ELSEVIER

Linear Algebra and its Applications 295 (1999) 61–72

LINEAR ALGEBRA AND ITS APPLICATIONS

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Embedding a regular subpencil into a general linear pencil

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Received 8 October 1998; accepted 12 February 1999

Submitted by J. Dias da Silva

Abstract

We study the possible strictly equivalence classes of a pencil when a regular subpencil is prescribed. We also study the possible invariant polynomials and the possible characteristic polynomials of $A + BY + XC + XDY$ when X and Y vary. © 1999 Elsevier Science Inc. All rights reserved.

1. Introduction

Throughout this paper F denotes an infinite field.

In [3], a necessary and sufficient condition for the existence of a regular pencil with prescribed Kronecker invariants and a prescribed subpencil was given. It was also given a necessary and sufficient condition for the existence of a square constant matrix with prescribed similarity invariants and a prescribed arbitrary submatrix. These results are reproduced in the next two theorems. In [1], the problem of embedding a regular subpencil into a regular pencil was solved, generalizing the well-known Sá-Thompson's interlacing theorem [4,8].

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PII: S0024-3795(99)00049-X

In this paper, we give a necessary and sufficient condition for the existence of a matrix pencil (not necessarily regular) with prescribed Kronecker invariants and a prescribed regular subpencil. See also [10].

As a consequence, we describe all the possible invariant polynomials of $A+BX+YC+YDX$ when X and Y vary. This problem had been solved for $C=0$ and $D=0$ in [11] and for $D=0$ in [7]. When $B=0$ and $C=0$, this result describes the possible invariant polynomials of $A+Z$ when Z varies and rank $Z \leqslant D$ (cf. [5,6]).

The problem of giving a necessary and sufficient condition for the existence of a matrix pencil with prescribed Kronecker invariants and a prescribed arbitrary subpencil remains open and seems to be a very difficult one. Note that theorems giving necessary and sufficient conditions for the existence of constant matrices with prescribed feedback equivalence invariants and a prescribed submatrix are particular solutions of this general problem.

Given a polynomial f, $d(f)$ denotes its degree.

Let $C(x) \in F[x]^{n\times h}$ be a matrix pencil, $\alpha_1(x,y) \mid \cdots \mid \alpha_w(x,y)$ its homogeneous invariant factors, $k_1 \geqslant \cdots \geqslant k_{n-w}$ its row minimal indices, $t_1 \geqslant \cdots \geqslant t_{h-w}$ its column minimal indices and $\epsilon = t_1 + \cdots + t_{h-w}$.

Theorem 1 [3]. *Let $D(x) \in F^{m\times m}$ be a regular pencil, with $n,h \leqslant m$. Let $\gamma_1(x,y) \mid \cdots \mid \gamma_m(x,y)$ be its homogeneous invariant factors. The following conditions are equivalent*:

(a_1) *There exists a pencil $E(x)$ strictly equivalent to $D(x)$containing $C(x)$ as a subpencil.*

(b_1) *There exist nonzero polynomials $\delta_1 \mid \cdots \mid \delta_n$ such that the following con ditions hold*:

- (i_1) $\operatorname{lcm}(\alpha_{i-n+w},\gamma_i) \mid \delta_i \mid \gcd(\alpha_i,\gamma_{i+2m-2n+w-h})$, $i \in \{1,\ldots,n\}$.
- (ii_1) $(k_1+1,\ldots,k_{n-w}+1) \prec (d(\sigma^{n-w})-d(\sigma^{n-w-1}),\ldots,d(\sigma^1)-d(\sigma^0))$, *where* $\sigma^j = \sigma^j_1 \cdots \sigma^j_{w+j-\epsilon}$ *and* $\sigma^j_i = \operatorname{lcm}(\alpha_{i-j+\epsilon},\delta_{i+\epsilon})$, $j \in \{0,\ldots,n-w\}$, $i \in \{1,\ldots,w+j-\epsilon\}$.
- (iii_1) $n+h-w \geqslant d(\eta^{h-w})$ *and* $(t_1+1,\ldots,t_{h-w}+1) \prec (n+h-w-d(\eta^{h-w-1}),\ d(\eta^{h-w-1})-d(\eta^{h-w-2}),\ldots,d(\eta^1)-d(\eta^0))$, *where* $\eta^j = \eta^j_1 \cdots \eta^j_{n+j}$ *and* $\eta^j_i = \operatorname{lcm}(\delta_{i-j},\gamma_i)$, $j \in \{0,\ldots,h-w\}$, $i \in \{1,\ldots,n+j\}$.

Convention. In the previous statement, we are making convention that, whenever a chain of polynomials $\beta_1 \mid \cdots \mid \beta_l$ is given, and β_i, $i \notin \{1,\ldots,l\}$, is not explicitly defined, then $\beta_i = 1$ for $i \leqslant 0$ and $\beta_i = 0$ for $i > l$. The symbol $\prec$ means majorization. We are also assuming that if $n-w=0$ then (ii_1) is true and if $h-w=0$ then (iii_1) is true and $\epsilon = 0$. This convention, with the adequate changes, applies throughout the paper in analogous situations.

Now assume that

$$C(x) = \begin{bmatrix} -A_{1,2} & -A_{1,3} \\ xI_q - A_{2,2} & -A_{2,3} \end{bmatrix} \in F[x]^{n\times h}, \tag{1}$$

where $A_{1,2}, A_{1,3}, A_{2,2}, A_{2,3}$ have their entries in F, and that $n = p + q$, $h = q + u$, $m = p + q + u + v$, where all the letters denote nonnegative integers.

Theorem 2 [3]. *Let $B \in F^{m\times m}$ and let $\gamma_1(x,y) \mid \cdots \mid \gamma_m(x,y)$ be the homogeneous invariant factors of $xI_m - B$. The following condition is equivalent to* (b_1):

(a_2) *There exist matrices $A_{1,1}(\in F^{p\times p})$, $A_{1,4}$, $A_{2,1}$, $A_{2,4}$, $A_{3,1}$, $A_{3,2}$, $A_{3,3}$, $A_{3,4}$, $A_{4,1}$, $A_{4,2}$, $A_{4,3}$, $A_{4,4}$, with entries in F, such that*

$$A = [A_{i,j}] \in F^{m\times m} \qquad (i,j \in \{1,2,3,4\}) \tag{2}$$

is similar to B.

For notational convenience, denote the condition (b_1) by

$$\Upsilon(\gamma;\alpha;k;t) = \Upsilon(\gamma_1,\ldots,\gamma_m;\alpha_1,\ldots,\alpha_w;k_1,\ldots,k_{n-w};t_1,\ldots,t_{h-w}).$$

With every monic polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in F[x],$$

associate the homogeneous polynomial

$$\tilde{f}(x,y) = x^n + a_{n-1}x^{n-1}y + \cdots + a_1xy^{n-1} + a_0y^n \in F[x,y].$$

Note that every nonzero homogeneous polynomial $h(x,y)$ has a unique factorization of the form $ay^r\tilde{f}$, where $a \in F\setminus\{0\}$, r is a nonnegative integer and $f \in F[x]$ is a monic polynomial. Also note that $\widetilde{fg} = \tilde{f}\tilde{g}$ and $f \mid g$ if and only if $\tilde{f} \mid \tilde{g}$, for every monic polynomials $f, g \in F[x]$. Let $\mathscr{T}$ be the set of all the polynomials of the form $y^r\tilde{f}$. Throughout this paper, we assume that homogeneous invariant factors of matrix pencils and gcd and lcm of homogeneous polynomials (in $F[x,y]$) belong to $\mathscr{T}$.

2. Embedding a regular subpencil into a general linear pencil

Theorem 3 [7]. *Let $A, A' \in F^{m\times m}$, $B \in F^{m\times r}$, $C \in F^{s\times m}$. Let $\zeta_1(x,y) \mid \cdots \mid \zeta_w(x,y)$ be the homogeneous invariant factors, $k_1 \geqslant \cdots \geqslant k_{m+s-w}$ the row minimal indices and $t_1 \geqslant \cdots \geqslant t_{m+r-w}$ the column minimal indices of*

$$\begin{bmatrix} xI_m - A & -B \\ -C & 0 \end{bmatrix}. \tag{3}$$

Let $\gamma_1(x,y) \mid \cdots \mid \gamma_m(x,y)$ *be the homogeneous invariant factors of* $xI_m - A'$. *Let*

$$\alpha_i = \frac{\zeta_i}{\gcd(\zeta_i, y^2)}, \qquad i \in \{1,\dots,w\}.$$

Let $v = w - m$, *the number of infinite elementary divisors of* (3). *Let* $u = \text{rank } B - v$, $p = \text{rank } C - v$. *Then there exist* $B' \in F^{m\times r}$, $C' \in F^{s\times m}$ *such that*

$$\begin{bmatrix} xI_m - A' & -B' \\ -C' & 0 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} xI_m - A & -B \\ -C & 0 \end{bmatrix}$$

are strictly equivalent if and only if

(b_3) $\Upsilon(\gamma_1,\dots,\gamma_m;\alpha_{p+u+2v+1},\dots,\alpha_w;k_1-1,\dots,k_p-1;t_1-1,\dots,t_u-1)$.

Note that all the infinite elementary divisors of (3) have degree $\geqslant 2$, u is the number of nonzero column minimal indices and p is the number of nonzero row minimal indices of (3).

The particular case of Theorem 3 where (3) does not have infinite elementary divisors is a lemma for the proof of the main result in this section. Note that, in this case, $w = m$, $\alpha_i = \zeta_i$, $i \in \{1,\dots,m\}$, $p = \text{rank } C$, $u = \text{rank } B$, and the condition (b_3) takes the form:

(b_3') There exist nonzero polynomials $\delta_1 \mid \cdots \mid \delta_{m-u}$ such that the following conditions hold:
- (i_3) $\text{lcm}(\alpha_{i+u},\gamma_i) \mid \delta_i \mid \gcd(\alpha_{i+p+u},\gamma_{i+u})$, $\quad i \in \{1,\dots,m-u\}$.
- (ii_3) $(k_1,\dots,k_p) \prec (d(\sigma^p) - d(\sigma^{p-1}),\dots,d(\sigma^1) - d(\sigma^0))$, where $\sigma^j = \sigma_1^j \cdots \sigma_{m-p+j-\epsilon}^j$ and $\sigma_i^j = \text{lcm}(\alpha_{i-j+\epsilon+p},\delta_{i+\epsilon-u})$, $\ j \in \{0,\dots,p\}$, $\ i \in \{1,\dots, m-p+j-\epsilon\}$, $\epsilon = t_1 + \cdots + t_u$.
- (iii_3) $m \geqslant d(\eta^u)$ and $(t_1,\dots,t_u) \prec (m - d(\eta^{u-1}), d(\eta^{u-1}) - d(\eta^{u-2}),\dots, d(\eta^1) - d(\eta^0))$, where $\eta^j = \eta_1^j \cdots \eta_{m-u+j}^j$ and $\eta_i^j = \text{lcm}(\delta_{i-j},\gamma_i)$, $j \in \{0,\dots,u\}$, $\ i \in \{1,\dots,m-u+j\}$.

Theorem 4. *Let* $A_{1,1} \in F^{h\times h}$ *and* $\beta_1 \mid \cdots \mid \beta_h$ *be the homogeneous invariant factors of* $xI_h - A_{1,1}$. *Let* $A \in F^{m\times m}$, $B \in F^{m\times r}$, $C \in F^{s\times m}$, $m \geqslant h$. *Suppose that* (3) *does not have infinite elementary divisors. Let* $\alpha_1 \mid \cdots \mid \alpha_m$ *be the homogeneous invariant factors,* $k_1 \geqslant \cdots \geqslant k_s$ *be the row minimal indices,* $t_1 \geqslant \cdots \geqslant t_r$ *be the column minimal indices of* (3). *Let* $\epsilon = t_1 + \cdots + t_r$, $p = \text{rank } C$, $u = \text{rank } B$. *The following conditions are equivalent*:

(a_4) *There exist matrices* $A_{1,2}, A_{1,3}, A_{2,1}, A_{2,2}, A_{2,3}, A_{3,1}$ *and* $A_{3,2}$, *with entries in* F, *such that*

$$\begin{bmatrix} xI_h - A_{1,1} & -A_{1,2} & -A_{1,3} \\ -A_{2,1} & xI_{m-h} - A_{2,2} & -A_{2,3} \\ -A_{3,1} & -A_{3,2} & 0 \end{bmatrix} \tag{4}$$

and (3) *are strictly equivalent.*

(b_4) *There exist nonzero polynomials* $\delta_1 \mid \cdots \mid \delta_{m-u}$ *such that the following conditions hold*:

(i_4) $\mathrm{lcm}(\alpha_{i+u}, \beta_{i-2m+2h}) \mid \delta_i \mid \gcd(\alpha_{i+p+u}, \beta_{i+u}), \quad i \in \{1, \ldots, m-u\}$.

(ii_4) $(k_1, \ldots, k_p) \prec (d(\sigma^p) - d(\sigma^{p-1}), \ldots, d(\sigma^1) - d(\sigma^0))$, *where* $\sigma^j = \sigma_1^j \cdots \sigma_{m-p+j-\epsilon}^j$ *and* $\sigma_i^j = \mathrm{lcm}(\alpha_{i-j+\epsilon+p}, \delta_{i+\epsilon-u}), \quad j \in \{0, \ldots, p\}, \quad i \in \{1, \ldots, m-p+j-\epsilon\}, \quad \epsilon = t_1 + \cdots + t_u$.

(iii_4) $m \geqslant d(\eta^u)$ *and* $(t_1, \ldots, t_u) \prec (m - d(\eta^{u-1}), d(\eta^{u-1}) - d(\eta^{u-2}), \ldots, d(\eta^1) - d(\eta^0))$, *where* $\eta^j = \eta_1^j \cdots \eta_{m-u+j}^j$ *and* $\eta_i^j = \mathrm{lcm}(\delta_{i-j}, \beta_{i-2m+2h}), j \in \{0, \ldots, u\}, \quad i \in \{1, \ldots, m-u+j\}$.

Proof. *Necessary condition*. Suppose that the matrices (3) and (4) are strictly equivalent. Let $\gamma_1 \mid \cdots \mid \gamma_m$ be the homogeneous invariant factors of

$$xI_m - \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}. \tag{5}$$

According to Theorem 3, the condition (b_3') is satisfied. It follows from the Sá-Thompson's interlacing theorem [4,8] that (also see [1])

$$\gamma_i \mid \beta_i, \qquad i \in \{1, \ldots, h\}, \tag{6}$$

$$\beta_i \mid \gamma_{i+2m-2h}, \qquad i \in \{1, \ldots, 2h-m\}. \tag{7}$$

Then (i_4) follows from (i_3), (6) and (7). Note that the conditions (ii_4) and (ii_3) coincide. Let us prove (iii_4). Let $\eta_i'^j = \mathrm{lcm}(\delta_{i-j}, \gamma_i)$, $\eta'^j = \eta_1'^j \cdots \eta_{m-u+j}'^j$, $j \in \{0, \ldots, u\}, i \in \{1, \ldots, m-u+j\}$. As in [4 Proposition 4.1], it can be shown that the sequence $(\eta'^u, \ldots, \eta'^0)$ has a convex degree function, that is

$$d(\eta'^{j+1}) - d(\eta'^j) \geqslant d(\eta'^j) - d(\eta'^{j-1}), \qquad j \in \{1, \ldots, u-1\}.$$

From (6) and (7) it follows that $d(\eta'^j) \geqslant d(\eta^j)$, $j \in \{0, \ldots, u\}$. Also note that $\eta'^0 = \eta^0 = \delta_1 \cdots \delta_{m-u}$. Then ($iii_4$) follows from ($iii_3$).

Sufficient condition. Suppose that (b_4) is satisfied. For $h = m$ the theorem had already been proved. Suppose now that $h < m$.

As the divisors of homogeneous polynomials are homogeneous, δ_i is homogeneous, for every $i \leqslant \max\{m-p-u, h-u\}$. As the pencils $xI_h - A_{1,1}$ and (3) do not have infinite elementary divisors, y does not divide $\beta_h \alpha_m$. Therefore y does not divide δ_i, for every $i \leqslant \max\{m-p-u, h-u\}$.

For $i > \max\{m-p-u, h-u\}$, we assume, without loss of generality, that δ_i is homogeneous and is not a multiple of y. Otherwise, suppose that $\delta_i = l_i \tilde{h}_i$, where l_i does not have homogeneous divisors different from y. Note that $\gcd(l_i, \alpha_m \beta_h) = 1$. As F is infinite, one can choose $a \in F$ such that $\gcd(x - ay, \alpha_m \beta_h) = 1$. Then the conditions that result from (i_4), (ii_4), (iii_4) on replacing δ_i by $\delta_i^* = (x - ay)^{d(l_i)} \tilde{h}_i$, $i > \max\{m-p-u, h-u\}$, are satisfied.

Let

$$\gamma_i = \mathrm{lcm}(\delta_{i-u}, \beta_{i-2m+2h}), \qquad i \in \{1, \ldots, m-1\}, \tag{8}$$

$$\gamma_m = \mathrm{lcm}(\delta_{m-u}, \beta_{2h-m})\tilde{\chi}, \tag{9}$$

where $\chi \in F[x]$ is a monic polynomial such that $d(\chi) = m - d(\eta^u)$. From the previous remarks it follows that the polynomials $\gamma_1, \ldots, \gamma_m$ are homogeneous and are not multiples of y. Therefore $\gamma_i = \tilde{s}_i$, for some monic polynomial s_i, $i \in \{1, \ldots, m\}$. Analogously, $\beta_i = \tilde{g}_i$, $i \in \{1, \ldots, h\}$, where $g_1 \mid \cdots \mid g_h$ are the (nonhomogeneous) invariant factors of $xI_h - A_{1,1}$.

From (i_4), (8) and (9) it follows that (6) and (7) are satisfied. Consequently,

$$s_i \mid g_i, \qquad i \in \{1, \ldots, h\},$$
$$g_i \mid s_{i+2m-2h}, \quad i \in \{1, \ldots, 2h-m\}.$$

According to the Sá-Thompson's interlacing theorem [4,8], there exist matrices $A_{1,2}$, $A_{2,1}$ and $A_{2,2}$ such that (5) has invariant factors $s_1 \mid \cdots \mid s_m$.

From (i_4), (8) and (9) it also follows that (i_3) is satisfied. (Note that if $u = 0$ then all the column minimal indices of (3) are zero and $\epsilon = 0$. Then, from (ii_4), it follows that $k_1 + \cdots + k_p = d(\delta_1 \cdots \delta_m) - d(\alpha_1 \cdots \alpha_m)$. As (3) does not have infinite elementary divisors, $k_1 + \cdots + k_p + d(\alpha_1 \cdots \alpha_m) = m$. Therefore $d(\eta^0) = d(\delta_1 \cdots \delta_m) = m$ and $\chi = 1$.)

From (i_4) and the definition of γ_i it follows that, for every $j \in \{0, \ldots, u\}$, $i \in \{1, \ldots, m-u+j\}$, with $i \neq m$,

$$\mathrm{lcm}(\delta_{i-j}, \beta_{i-2m+2h}) = \mathrm{lcm}(\delta_{i-j}, \gamma_i),$$

while

$$\mathrm{lcm}(\delta_{m-u}, \beta_{2h-m})\tilde{\chi} = \mathrm{lcm}(\delta_{m-u}, \gamma_m).$$

Therefore

$$d\left(\prod_{i=1}^{m} \mathrm{lcm}(\delta_{i-u}, \gamma_i)\right) = m$$

and (iii_3) follows from (iii_4).

As conditions (ii_3) and (ii_4) coincide, (b'_3) is satisfied. According to Theorem 3, there exist matrices $A_{1,3}, A_{2,3}, A_{3,1}, A_{3,2}$ such that (3) and (4) are strictly equivalent. □

The following two lemmas are easy to prove.

Lemma 5. *Let $S(x)$, $S'(x)$, $D(x)$ be matrix pencils, with $S(x)$ strictly equivalent to $S'(x)$.*

There exists a pencil $E(x)$ strictly equivalent to $D(x)$ containing $S(x)$ as a subpencil if and only if there exists a pencil $E'(x)$ strictly equivalent to $D(x)$ containing $S'(x)$ as a subpencil.

Lemma 6. *Let $A_{1,1} \in F^{h\times h}$, let $D(x)$ be a matrix pencil without infinite elementary divisors and* rank $D(x) = m \geqslant h$. *Then*:

(a_6) *$D(x)$ is strictly equivalent to a pencil of the form* (3).

(b_6) *There exists a pencil $E(x)$ strictly equivalent to $D(x)$ containing $xI_h - A_{1,1}$ as a subpencil if and only if there exist matrices $A_{1,2}, A_{1,3}, A_{2,1}, A_{2,2}, A_{2,3}, A_{3,1}$ and $A_{3,2}$, with entries in F, such that* (4) *and $D(x)$ are strictly equivalent.*

Let

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in F^{2\times 2}$$

be a nonsingular matrix. If $xA + B$ is a pencil, where A, B have entries in F, then

$$P_X(xA + B) = x(aA + cB) + (bA + dB).$$

If $f(x, y) \in F[x, y]$, then

$$\Pi_X(f) = f(xa + yb, xc + yd).$$

The transformations P_X and Π_X were introduced in [2]. The following lemmas are easy to prove. For details, see [2].

Lemma 7. (a_7) *P_X is invertible and $(P_X)^{-1} = P_{X^{-1}}$.*

(b_7) *Two pencils $D(x)$ and $E(x)$ are strictly equivalent if and only if $P_X(D)$ and $P_X(E)$ are strictly equivalent.*

(c_7) *Given two pencils, $D(x)$ and $S(x)$, there exists a pencil $E(x)$ strictly equivalent to $D(x)$ containing $S(x)$ as a subpencil if and only if there exists a pencil $E'(x)$ strictly equivalent to $P_X(D)$ containing $P_X(S)$ as a subpencil.*

Lemma 8. (a_8) *Π_X is invertible and $(\Pi_X)^{-1} = \Pi_{X^{-1}}$.*

(b_8) *$\Pi_X(fg) = \Pi_X(f)\Pi_X(g)$, for every $f, g \in F[x, y]$.*

(c_8) *$d(\Pi_X(f)) = d(f)$, for every $f \in F[x, y]$.*

Theorem 9. *Let $D(x)$ be an $m' \times n'$ matrix pencil, $\alpha_1 \mid \cdots \mid \alpha_w$ its homogeneous invariant factors, $k_1 \geqslant \cdots \geqslant k_{m'-w}$ its row minimal indices and $t_1 \geqslant \cdots \geqslant t_{n'-w}$ its column minimal indices. Let u be the number of nonzero column minimal indices and p be the number of nonzero row minimal indices of $D(x)$. Let $S(x) \in F^{h\times h}$, $h \leqslant w$, be a regular pencil and $\beta_1 \mid \cdots \mid \beta_h$ be its homogeneous invariant factors. The following conditions are equivalent*:

(a_9) *There exists a pencil $E(x)$ strictly equivalent to $D(x)$ containing $S(x)$ as a subpencil.*

(b_9) *There exist nonzero polynomials $\delta_1 \mid \cdots \mid \delta_{w-u}$ such that the following conditions hold:*

(i_9) $\operatorname{lcm}(\alpha_{i+u}, \beta_{i-2w+2h}) \mid \delta_i \mid \gcd(\alpha_{i+p+u}, \beta_{i+u}), \quad i \in \{1, \ldots, w-u\}.$

(ii_9) $(k_1, \ldots, k_p) \prec (d(\sigma^p) - d(\sigma^{p-1}), \ldots, d(\sigma^1) - d(\sigma^0))$, *where* $\sigma^j = \sigma_1^j \cdots \sigma_{w-p+j-\epsilon}^j$ *and* $\sigma_i^j = \operatorname{lcm}(\alpha_{i-j+\epsilon+p}, \delta_{i+\epsilon-u}), \quad j \in \{0, \ldots, p\}, \quad i \in \{1, \ldots, w-p+j-\epsilon\}, \quad \epsilon = t_1 + \cdots + t_u$.

(iii_9) $w \geqslant d(\eta^u)$ *and* $(t_1, \ldots, t_u) \prec (w - d(\eta^{u-1}), d(\eta^{u-1}) - d(\eta^{u-2}), \ldots, d(\eta^1) - d(\eta^0))$, *where* $\eta^j = \eta_1^j \cdots \eta_{w-u+j}^j$ *and* $\eta_i^j = \operatorname{lcm}(\delta_{i-j}, \beta_{i-2w+2h})$, $j \in \{0, \ldots, u\}, \quad i \in \{1, \ldots, w-u+j\}$.

Proof. *Case* 1. Suppose that $D(x)$ and $S(x)$ do not have infinite elementary divisors. According to Lemma 6, $D(x)$ is strictly equivalent to a pencil of the form (3), where $m = w$, and $S(x)$ is strictly equivalent to a pencil of the form $xI_h - A_{1,1}$, with $A_{1,1} \in F^{h \times h}$. The proof is a simple consequence of Lemmas 5, 6 and Theorem 4.

Case 2. Now consider the general case. As F is infinite, one can choose a nonsingular matrix $X \in F^{2\times 2}$ such that y does not divide $\Pi_X(\alpha_w)\Pi_X(\beta_h)$.

According to [2, Lemma 10], $\Pi_X(\alpha_1) \mid \cdots \mid \Pi_X(\alpha_w)$ and $\Pi_X(\beta_1) \mid \cdots \mid \Pi_X(\beta_h)$ are the homogeneous invariant factors of $P_X(D)$ and $P_X(S)$, respectively, while the minimal indices of $P_X(D)$ and $P_X(S)$ coincide with the minimal indices of D and S, respectively. Bearing in mind the choice of X, $P_X(D)$ and $P_X(S)$ do not have infinite elementary divisors.

From Lemma 8, it follows that (b_9) is equivalent to the condition (b_9') that results from it on replacing the polynomials α_i, β_i, δ_i by $\Pi_X(\alpha_i)$, $\Pi_X(\beta_i)$, $\Pi_X(\delta_i)$, respectively.

According to Case 1, (b_9') is satisfied if and only if there exists a pencil $E'(x)$ strictly equivalent to $P_X(D)$ containing $P_X(S)$ as a subpencil. According to Lemma 7, this last statement is equivalent to (a_9). □

3. The similarity class and the characteristic polynomial of $A + BY + XC + XDY$

Theorem 10. *Let* $A, A' \in F^{m\times m}$, $B \in F^{m\times r}$, $C \in F^{s\times m}$ *and* $D \in F^{s\times r}$. *Let* $\beta_1 \mid \cdots \mid \beta_m$ *be the homogeneous invariant factors of* $xI_m - A'$. *Let* $\alpha_1 \mid \cdots \mid \alpha_w$ *be the homogeneous invariant factors,* $k_1 \geqslant \cdots \geqslant k_{m+s-w}$ *be the row minimal indices and* $t_1 \geqslant \cdots \geqslant t_{m+r-w}$ *be the column minimal indices of*

$$\begin{bmatrix} xI_m - A & -B \\ -C & -D \end{bmatrix}. \tag{10}$$

Let u be the number of nonzero column minimal indices and p be the number of nonzero row minimal indices of (10). *The following conditions are equivalent*:

(a_{10}) *There exist* $X \in F^{m\times s}$, $Y \in F^{r\times m}$ *such that* $A + BY + XC + XDY$ *is similar to* A'.

(b_{10}) *There exist* $B' \in F^{m\times r}$, $C' \in F^{s\times m}$ *and* $D' \in F^{s\times r}$ *such that the matrices* (10) *and*

$$\begin{bmatrix} xI_m - A' & -B' \\ -C' & -D' \end{bmatrix} \tag{11}$$

are strictly equivalent.

(c_{10}) *There exist nonzero polynomials* $\delta_1 \mid \cdots \mid \delta_{w-u}$ *such that the following conditions hold*:

(i_{10}) $\mathrm{lcm}(\alpha_{i+u}, \beta_{i-2w+2m}) \mid \delta_i \mid \gcd(\alpha_{i+p+u}, \beta_{i+u}), \quad i \in \{1, \ldots, w-u\}$.

(ii_{10}) $(k_1, \ldots, k_p) \prec (d(\sigma^p) - d(\sigma^{p-1}), \ldots, d(\sigma^1) - d(\sigma^0))$, *where* $\sigma^j = \sigma_1^j \cdots \sigma_{w-p+j-\epsilon}^j$ *and* $\sigma_i^j = \mathrm{lcm}(\alpha_{i-j+\epsilon+p}, \delta_{i+\epsilon-u}), \quad j \in \{0, \ldots, p\}, i \in \{1, \ldots, w-p+j-\epsilon\}, \quad \epsilon = t_1 + \cdots + t_u$.

(iii_{10}) $w \geqslant d(\eta^u)$ *and* $(t_1, \ldots, t_u) \prec (w - d(\eta^{u-1}), d(\eta^{u-1}) - d(\eta^{u-2}), \ldots, d(\eta^1) - d(\eta^0))$, *where* $\eta^j = \eta_1^j \cdots \eta_{w-u+j}^j$ *and* $\eta_i^j = \mathrm{lcm}(\delta_{i-j}, \beta_{i-2w+2m}), j \in \{0, \ldots, u\}, \quad i \in \{1, \ldots, w-u+j\}$.

Proof. (a_{10}) *implies* (b_{10}). Suppose that (a_{10}) is satisfied. Let $N \in F^{m\times m}$ be a nonsingular matrix such that $A' = N(A + BY + XC + XDY)N^{-1}$. Then (10) is strictly equivalent to

$$\begin{bmatrix} N & NX \\ 0 & I_s \end{bmatrix} \begin{bmatrix} xI_m - A & -B \\ -C & -D \end{bmatrix} \begin{bmatrix} N^{-1} & 0 \\ YN^{-1} & I_s \end{bmatrix},$$

which has the prescribed form.

(b_{10}) *implies* (a_{10}). Suppose that (10) and (11) are strictly equivalent. Then there exist $P \in F^{m\times m}, R \in F^{m\times s}, S \in F^{s\times s}, Q \in F^{r\times m}, U \in F^{r\times r}$ such that P, S, U are nonsingular and

$$\begin{bmatrix} xI_m - A' & -B' \\ -C' & -D' \end{bmatrix} = \begin{bmatrix} P & R \\ 0 & S \end{bmatrix} \begin{bmatrix} xI_m - A & -B \\ -C & -D \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ Q & U \end{bmatrix}.$$

Then $A' = P(A + P^{-1}RC + BQP + P^{-1}RDQP)P^{-1}$.

It follows immediately from Theorem 9 that (b_{10}) implies (c_{10}).

(c_{10}) *implies* (b_{10}). According to Theorem 9, there exists a pencil of the form

$$\begin{bmatrix} xI_m - A' & B'(x) \\ C'(x) & D'(x) \end{bmatrix} \tag{12}$$

strictly equivalent to (10). As the coefficient of x in (12) has rank equal to m, it is not hard to deduce that B', C', D' may be taken constant. □

Theorem 11. *Let f be a monic polynomial of degree m. Let* $A \in F^{m\times m}$, $B \in F^{m\times r}$, $C \in F^{s\times m}$ *and* $D \in F^{s\times r}$. *Let* $s_1 \mid \cdots \mid s_w$ *be the nonhomogeneous invariant factors of* (10). *The following conditions are equivalent*:

(a_{11}) *There exist* $X \in F^{m\times s}$, $Y \in F^{r\times m}$ *such that* $A + BY + XC + XDY$ *has characteristic polynomial f.*

(b_{11}) *There exist* $A' \in F^{m\times m}$, $B' \in F^{m\times r}$, $C' \in F^{s\times m}$ *and* $D' \in F^{s\times r}$ *such that* A' *has characteristic polynomial f and the matrices* (10) *and* (11) *are strictly equivalent.*

(c_{11}) *The following conditions hold*:

(i_{11}) $s_1 \cdots s_m \mid f$.

(ii_{11}) *If* $w = m$, *then there exists* $g \in F[x]$ *such that* $s_1 \cdots s_m g \mid f$ *and* $d(s_1 \cdots s_m g) = m - \epsilon$, *where* ϵ *is the sum of the column minimal indices of* (10).

Proof. The equivalence between (a_{11}) and (b_{11}) can be proved with arguments analogous to the ones used to prove that (a_{10}) and (b_{10}) are equivalent.

(b_{11}) *implies* (c_{11}). Use the notation of Theorem10. According to that theorem, (c_{10}) is satisfied. Note that $\tilde{s}_i \mid \alpha_i$, $i \in \{1, \ldots, w\}$. From (i_{10}) it follows that

$$\widetilde{s_1 \cdots s_m} \mid \alpha_1 \cdots \alpha_m \mid \delta_1 \cdots \delta_{m-u} \mid \beta_{u+1} \cdots \beta_m \mid \tilde{f}. \tag{13}$$

Then $s_1 \cdots s_m \mid f$.

Now suppose that $w = m$. From (13) it follows that $\delta_1 \cdots \delta_{m-u}$ is a polynomial of the form $a\tilde{h}$, where a is a nonzero constant and $h \in F[x]$ is a monic polynomial. Therefore $h = s_1 \cdots s_m g \mid f$, for some $g \in F[x]$. From (iii_{10}) it follows that $m - \epsilon = d(\eta^0) = d(\delta_1 \cdots \delta_{m-u}) = d(h)$.

(c_{11}) *implies* (b_{11}). Use the notation of Theorem 10 for the Kronecker invariants of (10). Let $A' \in F^{m\times m}$ be a matrix such that $xI_m - A'$ has homogeneous invariant factors $\beta_1 \mid \cdots \mid \beta_m$, where

$$\beta_i = \alpha_i, \qquad i \in \{1, \ldots, m-1\},$$
$$\beta_m = \frac{\tilde{f}}{\alpha_1 \cdots \alpha_{m-1}}.$$

Note that $\alpha_i = \tilde{s}_i$ for $i \in \{1, \ldots, m\}$. Let

$$\delta_i = \alpha_{i+u}, \qquad i \in \{1, \ldots, w-u-1\},$$
$$\delta_{w-u} = \alpha_w \tilde{\chi},$$

where $\chi \in F[x]$ is a monic polynomial of degree $k_1 + \cdots + k_p$ and $\chi = g$ if $w = m$.

It is not hard to see that the conditions (i_{10}), (ii_{10}) and (iii_{10}) are satisfied. According to Theorem 10, (b_{10}) is also satisfied. Hence (b_{11}) holds. □

The next theorem can be proved with similar arguments.

Theorem 12. *Let f be a monic polynomial of degree m. Let* $A \in F^{m\times m}$, $B \in F^{m\times r}$, $C \in F^{s\times m}$ *and* $D \in F^{s\times r}$. *Let* $s_1 \mid \cdots \mid s_w$ *be the nonhomogeneous invariant factors of* (10). *The following conditions are equivalent* :

(a_{12}) *There exist* $X \in F^{m\times s}, Y \in F^{r\times m}$ *such that* $A + BY + XC + XDY$ *is nonderogatory and has characteristic polynomial f.*

(b_{12}) *There exist* $A' \in F^{m\times m}$, $B' \in F^{m\times r}$, $C' \in F^{s\times m}$ *and* $D' \in F^{s\times r}$ *such that* A' *is nonderogatory, has characteristic polynomial f and the matrices* (10) *and* (11) *are strictly equivalent.*

(c_{12}) *The following conditions are satisfied*:

 (i_{12}) $s_1 = \cdots = s_{m-1} = 1$ *and* $s_m \mid f$.

 (ii_{12}) *If* $w = m$, *then there exists* $g \in F[x]$ *such that* $s_m g \mid f$ *and* $d(s_m g) = m - \epsilon$, *where* ϵ *is the sum of the column minimal indices of* (10).

Theorem 13 [5,6]. *Let* $A, A' \in F^{m\times m}$. *Let* $g_1 \mid \cdots \mid g_m$ *be the nonhomogeneous invariant factors of* $xI_m - A$. *Let* $h_1 \mid \cdots \mid h_m$ *be the nonhomogeneous invariant factors of* $xI_m - A'$. *The following conditions are equivalent*:

(a_{13}) *There exists a matrix* $Z \in F^{m\times m}$ *such that* rank $Z \leqslant t$ *and* $A + Z$ *is similar to* A'.

(b_{13}) $g_i \mid h_{i+t}$ *and* $h_i \mid g_{i+t}, i \in \{1, \ldots, m-t\}$.

Proof. Consider the pencil of the form (10), where $r = s = t$, $B = 0$, $C = 0$ and $D = I_t$. Then (a_{13}) $\iff$ (a_{10}) $\iff$ (c_{10}). In this case, $p = u = 0$ and (c_{10}) takes the form:

(c'_{10}) $\mathrm{lcm}(\alpha_i, \beta_{i-2t}) \mid \gcd(\alpha_i, \beta_i), \quad i \in \{1, \ldots, m+t\}$,

where

$$\begin{aligned} \beta_i &= \tilde{h}_i, & i &\in \{1, \ldots, m\}, \\ \alpha_i &= 1, & i &\in \{1, \ldots, t\}, \\ \alpha_i &= \tilde{g}_{i-t}, & i &\in \{t+1, \ldots, m\}, \\ \alpha_i &= \tilde{g}_{i-t} y, & i &\in \{m+1, \ldots, m+t\}. \end{aligned}$$

It is easy to see that (c'_{10}) is equivalent to (b_{13}). $\square$.

Acknowledgements

Work done within the activities of the Thematic Term on Linear Algebra and Control Theory (C.I.M.). Supported by Fundação Calouste Gulbenkian, Centro de Álgebra da Universidade de Lisboa and Praxis Program (praxis/2/2.1/mat/73/94).

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