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On binary reflected Gray codes and functions

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Abstract

The binary reflected Gray code function b is defined as follows: If m is a nonnegative integer, then $b(m)$ is the integer obtained when initial zeros are omitted from the binary reflected Gray code of m.

This paper examines this Gray code function and its inverse and gives simple algorithms to generate both. It also simplifies Conder's result that the jth letter of the kth word of the binary reflected Gray code of length n is

$$\binom{2^n - 2^{n-j} - 1}{\lfloor 2^n - 2^{n-j-1} - k/2 \rfloor} \bmod 2$$

by replacing the binomial coefficient by

$$\left\lfloor \frac{k-1}{2^{n-j+1}} + \frac{1}{2} \right\rfloor.$$

Keywords: Binary reflected Gray codes

1. Introduction

A binary Gray code of length n is a sequence $s_0, s_1, \ldots, s_{2^n-1}$ of the 2^n distinct n-bit strings (or words) of 0s and 1s, with the property that each s_i differs from s_{i+1} in only one digit. Gray codes were first designed to speed up telegraphy, but now have numerous applications such as in addressing microprocessors, hashing algorithms, distributed systems, detecting/correcting channel noise and in solving problems such as the Towers of Hanoi, Chinese Ring and Brain and Spinout. Cyclic binary Gray codes of length n also describe Hamiltonian paths around an n-dimensional hypercube.

For the binary reflexive Gray code of length n, defined below, we will write $b(m)$ for s_m represented as an integer; $b(m)$ will be independent of n.

In this paper we study the function b, its orbits and the decoding function b^{-1}, and give new simple methods for evaluating $b(m)$ and $b^{-1}(m)$.

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Table 1

m	m in binary	BRGC of m	$b(m)$
0	0	0	0
1	1	1	1
2	10	11	3
3	11	10	2
4	100	110	6
5	101	111	7
6	110	101	5
7	111	100	4

One result, in Theorem 6(ii), is that

$$(b(m))_i = \left\lfloor \frac{m}{2^{i+1}} + \frac{1}{2} \right\rfloor \bmod 2$$

(where n_i denotes the coefficient of 2^i in the binary expansion of n); this is a major simplification of a result of Conder in [2].

Also we show that $b(m)$ can be represented using Nim sums.

2. Binary reflexive Gray codes

Table 1 above will help illustrate the construction of the binary reflexive Gray code (BRGC) of any length n. Decimal values of m and of the BRGC of m, written as $b(m)$, are also given.

The initial line, representing: 0 is the BRGC (of length 1) of 0, is given. Further lines are then generated by drawing (for successively $k = 0, 1, 2, \ldots, n-1$) a line below $m = 2^k - 1$ and doing a reflection about the line of all the numbers in the BRGC of m column. Then 2^k (i.e. a 1 in the currently empty $k+1$th place) is added. Finally, after the $k = n-1$ case of this algorithm, initial 0s can be added to make the words of length n, giving binary reflexive Gray codes of length n (BRGC(n)). For each n, the top half of the table for the BRGC(n) of m (that is for $0 \leqslant m < 2^{n-1}$), with the initial zero deleted, will show the BRGC($n-1$) of m.

Table 1, once the initial 0s are added, gives BRGC(3).

3. The function b

Clearly from the above description b is given by

Definition 1. $b(0) = 0, \quad b(2^k + i) = b(2^k - i - 1) + 2^k \quad (0 \leqslant i < 2^k)$.

The Gray Code properties, given this definition, will be proved in Section 4. For this we need some notation and some lemmas.

Notation: We will sometimes write a nonnegative integer m as $m_k m_{k-1} \ldots m_0$ where m_i (0 or 1) is the coefficient of 2^i in the binary expansion of m. We assume $m_k = 1$ unless $k = 0$ and $m_0 = 0$. If $k > 0$ we will let m_p denote the first 0 (if any) from the left in $m_k m_{k-1} \ldots m_0$.

Lemma 1. (i) $b(m_k m_{k-1} \ldots m_0) = 2^k + 2^p + b(m_{p-1} \ldots m_0)$.

(ii) *If* $m_k = m_{k-1} \ldots = m_0 = 1$ *then* $b(m_k m_{k-1} \ldots m_0) = 2^k$, *that is* $b(2^{k+1} - 1) = 2^k$ *for all* $k \geqslant 0$.

Proof. By Definition 1:

(i)
$$
\begin{aligned}
&b(2^k+2^{k-1}+\cdots+2^{p+1}+m_{p-1}2^{p-1}+\cdots+m_0)\\
&\quad=2^k+b(2^k-2^{k-1}\cdots-2^{p+1}-m_{p-1}2^{p-1}-\cdots m_0-1)\\
&\quad=2^k+b(2^p+2^p-m_{p-1}2^{p-1}-\cdots-m_0-1)\\
&\quad=2^k+2^p+b(m_{p-1}2^{p-1}+\cdots+m_0).
\end{aligned}
$$

(ii)
$$
\begin{aligned}
b(2^k+2^{k-1}+\cdots+2+1)&=2^k+b(2^k-2^{k-1}-\cdots-1-1)\\
&=2^k+b(0)=2^k. \qquad \square
\end{aligned}
$$

Corollary 2. *If* $2^k-2^{p+1}\leqslant j<2^k-2^p$, *then* $b(2^k+j)=b(j+2^{p+1}-2^k)+2^p+2^k$.

Corollary 3. *If* $0\leqslant j<2^{k-1}$, *then* $b(2^k+j)=b(j)+2^k+2^{k-1}$.

Lemma 4. *If* $2^k\leqslant m<2^{k+1}$ *then* $2^k\leqslant b(m)<2^{k+1}$.

Proof. By an easy induction, using Lemma 1. □

Lemma 5.

(i) *If* $0\leqslant p<k$ *and* $2^k-2^{p+1}\leqslant j<2^k-2^p$, *then*
  (a) *If* $i=k$ *or* p, $b(2^k+j))_i=1$.
  (b) *If* $p<i<k$, $(b(2^k+j))_i=0$.
  (c) *If* $i<p$, $(b(2^k+j))_i=(b(j))_i$.
(ii) *If* $j=2^k-1$, *then* $(b(2^k+j))_k=1$, *while* $(b(2^k+j))_i=0$ *for* $i<k$.

Proof. (i) (a) and (b) follow from Corollary 2 and Lemma 4.
(c) If $i<p=k-1$, by Corollary 3

$$(b(2^k+j))_i=(b(j))_i.$$

If $i<p<k-1$ and $2^k-2^{p+1}\leqslant j<2^k-2^p$ we have $0\leqslant 2^{k-1}-2^{p+1}\leqslant j-2^{k-1}<2^{k-1}-2^p$ and by Corollary 2:

$$
\begin{aligned}
b(j)&=b(2^{k-1}+(j-2^{k-1}))\\
&=2^{k-1}+2^p+b(j+2^{p+1}-2^k),
\end{aligned}
$$

$$b(j)+2^{k-1}=b(2^k+j).$$

So if $i<p<k-1$, then $(b(j))_i=(b(2^k+j))_i$.
(ii) By Lemma 1(ii). □

The following theorem gives three ways of quickly evaluating $b(m)$.

Theorem 6. *For* $m\geqslant 0$,

(i) $(b(m))_i=(m+2^i)_{i+1}$,
(ii) $(b(m))_i=\lfloor m/(2^{i+1})+\frac{1}{2}\rfloor \bmod 2$,
(iii) $(b(m))_i=(m_{i+1}+m_i)\bmod 2$.

Proof. (i) Let $0\leqslant m<2^{k+1}$, then by Lemma 4, $0\leqslant b(m)<2^{k+1}$ and if $i>k$, $0\leqslant m+2^i<2^{i+1}$ and so $(m+2^i)_{i+1}=(b(m))_i=0$.

We now assume $i \leqslant k$ and proceed by induction on m.
Case 1: $m = 0$ $(b(0))_i = 0_i = 0 = (0 + 2^i)_{i+1}$.
Case 2: $m = 2^{k+1} - 1$, $k \geqslant 0$.

$$m + 2^i = 2^{k+1} + 2^{i-1} + 2^{i-2} + \cdots + 1,$$

so

$$(m + 2^i)_{i+1} = 0 \quad \text{if } i \neq k,$$
$$(m + 2^k)_{k+1} = 1$$

so we have the result, by Lemma 1 (ii),
Case 3: $m = 2^k + j$ where $2^k - 2^{p+1} \leqslant j < 2^k - 2^p$ and $0 \leqslant p < k$
By Lemma 5(i)(a)

$$(b(m))_k = 1 = (2^k + j + 2^k)_{k+1},$$

$$(b(m))_p = 1 = (2^k + j + 2^p)_{p+1}$$

as $2^{k+1} - 2^p \leqslant 2^k + j + 2^p < 2^{k+1}$.
By Lemma 5(i)(b) if $p < i < k$,

$$(b(m))_i = 0 = (2^k + j + 2^i)_{i+1}$$

as $2^{k+1} + 2^i - 2^{p+1} \leqslant 2^k + j + 2^i < 2^{k+1} + 2^i - 2^p$.
By Lemma 5(i)(c) if $i < p$, by the induction hypothesis

$$(b(m))_i = (b(j))_i = (j + 2^i)_{i+1} = (j + 2^k + 2^i)_{i+1}$$

as $i < p < k$.

(ii) By (i) and $n_{i+1} = \lfloor n/(2^{i+1}) \rfloor \bmod 2$.
(iii) By (i) if $m_i = 0$, $(b(m)) = m_{i+1} = m_{i+1} + m_i \bmod 2$.

$$\text{If } m_i = 1 \ (b(m))_i = (m_{i+1} + 1) \bmod 2$$
$$= (m_{i+1} + m_i) \bmod 2. \qquad \square$$

Note that, in Sharma and Khanna [5], part (ii) of our Theorem 6 is used as the definition of the BRGC; our Definition 1 is later proved as a theorem.

4. b has BRGC properties

We require b to be a one to one and onto map and, for each m, $b(m)$ and $b(m + 1)$, in binary notation (i.e. the BRGC of m and $m + 1$) to differ by one digit.

Lemma 7. $b : \{0, 1, \ldots, 2^n - 1\} \to \{0, 1, \ldots, 2^n - 1\}$ *is one to one and onto*.

Proof. We prove by induction on i that

$$b(k) = b(m) \Rightarrow k_{n-i} = m_{n-i},$$

which proves that b is one to one. It then follows by Lemma 4 that b is onto.
$i = 0$. If $b(k) = b(m)$ by Theorem 6(i) and $m, k < 2^n$,

$$(b(k))_n = (b(m))_n = (m + 2^n)_{n+1} = (k + 2^n)_{n+1} = 0 = m_n = k_n.$$

$i > 0$. We assume $b(k) = b(m)$ and $k_{n-i+1} = m_{n-i+1}$.

By Theorem 6(iii)

$$k_{n-i+1} + k_{n-i} = (m_{n-i+1} + m_{n-i}) \bmod 2$$

and so

$$k_{n-i} = m_{n-i}. \qquad \square$$

Theorem 8. *There is exactly one i such that* $(b(m))_i \neq (b(m+1))_i$.

Proof. By Theorem 6(ii) and the fact that

$$\left\lfloor \frac{m}{2^{i+1}} + \frac{1}{2} \right\rfloor \quad \text{and} \quad \left\lfloor \frac{m+1}{2^{i+1}} + \frac{1}{2} \right\rfloor$$

cannot differ by more than 1, we have $(b(m))_i \neq (b(m))_{i+1}$ if and only if

$$\left\lfloor \frac{m+1}{2^{i+1}} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{m}{2^{i+1}} + \frac{1}{2} \right\rfloor = 1.$$

Letting $m = 2^{i+1}\ell + k$, where $0 \leqslant k < 2^{i+1}$ this condition becomes

$$\left\lfloor \frac{k+1}{2^{i+1}} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{k}{2^{i+1}} + \frac{1}{2} \right\rfloor = 1,$$

which holds if and only if $k = 2^i - 1$, that is, if and only if $m + 1 = 2^i(2\ell + 1)$. $\square$

Note: The i for which $(b(m))_i \neq (b(m+1))_i$ is the highest power of 2 to divide $m + 1$.

5. BRGC algorithms

A standard algorithm (given in Nashelsky [4]) for generating a BRGC is:

Algorithm 1 (*For b*). For each digit in an n-digit word m, starting from the right, if the digit to its left is 0 leave it as it is, while if the digit to its left is 1, change the digit.

This is effectively what we get from Theorem 6(iii):

Algorithm 2 (*For b*). $(b(m))_i$ is $(m_{i+1} + m_i) \bmod 2$.

Even simpler is Algorithm 3, which follows from Algorithm 1 or 2.

Algorithm 3 (*For b*). For $m > 0$, $b(m)$ is m, in binary, with the first of any sequence of 1s or 0s becoming a 1 and every other digit a 0.

Example. $m = 111101101111$. The 1st, 5th, 6th, 8th and 9th digits start new sequences of 1s or 0s, so $b(m) = 100011011000$.

6. Some recurrence relations for b

The following two lemmas give some interesting recurrence relations for b:

Lemma 9. $b(2m + 1) = b(2m) + (-1)^m$.

Proof. As, for $i > 0$, $(2m)_i = (2m+1)_i$ by Theorem 6(iii),

$$\begin{aligned}(b(2m))_i &= ((2m)_{i+1} + (2m)_i) \bmod 2\\ &= ((2m+1)_{i+1} + (2m+1)_i) \bmod 2\\ &= (b(2m+1))_i.\end{aligned}$$

Hence, by Theorem 6(i),

$$\begin{aligned}b(2m+1) - b(2m) &= (b(2m+1))_0 - (b(2m))_0\\ &= (2m+2)_1 - (2m+1)_1\\ &= (-1)^m\end{aligned}$$

as $(2m+2)_1 = 1$ and $(2m+1)_1 = 0$ if m is even, and $(2m+2)_1 = 0$ and $(2m+1)_1 = 1$ if m is odd. □

Lemma 10. $b(2m) = 2b(m) + (1 - (-1)^m)/2$.

Proof. By Theorem 6(iii) and (ii),

$$\begin{aligned}(b(2m))_{i+1} &= ((2m)_{i+2} + (2m)_{i+1}) \bmod 2\\ &= (m_{i+1} + m_i) \bmod 2\\ &= (b(m))_i = (2b(m))_{i+1}.\end{aligned}$$

So

$$\begin{aligned}b(2m) &= 2b(m) + (b(2m))_0\\ &= 2b(m) + \lfloor m + \tfrac{1}{2} \rfloor \bmod 2\\ &= 2b(m) + \frac{1-(-1)^m}{2}.\end{aligned}$$

A number of other recurrence relations can be obtained from these. For example

$$\begin{aligned}&b(8m+2) = 4b(2m) + 3,\\ &b(2^k) = 3.2^{k-1} \quad \text{if } k > 0,\\ &b(2^k + 1) = 3.2^{k-1} + 1 \quad \text{if } k > 1.\end{aligned}$$

We can also get a more general expression for $b(m+1)$ in terms of $b(m)$. □

Lemma 11. *If* $m + 1 = 2^k(2\ell + 1)$ *then* $b(m+1) - b(m) = 2^k(-1)^\ell$.

Proof. By induction on k. If $k = 0$ we have the result by Lemma 9.
If $k > 0$, we have by Lemmas 9 and 10,

$$\begin{aligned}b(m+1) - b(m) &= 2b\left(\frac{m+1}{2}\right) + \frac{1-(-1)^{(m+1)/2}}{2} - b(m-1) - (-1)^{(m-1)/2}\\ &= 2\left(b\left(\frac{m+1}{2}\right) - b\left(\frac{m-1}{2}\right)\right) + \frac{1-(-1)^{(m+1)/2}}{2}\\ &\quad - (-1)^{(m-1)/2} - \left(\frac{1-(-1)^{(m-1)/2}}{2}\right)\\ &= 2^k(-1)^\ell\end{aligned}$$

by the induction hypothesis. □

7. b and Nim sums

The Nim sum $m\#k$ of two nonnegative binary integers is the addition of these numbers without carry over. This is used in the study of the game of Nim in Berlekamp et al. [1].

i.e. $(m\#k)_i \equiv m_i + k_i \bmod 2$.

This, using $\lfloor m/2 \rfloor_i = m_{i+1}$ and Theorem 6(iii) proves:

Theorem 12. $b(m) = m\#\lfloor m/2 \rfloor$.

8. Orbits of b

An orbit of b is a set consisting of a number m and its successive images under powers of b, and we are interested in finding the size (or length) of this set for each m, viz. the smallest positive integer k for which $b^k(m) = m$.

First we need two lemmas.

Lemma 13.

(i) $\binom{j}{k}$ *is odd iff*

$$\sum_{i=1}^{\infty} \left\lfloor \frac{j}{2^i} \right\rfloor - \left\lfloor \frac{j-k}{2^i} \right\rfloor - \left\lfloor \frac{k}{2^i} \right\rfloor = 0.$$

(ii) $\binom{j}{k}$ *is even for* $1 \leqslant k \leqslant p < j$ *iff* $2^{\lfloor \log_2 p \rfloor + 1} | j$.

Proof.

(i) This follows from the well-known result (see for example Griffin [3] Theorem 3.16) that the highest power of 2 to divide $n!$ is $\sum_{i=1}^{\infty} \lfloor n/2^i \rfloor$.

(ii) Let $u = \lfloor \log_2 p \rfloor + 1$ and $j = 2^{u-1}w + v$, where $0 \leqslant v < 2^{u-1} \leqslant p$. Then if $i < u$,

$$\left\lfloor \frac{j}{2^i} \right\rfloor = 2^{u-1-i}w + \left\lfloor \frac{v}{2^i} \right\rfloor = \left\lfloor \frac{j-v}{2^i} \right\rfloor + \left\lfloor \frac{v}{2^i} \right\rfloor$$

and if $i \geqslant u$, as $v < 2^{u-1} < 2^i$,

$$\left\lfloor \frac{j}{2^i} \right\rfloor = \left\lfloor \frac{w}{2^{i+1-u}} \right\rfloor = \left\lfloor \frac{j-v}{2^i} \right\rfloor + \left\lfloor \frac{v}{2^i} \right\rfloor.$$

So by (i), $\binom{j}{v}$ is odd.

If $\binom{j}{1}, \binom{j}{2} \ldots, \binom{j}{p}$ are all even, it follows that $v = 0$.

If $w = 2r + 1$ and $v = 0$, $j = 2^u r + 2^{u-1}$ and we can show, exactly as above, that $\binom{j}{2^{u-1}}$ is odd.

Hence as $2^{u-1} \leqslant p$, if $\binom{j}{1}, \binom{j}{2}, \ldots, \binom{j}{p}$ are all even, $v = 0$ and w must be even so that $2^{\lfloor \log_2 p \rfloor + 1} | j$.

If $j = 2^u r$ and $k = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_h}$, where $k_1 > k_2 > \cdots > k_h \geqslant 0$, $h \geqslant 1$ and $0 < k \leqslant p$, then $k_1 \leqslant u - 1$,

$$j - k = 2^u(r-1) + 2^{u-1} + \cdots + 2^{k_1+1} + 2^{k_1-1} + \cdots 2^{k_2+1} + \cdots + 2^{k_{h-1}-1} + \cdots + 2^{k_h}$$

and $\lfloor \frac{j}{2^u} \rfloor - \lfloor \frac{j-k}{2^u} \rfloor - \lfloor \frac{k}{2^u} \rfloor = r - (r-1) > 0$.

Hence by (i) $\binom{j}{1}, \binom{j}{2}, \ldots, \binom{j}{p}$ are all even. □

Lemma 14. *If m, i and j are nonnegative integers, then* $(b^j(m))_i = \sum_{k=0}^{j} \binom{j}{k} m_{i+k} \bmod 2$.

Proof. By induction on j.

$j=0$. Obvious.

$j>0$. Assume the lemma holds for j, then by Theorem 6(iii),

$$\begin{aligned}(b^{j+1}(m))_i &= \sum_{k=0}^{j}\binom{j}{k}(b(m))_{i+k} \bmod 2\\ &= \sum_{k=0}^{j}\binom{j}{k}(m_{i+k+1}+m_{i+k}) \bmod 2\\ &= \sum_{k=0}^{j-1}\left(\binom{j}{k}+\binom{j}{k+1}\right)m_{i+k+1}+\binom{j}{0}m_i+\binom{j}{j}m_{i+j+1} \bmod 2\\ &= \sum_{k=0}^{j-1}\binom{j+1}{k+1}m_{i+k+1}+\binom{j+1}{0}m_i+\binom{j+1}{j+1}m_{i+j+1} \bmod 2\\ &= \sum_{k=0}^{j+1}\binom{j+1}{k}m_{i+k} \bmod 2.\end{aligned}$$

Hence, by induction the lemma holds. □

Theorem 15. *$b^j(m)=m$ iff $m=0$ or 1 or $2^{\lfloor\log_2\lfloor\log_2 m\rfloor\rfloor+1}|j$.*

Proof. The result holds for $j=0$, so assume $j>0$.

By Lemma 14, $b^j(m)=m$ iff for all $i\geqslant 0$, $\sum_{k=1}^{j}\binom{j}{k}m_{i+k}=0 \bmod 2$.

If $m=0$ or 1, this is true for all j. If $m>1$, for $q=\lfloor\log_2 m\rfloor$, $2^q\leqslant m<2^{q+1}$ and $m_q=1$. For $i+k>q$, $m_{i+k}=0$. Hence

$$b^j(m)=m \quad \text{iff for } q\geqslant i\geqslant 0, \quad \sum_{k=1}^{\min(j,q-i)}\binom{j}{k}m_{i+k}=0 \bmod 2.$$

If the statement to the right of the iff, which we will call (*), holds, we have, for $i=q-1$, $q-i=1\leqslant j$ and

$$\binom{j}{1}m_q\equiv\binom{j}{1}\equiv 0 \bmod 2.$$

Now assume $\binom{j}{t}\equiv 0 \bmod 2$, for $1\leqslant t<r\leqslant \min(j,q)$. Then if (*) holds we have, for $i=q-r$,

$$\sum_{k=1}^{r}\binom{j}{k}m_{i+k}=\binom{j}{r}m_q\equiv\binom{j}{r}\equiv 0 \bmod 2.$$

Hence, by induction, if (*) holds,

$$\binom{j}{1},\binom{j}{2},\ldots,\binom{j}{\min(j,q)}$$

are all even and as $\binom{j}{j}=1$, $\binom{j}{\min(j,q)}=\binom{j}{q}$.

If $\binom{j}{1},\ldots,\binom{j}{q}$ are all even (*) holds.

Hence, by Lemma 13, as $q=\lfloor\log_2 m\rfloor$, $b^j(m)=m$ iff $2^{\lfloor\log_2\lfloor\log_2 m\rfloor\rfloor+1}|j$. □

Corollary 16. *If $2^{2^k}\leqslant m<2^{2^{k+1}}$, $b^j(m)=m$ iff $2^{k+1}|j$.*

Corollary 17. *For $m > 1$, the length of the orbit of b is* $2^{\lfloor \log_2 \lfloor \log_2 m \rfloor \rfloor + 1}$.

9. The decoding function $d = b^{-1}$

We define a new function d recursively and then show that this is the inverse of b.

Definition 2.

$$d(0) = 0,$$
$$d(2^k + i) = 2^{k+1} - 1 - d(i) \quad (0 \leqslant i < 2^k).$$

We now prove lemmas similar to those for b.

Lemma 18.

(i) $d(1) = 1$.
(ii) *If m_p is the second 1 from the left in $m_k m_{k-1} \ldots m_0$, where $m_k = 1$, then*

$$d(m_k m_{k-1} \ldots m_0) = 2^{k+1} - 2^{p+1} + d(m_{p-1} \ldots m_0).$$

(iii) *If $m_k = 1$ and $m_{k-1} = m_{k-2} = \ldots = m_0 = 0$, then*

$$d(m_k m_{k-1} \ldots m_0) = 2^{k+1} - 1,$$

that is $d(2^k) = 2^{k+1} - 1$ for all $k \geqslant 0$.

Proof.

(i) From Definition 2.
(ii) By Definition 2,

$$\begin{aligned} d(2^k + 2^p + m_{p-1}2^{p-1} \ldots + m_0) &= 2^{k+1} - 1 - d(2^p + m_{p-1}2^{p-1} + \cdots + m_0) \\ &= 2^{k+1} - 1 - (2^{p+1} - 1 - d(m_{p-1}2^{p-1} + \cdots + m_0)) \\ &= 2^{k+1} - 2^{p+1} + d(m_{p-1} \ldots m_0). \end{aligned}$$

(iii) $d(2^k) = 2^{k+1} - 1 - d(0) = 2^{k+1} - 1$. □

Corollary 19. *If $2^p \leqslant j < 2^{p+1} \leqslant 2^k$, then $d(2^k + j) = 2^{k+1} - 2^{p+1} + d(j - 2^p)$.*

Lemma 20. *If $2^k \leqslant m < 2^{k+1}$ then $2^k \leqslant d(m) < 2^{k+1}$.*

Proof. By induction on m. □

We can now show that d is the inverse of b.

Theorem 21. $d = b^{-1}$.

Proof.

(i) We show, by induction on j, that $d(b(j)) = j$. This is obvious for $j = 0$.
If $j > 0$, we let $j = 2^k + i$ for $0 \leqslant i < 2^k$.

Then $b(j) = b(2^k - i - 1) + 2^k$ and as, by Lemma 4, $0 \leqslant b(2^k - i - 1) < 2^k$, we have by the induction hypothesis and Definition 2:

$$d(b(j)) = 2^{k+1} - 1 - d(b(2^k - i - 1))$$
$$= 2^{k+1} - 1 - (2^k - i - 1)$$
$$= 2^k + i = j.$$

(ii) We prove, by induction on j, that $b(d(j)) = j$. This is obvious for $j = 0$.
If $j > 0$, we let $j = 2^k + i$ for $0 \leqslant i < 2^k$, then

$$d(j) = 2^{k+1} - 1 - d(i)$$
$$= 2^k + (2^k - 1 - d(i)).$$

As by Lemma 20, $0 \leqslant 2^k - 1 - d(i) < 2^k$, by Definition 1 and the induction hypothesis,

$$b(d(j)) = b(2^k - 1 - (2^k - 1 - d(i))) + 2^k$$
$$= b(d(i)) + 2^k$$
$$= i + 2^k = j. \qquad \square$$

We now write down a lemma for d, similar to Lemma 5 for b.

Lemma 22.

(i) *If* $2^p \leqslant j < 2^{p+1} \leqslant 2^k$ *then*
(a) $(d(2^k + j))_p = 0$
(b) $(d(2^k + j))_i = 1$ *for* $p + 1 \leqslant i \leqslant k$
(c) $(d(2^k + j))_i = (d(j))_i$ *if* $0 \leqslant i < p$.
(ii) $(d(2^k))_i = 1$ *if* $0 \leqslant i \leqslant k$.

Proof. By Lemma 18. $\square$

Using Lemma 5, we were able to prove Theorem 6 which gave simple methods for finding $(b(m))_i$. The formula for $(d(m))_i$ given below is not quite as simple and its proof does not use Lemma 22.

Theorem 23. $(d(m))_i = \sum_{j=i}^{k} m_j \bmod 2$, *where k is the largest value of j for which* m_j *is non-zero.*

Proof. By induction on i.
Let the non-zero values of m_i be

$$m_{k_1}, m_{p_1}, m_{k_2}, m_{p_2}, \ldots, m_{p_r} \text{ (and } m_{k_{r+1}}),$$

where $k = k_1 > p_1 > k_2 \ldots > p_r$ $(> k_{r+1})$.
$\underline{i = 0}$. If there is an even number of these non-zero m_i's then by Lemma 18

$$d(m) = 2^{k_1+1} - 2^{p_1+1} + 2^{k_2+1} - 2^{p_2+1} + \cdots 2^{k_r+1} - 2^{p_r+1}$$

so $(d(m))_0 = 0 = \sum_{j=0}^{k} m_j \bmod 2$.
If this number is odd

$$d(m) = 2^{k_i+1} - 2^{p_i+1} + \cdots 2^{k_r+1} - 2^{p_r+1} + 2^{k_{r+1}} - 1$$

so $(d(m))_0 = 1 = \sum_{j=0}^{k} m_j \bmod 2$.

$\underline{i > 0}$. By Theorems 6(iii) and 21

$$m_{i-1} = (d(m))_i + (d(m))_{i-1} \bmod 2.$$

That is, using the induction hypothesis,

$$\begin{aligned}(d(m))_i &= (d(m))_{i-1} + m_{i-1} \bmod 2\\ &= \sum_{j=i-1}^{k} m_j + (m)_{i-1} \bmod 2\\ &= \sum_{j=i}^{k} m_j \bmod 2. \qquad \square\end{aligned}$$

Corollary 24. $(d(m))_i = 0$ *if there is an even number of* 1*s to the left of* m_{i-1} *in the binary representation of* m, *and* $d(m)_i = 1$ *otherwise*.

From this we have two forms of an algorithm to evaluate $d(m)$.

Algorithm 1 (*For d*). For each digit in the binary representation of *m*, put a 0 if there is an even number of 1s from this digit (including it) to the left and a 1 otherwise.

Algorithm 2 (*For d*). To form $d(m)$ from the binary representation of *m* replace the 1st, 3rd, 5th, etc. occurrences of 1 and any subsequent 0s by 1s and replace the 2nd, 4th, etc. occurrences of 1 and any subsequent 0s by 0s.

Example. $d(1100010110001) = 1000011011110$.

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