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Some Observations on Mixed Methods for Fully Nonlinear Parabolic Problems in Divergence Form

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Abstract—Mixed finite element methods are considered to approximate the solution of fully nonlinear second order parabolic problems in divergence form in $\mathbb{R}^d$, $d \leq 3$. Existence and uniqueness of the approximation are proved. Optimal order error estimates in $L^\infty(J; L^2(\Omega))$ and in $L^\infty(J; H(\text{div}; \Omega))$ are demonstrated for the relevant variables.

Keywords—Nonlinear parabolic problems, Mixed methods, Error estimates.

1. INTRODUCTION

Many physical phenomena are described by elliptic and parabolic partial differential equations in divergence form. Traditionally, the models used have been linearizations of strongly nonlinear phenomena [1]. During the last ten years or so, with very powerful new theoretical techniques and the ubiquity of high powered computing, the tendency has changed towards the use of increasingly more complex—and more strongly nonlinear—models. A choice method for the numerical approximation of solutions of elliptic and parabolic problems is the finite element method, frequently in the form of Galerkin or Ritz-Galerkin methods [2–4]. A more modern form of the finite element method was conceived by Brezzi twenty years ago [5] for the approximation of solutions of saddle point problems, rather than minimization problems—as in the case of Galerkin methods. The new methods were named mixed finite element methods, and a considerable literature was produced focusing on their analysis, especially for linear second order elliptic differential equations and systems [6–23]. Much less abundant in the literature on mixed methods applied to nonlinear elliptic problems [12,21,24,25]. Only a few papers dealt with parabolic problems using mixed finite element methods [26–29]. Very recently, the method was applied to a strongly nonlinear parabolic problem for the first time, generalized Forchheimer flow in porous

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Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX media [30,31]. In the present paper, we show how the methods employed in [31] can be extended to the analysis of the fully nonlinear second order parabolic problem in divergence form.

In the next section, we give the details of the formulation of the continuous-time mixed finite element method. In Section 3, we prove existence and uniqueness of the finite element solution and in Section 4, we derive L^2-error estimates.

2. THE MIXED FINITE ELEMENT PROCEDURE

We shall consider a time interval $J = [0, T]$ and a d-dimensional domain $\Omega \subset\subset \mathbb{R}^d$, $d \le 3$, with C^2-boundary $\partial\Omega$ (or smoother if necessary for the regularity of the solution of (2.1) below), and the following initial-boundary value problem defined in it:

$$\begin{aligned} c(\mathbf{x}, p)\, \frac{\partial p}{\partial t} - \operatorname{div}\left[\mathbf{a}(\mathbf{x}, p, \nabla p)\right] + \alpha(\mathbf{x}, p, \nabla p) &= 0, && (\mathbf{x}, t) \in \Omega \times J, \\ p &= -g, && (\mathbf{x}, t) \in \partial\Omega \times J, \\ p(0) &= p_0, && \mathbf{x} \in \Omega. \end{aligned} \tag{2.1}$$

A mixed weak form of (2.1) will be derived by introducing the flux

$$\mathbf{u} = -\mathbf{a}(\mathbf{x}, p, \nabla p). \tag{2.2}$$

We shall assume that this relation can be inverted based on the implicit function theorem as

$$\nabla p = -\mathbf{b}(\mathbf{x}, p, \mathbf{u}). \tag{2.3}$$

Next, let

$$f(p, \mathbf{u}) = -\alpha(p, \nabla p) = -\alpha(p, -\mathbf{b}(p, \mathbf{u})) \tag{2.4}$$

and introduce the enthalpy, $H(p)$, as a new dependent variable,

$$H(p) = \int_0^p c(s)\, ds. \tag{2.5}$$

Then, (2.1)–(2.5),

$$\begin{aligned} H(p)_t + \operatorname{div} \mathbf{u} &= f(p, \mathbf{u}), && \text{in } \Omega \times J, \\ \mathbf{b}(p, \mathbf{u}) + \nabla p &= 0, && \text{in } \Omega \times J, \\ p &= -g, && \text{on } \partial\Omega \times J, \\ p &= p_0, && \text{in } \Omega \times \{0\}. \end{aligned} \tag{2.6}$$

Here and in the sequel, the subindex t is used to denote differentiation in time. We shall make the following assumptions on the coefficients of (2.1) and (2.5):

(A1) $c = c(p) \in W^{2,\infty}(\mathbb{R})$, $0 < 1/K \le c$, and $p_0 \in L^2(\Omega)$.
(A2) $(\mathbf{b}(p, \mathbf{z}), \mathbf{z}) \ge C_1 \|\mathbf{z}\|^2 - C_0 \|p\|^2$.
(A3) $(\mathbf{b}(p, \mathbf{z}_1) - \mathbf{b}(p, \mathbf{z}_2), \mathbf{z}_1 - \mathbf{z}_2) \ge \delta_0 \|\mathbf{z}_1 - \mathbf{z}_2\|^2$ or $\delta_0 I \le \frac{\partial \mathbf{b}}{\partial \mathbf{z}}(p, \mathbf{z})$.
(A4) $\mathbf{b}(p, \mathbf{z})$ and $f(p, \mathbf{z})$ are C_B^2 in their arguments.
(A5) $\{\mathbf{u}, p\} \in W^{1,\infty}(J; H(\operatorname{div}; \Omega) \times L^2(\Omega))$ is the unique solution of the mixed form (2.6).
(A6) $f(p(t, \mathbf{x}), \mathbf{u}(t, \mathbf{x})) \in W^{2,\infty}(J; H^{k+2}(\Omega))$, $\mathbf{u} \in W^{2,\infty}(J; H^{k+2}(\Omega)^d)$, and $\operatorname{div} \mathbf{u}$, $p \in W^{2,\infty}(J; H^{k+2}(\Omega))$.

REMARK 1. (A2) is a Gårding-type inequality. (A3) amounts to the ellipticity of the associated elliptic operator. (A6) is needed for higher order approximations.

We now introduce the Hilbert spaces

$$\mathbf{V} = H(\operatorname{div}; \Omega) \quad \text{and} \quad W = L^2(\Omega),$$

and consider the following mixed weak formulation of (2.1).

Find $(\mathbf{u}, p) : J \to \mathbf{V} \times W$ such that

$$\begin{aligned} (\mathbf{b}(p,\mathbf{u}),\mathbf{v}) - (\operatorname{div}\mathbf{v}, p) &= \langle g, \mathbf{v}\cdot\boldsymbol{\nu}\rangle, && \mathbf{v}\in\mathbf{V},\\ (H(p)_t, w) + (\operatorname{div}\mathbf{u}, w) &= (f(p,\mathbf{u}), w), && w\in W,\\ p &= p_0, && \text{for } t=0. \end{aligned} \tag{2.7}$$

The notations $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ stand, respectively, for the standard inner product in W or W^d, and in the Hilbert space $L^2(\partial\Omega)$.

We consider now a quasi-uniform family of decomposition of Ω, $\mathcal{T}_h$, with boundary elements allowed to have one curved edge or side. Associated with it, consider the Raviart-Thomas-Nedelec mixed finite space [20,22], or the Brezzi-Douglas-Marini [6] space of index $k > 0$, $\mathbf{V}_h \times W_h$. Then, the continuous-time mixed finite element method we shall analyze is the following.

Find $(\mathbf{u}_h, p_h) : J \to \mathbf{V}_h \times W_h$ such that

$$\begin{aligned} (\mathbf{b}(p_h,\mathbf{u}_h),\mathbf{v}) - (\operatorname{div}\mathbf{v}, p_h) &= \langle g, \mathbf{v}\cdot\boldsymbol{\nu}\rangle, && \mathbf{v}\in\mathbf{V}_h,\\ (H(p_h)_t, w) + (\operatorname{div}\mathbf{u}_h, w) &= (f(p_h,\mathbf{u}_h), w), && w\in W_h,\\ p_h &= P_h p_0, && \text{for } t=0, \end{aligned} \tag{2.8}$$

where P_h denotes the L^2 orthogonal projection of W onto W_h.

3. EXISTENCE AND UNIQUENESS

We have the following theorem concerning the stability of the mixed method (2.8).

THEOREM 3.1. *There exists a constant C, which depends on the characteristic parameter h, such that*

$$\|p_h\|_{L^\infty(J;L^2(\Omega))} + \|\mathbf{u}_h\|_{L^\infty(J;L^2(\Omega)^d)} \le C\left[\|f\|_{L^2(J;L^2(\Omega))} + \|g\|_{L^2(J;L^2(\partial\Omega))} + \|p_0\|_{L^2(\Omega)}\right].$$

Here we have used the notation $f = f(p(t,\mathbf{x}), \mathbf{u}(t,\mathbf{x}))$.

The proof of this theorem is essentially identical with that of [31] for Forchheimer flows and we shall omit it here.

Next note that this is an *a priori* estimate sufficient to guarantee the existence of a solution of (2.8). In order to establish uniqueness, assume $(\mathbf{u}_h^{(i)}, p_h^{(i)}) \in \mathbf{V}_h \times W_h$ is a solution of (2.8), $i = 1$ or 2. Consider now $\mathbf{U} = \mathbf{u}_h^{(1)} - \mathbf{u}_h^{(2)} \in \mathbf{V}_h$ and $P = p_h^{(1)} - p_h^{(2)} \in W_h$. Subtract the relations (2.8) for $\mathbf{u}_h = \mathbf{u}_h^{(2)}$ and $p_h = p_h^{(2)}$ from those obtained for $\mathbf{u}_h = \mathbf{u}_h^{(1)}$ and $p_h = p_h^{(1)}$. In the resulting relations, let $\mathbf{v} = \mathbf{U}$ and $w = P$. Then, we have

$$\begin{aligned} &\left(\mathbf{b}\left(p_h^{(1)}, \mathbf{u}_h^{(1)}\right) - \mathbf{b}\left(p_h^{(2)}, \mathbf{u}_h^{(2)}\right), \mathbf{U}\right) - (\operatorname{div}\mathbf{U}, P) = 0,\\ &\quad\left(H\left(p_h^{(1)}\right)_t - H\left(p_h^{(2)}\right)_t, P\right) + (\operatorname{div}\mathbf{U}, P) = \left(f\left(p_h^{(1)}, \mathbf{u}_h^{(1)}\right) - f\left(p_h^{(2)}, \mathbf{u}_h^{(2)}\right), P\right), \end{aligned}$$

and, adding these relations and using the mean value theorem and hypothesis (A1), it can be seen from Gronwall's lemma, just as in [31], that

$$\|P\|_{L^\infty(J;L^2(\Omega))} + \|\mathbf{U}\|_{L^2(J;L^2(\Omega)^d)} \le C_3\|P(0)\|_W = 0,$$

which means $P = 0$ and $\mathbf{U} = 0$.

Therefore, we have demonstrated the following result.

THEOREM 3.2. *The mixed finite element problem (2.8) admits a unique solution $(\mathbf{u}_h, p_h) : J \to \mathbf{V}_h \times W_h$.*

4. L^2-ERROR ESTIMATES

We shall make use of the elliptic projection of the solution of (2.6), for each $t \in J$, onto $\mathbf{V}_h \times W_h$. Choose $\lambda > \delta_1^2/2\delta_0$, where δ_0 is the constant in hypothesis (A3) and δ_1 is the supreme of $\frac{\partial \mathbf{b}}{\partial p}$. Next, define $(\mathbf{u}^*, p^*) : J \to \mathbf{V}_h \times W_h$ pointwise through the relations

$$\begin{aligned} (\mathbf{b}(p^*, \mathbf{u}^*) - \mathbf{b}(p, \mathbf{u}), \mathbf{v}) - (\operatorname{div} \mathbf{v}, p^* - p) &= 0, \qquad \mathbf{v} \in \mathbf{V}_h, \\ (\operatorname{div}(\mathbf{u}^* - \mathbf{u}), w) + \lambda(p^* - p, w) &= 0, \qquad w \in W_h. \end{aligned} \tag{4.1}$$

The existence, for each $t \in J$, of a unique $(\mathbf{u}^*(t), p^*(t)) \in \mathbf{V}_h \times W_h$ follows from [25,32], since (4.1) corresponds to the mixed method for the elliptic problem

$$-\operatorname{div}[\mathbf{a}(\mathbf{x}, p, \nabla p)] + \lambda p = 0.$$

Furthermore, the following error estimates follow from [25,32].

LEMMA 4.1. *Let $k \geq 1$. Then, for h sufficiently small,*

(a) $\|p - p^*\| \leq K h^{k+1}(\|p\|_{k+1} + \|\mathbf{u}|_k)$,
(b) $\|\mathbf{u} - \mathbf{u}^*\| \leq K h^{k+1}(\|p\|_{k+1} + \|\mathbf{u}\|_{k+1})$,
(c) $\|\operatorname{div}(\mathbf{u} - \mathbf{u}^*)\| \leq K h^{k+1}(\|p\|_{k+2} + \|\mathbf{u}\|_{k+2})$,
(d) $\|p - p^*\|_{0,\infty} \leq K h^{k+1-(1/2)\delta_{d3}}(\|p\|_{k+1-(1/2)\delta_{d3},\infty} + \|\mathbf{u}\|_{k+1-(1/2)\delta_{d3},\infty})$,

where the constant K is independent of h.

Since the elliptic projection commutes with differentiation in time, we also have the following estimates.

LEMMA 4.2. *Let $k \geq 1$. Then, for h sufficiently small,*

(a) $\|(p - p^*)_t\| + \|(\mathbf{u} - \mathbf{u}^*)_t\| \leq C h^s, \quad 1 \leq s \leq k + 1$,
(b) $\|(p - p^*)_{tt}\| + \|(\mathbf{u} - \mathbf{u}^*)_{tt}\| \leq C h^s, \quad 1 \leq s \leq k + 1$,

where the constant C is independent of h.

REMARK 2. Note that, using Lemma 4.2 and hypothesis (A6), we see that

$$\begin{aligned} &\|p_t^*\|_{L^\infty(J;L^\infty(\Omega))}, \qquad \|p_{tt}^*\|_{L^\infty(J;L^\infty(\Omega))}, \\ &\|\mathbf{u}_t^*\|_{L^\infty(J;L^\infty(\Omega)^d)}, \qquad \|\mathbf{u}_{tt}^*\|_{L^\infty(J;L^\infty(\Omega)^d)} \end{aligned}$$

are finite.

We obtain now our first error equations by subtracting (2.8) from (2.7).

$$\begin{aligned} &(\mathbf{b}(p_h, \mathbf{u}_h) - \mathbf{b}(p, \mathbf{u}), \mathbf{v}) - (\operatorname{div} \mathbf{v}, p_h - p) = 0, && \mathbf{v} \in \mathbf{V}_h, \\ &(H(p_h)_t - H(p)_t, w) + (\operatorname{div}(\mathbf{u}_h - \mathbf{u}), w) = (f(p_h, \mathbf{u}_h) - f(p, \mathbf{u}), w), && w \in W_h. \end{aligned} \tag{4.2}$$

Using (4.1), we rewrite (4.2) in the following form:

$$\begin{aligned} &(\mathbf{b}(p_h, \mathbf{u}_h) - \mathbf{b}(p^*, \mathbf{u}^*), \mathbf{v}) - (\operatorname{div} \mathbf{v}, p_h - p^*) = 0, && \mathbf{v} \in \mathbf{V}_h, \\ &\quad (H(p_h)_t - H(p^*)_t, w) + (\operatorname{div}(\mathbf{u}_h - \mathbf{u}^*), w) = (H(p)_t - H(p^*)_t, w) \\ &\qquad + \lambda(p^* - p, w) + (f(p_h, \mathbf{u}_h) - f(p, \mathbf{u}), w), && w \in W_h. \end{aligned} \tag{4.3}$$

In order to simplify the notation, let

$$\begin{aligned} \boldsymbol{\zeta} &= \mathbf{u}_h - \mathbf{u}^*, \qquad \boldsymbol{\sigma} = \mathbf{u}^* - \mathbf{u}, \\ \xi &= p_h - p^*, \qquad \eta = p^* - p. \end{aligned} \tag{4.4}$$

The following theorem gives the first error estimates.

THEOREM 4.3. *For h sufficiently small,*

$$\|\mathbf{u}_h - \mathbf{u}\|_{L^2(J;L^2(\Omega)^d)} + \|p_h - p\|_{L^\infty(J;L^2(\Omega))} \le Ch^{k+1}, \qquad k \ge 1,$$

where the constant C is independent of h, but depends on norms of $\mathbf{u}$ and p.

PROOF. It is easy to see that

$$(f(p_h, \mathbf{u}_h) - f(p, \mathbf{u}), \xi) \le \varepsilon \|\boldsymbol{\zeta}\|^2 + C\left[\|\boldsymbol{\sigma}\|^2 + \|\xi\|^2 + \|\eta\|^2\right]. \tag{4.5}$$

Following again [31], the result follows since the left-hand side of this inequality is the only additional term resulting from our general parabolic operator, and all the terms on the right-hand side of (4.5) were already on the right-hand side of the bounding inequality in [31]. ∎

We can now prove our main result.

THEOREM 4.4. *Let $k \ge 1$. Then, for h sufficiently small,*

$$\|\mathbf{u}_h - \mathbf{u}\|_{L^\infty(J;L^2(\Omega)^d)} + \|(\mathbf{u}_h - \mathbf{u})_t\|_{L^2(J;L^2(\Omega)^d)} + \|(p_h - p)_t\|_{L^\infty(J;L^2(\Omega))} \le Ch^{k+1},$$

where the constant C is independent of h.

PROOF. Comparing again with [31], we see that the only new terms arising in the proof can be treated as follows:

$$(f(p_h, \mathbf{u}_h) - f(p, \mathbf{u}), \xi_t) \le \varepsilon \|\xi_t\|^2 + C\left[\|\xi\|^2 + \|\eta\|^2 + \|\boldsymbol{\zeta}\|^2 + \|\boldsymbol{\sigma}\|^2\right], \tag{4.6}$$

and

$$\begin{aligned}(f(p_h, \mathbf{u}_h)_t - f(p, \mathbf{u})_t, \xi_t) &= (f_{\mathbf{u}_h} \cdot (\boldsymbol{\zeta}_t + \boldsymbol{\sigma}_t) + (f_{\mathbf{u}_h} - f_{\mathbf{u}}) \cdot \mathbf{u}_t + f_{p_h}(\xi_t + \eta_t) + (f_{p_h} - f_p)p_t, \xi_t) \\ &\le \varepsilon \|\boldsymbol{\zeta}_t\|^2 + C\left[\|\boldsymbol{\sigma}_t\|^2 + \|\boldsymbol{\sigma}\|^2 + \|\boldsymbol{\zeta}\|^2 + \|\eta_t\|^2 + \|\eta\|^2 + \|\xi\|^2 + \|\xi_t\|^2\right].\end{aligned} \tag{4.7}$$

Proceeding as in [31], we combine (4.6) and (4.7) with the rest of the terms to complete the proof. ∎

Finally, for completeness, we can derive an optimal error estimate for the divergence of the flux.

THEOREM 4.5. *Let $k \ge 1$. Then, for h sufficiently small,*

$$\|\mathrm{div}\,(\mathbf{u} - \mathbf{u}_h)\|_{L^\infty(J;L^2(\Omega))} \le Ch^{k+1},$$

where the constant C is independent of h.

PROOF. Combine Theorems 4.3 and 4.4 with the second equation in (4.3). ∎

5. L^∞-ERROR ESTIMATES

We can also establish pointwise error estimates.

THEOREM 5.1. *Let $k \ge 1$. Then, for h sufficiently small,*

(a) $\|p - p_h\|_{L^\infty(J;L^\infty(\Omega))} \le Ch^{k+1}(\log h^{-1})$, $(d = 2)$,
(d) $\|p - p_h\|_{L^\infty(J;L^\infty(\Omega))} \le Ch^{k+(1/2)}$, $(d = 3)$,

where the constant C is independent of h.

PROOF. Apply Lemma 2.1 of [26] (which is also true for the case $d = 3$) to the first equation in (4.3). ∎

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