}} \prod_{i=0}^{q-1} \Gamma\left[\frac{1}{2q}(n_3 + 2i) + z - 1\right]. \tag{35}$$

On the other hand,

$$\Gamma_{rk(X)}\left(\frac{1}{2}n_1 + z - 1\right) = \pi^{rk(X)(rk(X)-1)/4} \prod_{i=1}^{rk(X)} \Gamma\left[\frac{1}{2}(n_1 - i - 1) + z\right], \tag{36}$$

$$\Gamma_{q-rk(X)}\left(\frac{1}{2}n_2+z-1\right)=\pi^{(q-rk(X))(q-rk(X)-1)/4}\prod_{i=1}^{q-rk(X)}\Gamma\left[\frac{1}{2}(n_2-i-1)+z\right]. \quad (37)$$

Substitute (35)–(37) into (34) to get

$$g_{\tilde{\Lambda}}(z) = \frac{(2\pi)^{(q-1)/2}}{q^{(n_3-1)/2}}\frac{\pi^{[rk(X)(rk(X)-1)+(q-rk(X))(q-rk(X)-1)]/4}\Gamma(n_3/2)}{\Gamma_{rk(X)}(n_1/2)\Gamma_{q-rk(X)}(n_2/2)} \times \frac{\prod_{i=1}^{rk(X)}\Gamma[(n_1-i-1)/2+z]\prod_{i=1}^{q-rk(X)}\Gamma[(n_2-i-1)/2+z]}{\prod_{i=0}^{q-1}\Gamma[(n_3+2i)/(2q)+z-1]}, \quad \mathrm{Re}(z)\geq 1.$$

Apply the definition of Meijer's G-function [9], the above equality and

$$f_{\tilde{\Lambda}}(\tilde{\lambda})=\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\tilde{\lambda}^{-z}g_{\tilde{\Lambda}}(z)\mathrm{d}z, \quad 0<\tilde{\lambda}<1,$$

we obtain (33). □

Remark 2. Davis [10] provided an effective algorithm for computing the quantile of distribution with the density expressed by Meijer's $G^{p,0}_{p,p}$ function.

Furthermore, we have

Theorem 4. *When the sphericity hypothesis (3) holds,*

$$-2\log\Lambda \xrightarrow{\mathcal{L}} \chi^2_{q(q+1)/2-1}, \quad n\to\infty, \quad (38)$$

where $\xrightarrow{\mathcal{L}}$ *denotes convergence in distribution.*

Proof. It follows from Theorem 2 that the characteristic function of $-2\log\Lambda$ under hypothesis H is

$$\begin{aligned}\varphi_{-2\log\Lambda}(t) &= E(\Lambda^{-2\mathrm{i}t})\\ &= q^{-\mathrm{i}qnt}\frac{\Gamma_{rk(X)}(n_1/2-\mathrm{i}nt)}{\Gamma_{rk(X)}(n_1/2)}\frac{\Gamma_{q-rk(X)}(n_2/2-\mathrm{i}nt)}{\Gamma_{q-rk(X)}(n_2/2)}\frac{\Gamma(n_3/2)}{\Gamma(n_3/2-\mathrm{i}qnt)}, \quad t\in(-\infty,+\infty),\end{aligned}$$

which shows that

$$\begin{aligned}\log\varphi_{-2\log\Lambda}(t) &= -\mathrm{i}qnt\log q+\sum_{k=1}^{rk(X)}\left\{\log\Gamma\left[\frac{1}{2}(n_1-k+1)-\mathrm{i}nt\right]-\log\Gamma\left[\frac{1}{2}(n_1-k+1)\right]\right\}\\ &\quad+\sum_{k=1}^{q-rk(X)}\left\{\log\Gamma\left[\frac{1}{2}(n_2-k+1)-\mathrm{i}nt\right]-\log\Gamma\left[\frac{1}{2}(n_2-k+1)\right]\right\}\\ &\quad+\log\Gamma\left(\frac{1}{2}n_3\right)-\log\Gamma\left(\frac{1}{2}n_3-\mathrm{i}qnt\right), \quad t\in(-\infty,+\infty). \quad (39)\end{aligned}$$

Use the asymptotic formula for $\log(z+a)$ [3],

$$\log\Gamma(z+a)=\left(z+a-\frac{1}{2}\right)\log z-z+\frac{1}{2}\log(2\pi)+O(z^{-1})$$

it is a simple matter from (39) to show that

$$\log\varphi_{-2\log\Lambda}(t)\to-\frac{1}{2}\left[\frac{1}{2}q(q+1)-1\right]\log(1-2\mathrm{i}t), \quad n\to\infty,$$

which indicates that (38) is true. □

In order to obtain and improve the order by approximating the null distribution of $-\log\Lambda$ with $\chi^2_{q(q+1)/2-1}$ (as $n\to\infty$) based on Theorem 4, it follows from Theorem 2 that, under the null hypothesis (3), we have

$$E(\Lambda^z)=E(\tilde{\Lambda}^{nz/2})$$

$$= C\left[\frac{{y_1}^{y_1}}{\prod_{k=1}^{q} {x_k}^{x_k}}\right]^z \frac{\prod_{k=1}^{q}\Gamma[x_k(1+z)+\xi_k]}{\Gamma[y_1(1+z)+\eta_1]}, \quad \mathrm{Re}(z) \geq 0,$$

where C is a constant determined by $E(\Lambda^0) = 1$, $x_k = \frac{n}{2}$, $k = 1, \ldots, q$, $y_1 = \frac{qn}{2}$, $\xi_k = -\frac{1}{2}(rk(Z) + q - rk(X) + k - 1)$, $k = 1, \ldots, rk(X)$, $\xi_{rk(X)+k} = -\frac{1}{2}(rk(Z_2)+k-1)$, $k = 1, \ldots, q-rk(X)$, $\eta_1 = -\frac{1}{2}[rk(X)rk(Z)+(q-rk(X))rk(Z_2)]$. Therefore, based on the discussions in pp. 304–307 of Muirhead [3] or Box [11], we immediately obtain

Theorem 5. *When the sphericity hypothesis (3) holds,*

$$P(-2\log\Lambda \leq u) = P(\chi^2_{q(q+1)/2-1} \leq u) + O(n^{-1}),$$

and

$$P(-2\rho\log\Lambda \leq u) = P(\chi^2_{q(q+1)/2-1} \leq u) + O(n^{-2}),$$

where

$$\rho = 1 - \frac{1}{[q(q+1)-2]n}(A_1 - A_2),$$

$$A_1 = rk(X)\left[(rk(Z)+q-rk(X))(rk(Z)+q+1) + rk(X)\left(\frac{rk(X)}{3}+\frac{1}{2}\right)\right] + (q-rk(X))\left[rk(Z_2)(rk(Z_2)+q-rk(X)+1) + (q-rk(X))\left(\frac{1}{3}(q-rk(X))+\frac{1}{2}\right)\right] - \frac{q}{6},$$

$$A_2 = \frac{1}{q}\left[(rk(X)rk(Z)+(q-rk(X))rk(Z_2)+1)^2 - \frac{1}{3}\right].$$

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