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Non-commutative duality: high spin fields and CP^1 model with Hopf term

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Abstract

We show that the non-commutative CP^1 model coupled with Hopf term in 3 dimensions is equivalent to an interacting spin-s theory where the spin-s of the dual theory is related to the coefficient of the Hopf term. We use the Seiberg–Witten map in studying this non-commutative duality equivalence, keeping terms to order θ and show that the spin of the dual theory do not get any θ-dependant corrections. The map between current correlators shows that topological index of the solitons in the non-commutative CP^1 model is unaffected by θ where as the Noether charge of the corresponding dual particle do get a θ dependence. We also show that this dual theory smoothly goes to the limit $\theta \to 0$ giving dual theory in the commutative plane.

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1. Introduction

The recent developments in non-commutative (NC) geometry [1] and string theory [2] have motivated the study of different features of field theory models constructed on NC space–time [3]. The non-commutativity of the space–time introduces non-linear and non-local effect and hence the field theory models constructed on such spaces have many interesting features which their commutative counterparts do not share, like the possibility of novel soliton solutions [4], UV/IR mixing [5], etc. The UV/IR mixing which is a characteristic feature of non-commutative (NC) field theories affect their renormalisability [5,6]. Recently renormalisability of super-symmetric field theory models

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in NC setting have also been studied and it has been argued that the non-commutative (NC) super-symmetric gauge theories have better renormalisability [7]. NC super-symmetric quantum mechanical models [8] have also been constructed and studied. Recently fermionic field theory models have been studied on NC space–time which avoids the fermion doubling problem and serves as alternative to lattice regularisation [9]. Quantum theories with space–time non-commutativity have also been considered recently with potential applications [10].

Seiberg–Witten (SW) map [2] allows to re-express the NC gauge theoretic models in terms of the ordinary gauge fields and the NC parameter θ and has been employed to study various aspects of NC field theoretic models [11,12]. The SW map is derived by demanding that the ordinary gauge fields which are connected by a gauge transformation are mapped to NC fields which are likewise related by the corresponding NC gauge transformation and this map smoothly reduces to the commutative limit when $\theta \to 0$. Using SW map, it has been shown recently that the NC Chern–Simons term get mapped to standard Chern–Simons term in the commutative plane [12]. It has been argued that the commutative limit (i.e., $\theta \to 0$) of NC models may not be smooth [5,13]. Therefore it is of interest to see how some of the well-established field theoretic notions in the commutative spaces generalises to NC settings. In this Letter we investigate one such problem, namely the dualisation of CP^1 model with Hopf term in NC plane.

Study of the duality between bosonic and fermionic theories in commutative spaces has a long history. In [14] the equivalence between sine-Gordon and massive Thirring model in $1+1$ dimension has been studied. Following [15], boson–fermion transmutation $(2+1)$-dimensional field theoretic models were studied in [16,17] and also perturbatively in [19]. In [17] it has been shown that the non-linear sigma model when coupled to Hopf term (written in CP^1 language) is equivalent to an interacting spin-s ($s = \frac{1}{2}, 1, \ldots$) theory and the mapping between the dual fields has been obtained. Duality and bosonisation of non-linear and non-Abelian theories has also been studied recently [20,21].

The duality between Maxwell–Chern–Simons theory and self-dual model in $2+1$ dimensions [22] (which is a crucial ingredient in obtaining the 'bosonisation' rules for massive Thirring model in $2+1$ dimensions) has been recently analysed in the NC settings [23] using SW map to the order θ. Following this it has been shown that the equivalence between the massive Thirring model and Maxwell–Chern–Simons theory (to the *leading* order in the inverse fermion mass) is (not) valid in the NC space where as the $(1+1)$-dimensional bosonisation is intact in NC settings [24]. The study of NC duality and bosonisation is also of interest as these studies can shed further light to the similar problems in the non-Abelian gauge theories since later have a similar gauge structure as NC gauge theories. In this Letter we study the dualisation of NC CP^1 model coupled with Hopf term. The CP^1 model in NC plane has been studied and soliton solutions were obtained recently. It has been argued that the equivalence of non-linear sigma model and CP^1 model in the commutative plane do not hold good in the NC settings [28].

In this Letter we show that the NC CP^1 model coupled with Hopf term is equivalent to NC spin-s theory. We obtain this duality equivalence using the path integral method developed [16,17] in implementing the approach of [15] in $(2+1)$-dimensional field theoretic models. We apply this method, after re-expressing the NC CP^1 model coupled with Hopf term in terms of the commutative fields and NC parameter θ using SW map. We obtain the dual interacting spin-s theory where the spin-s is given by $s = \frac{\pi}{2\lambda}$ where λ is the coefficient of the Hopf term. Here we obtain exact duality equivalence between NC CP^1 model coupled with Hopf term and NC spin-s theory. We also obtain the mapping between the current correlators of these two equivalent NC models.

2. NC CP^1 model and SW map

The CP^1 model in commutative plane is described by the action

$$S = \int d^3x \left|(\partial_\mu \Phi_a - i A_\mu \Phi_a)\right|^2, \quad a = 1, 2, \tag{1}$$

where the complex doublet field Φ_a satisfies the conditions

$$|\Phi_1|^2 + |\Phi_2|^2 = 4g^2, \tag{2}$$

$$-i\Phi_a^* \partial_\mu \Phi_a = 4g^2 A_\mu. \tag{3}$$

It has been shown that the above local $U(1)$ invariant action when coupled to the Hopf term

$$H = -\frac{i\lambda}{4\pi^2} \int d^3x\, \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \tag{4}$$

is equivalent to spin-s theory [17]. Here the spin-s is related to the coupling strength λ of the Hopf term. In this Letter we investigate this equivalence in the NC plane. The NC space–time is defined by the coordinates obeying

$$[X_\mu, X_\nu]_* = i\theta_{\mu\nu}, \tag{5}$$

where the $*$ product is defined as

$$f(x) * g(x) = e^{\frac{i}{2}\theta^{ij}\partial_i^x \partial_j^y} f(x)g(y)\big|_{x=y}. \tag{6}$$

In the following we take the anti-symmetric tensor $\theta_{\mu\nu}$ to be a constant.

We start with the NC CP^1 model action coupled to the Hopf term

$$\hat{\mathcal{S}} = \int d^3x \left[(\hat{D}_\mu \hat{\Phi}_a)^\dagger (\hat{D}_\mu \hat{\Phi}_a) - \frac{i\lambda}{4\pi^2} \epsilon_{\mu\nu\lambda} \left(\hat{A}_\mu \partial_\nu \hat{A}_\lambda + \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda \right) \right], \tag{7}$$

where the covariant derivative is defined as $\hat{D}_\mu \hat{\Phi} = \partial_\mu \hat{\Phi} - i\hat{A}_\mu \hat{\Phi}$ and all the products in the above are $*$ products. This action is invariant under the NC $U(1)$ transformations

$$\hat{\Phi} \to \hat{U}\hat{\Phi}, \qquad \hat{A}_\mu \to \hat{U}\hat{A}_\mu \hat{U}^\dagger - i\partial_\mu \hat{U}\hat{U}^\dagger. \tag{8}$$

We re-express this action in terms of the commutative fields and the non-commutative parameter θ using Seiberg–Witten (SW) map. The SW map for the complex scalar field and the gauge field, to the order θ is given by

$$\hat{\Phi} = \Phi - \frac{1}{2}\theta_{\mu\nu} A_\mu \partial_\nu \Phi, \tag{9}$$

$$\hat{A}_\mu = A_\mu - \frac{1}{2}\theta_{\nu\lambda} A_\nu (\partial_\lambda A_\mu + F_{\lambda\mu}), \tag{10}$$

respectively.

Since the NC Chern–Simons term get mapped to the standard Chern–Simons term in the commutative plane under the SW map, all the θ-dependant terms come from the first term when we apply SW map to the action in Eq. (7). To the order θ, the SW mapped action is

$$S = \int d^3x\, |D_\mu \Phi_a|^2 - h_{\mu\nu} (D_\nu \Phi_a)^* (D_\mu \Phi_a) - \frac{i\lambda}{4\pi^2} \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \tag{11}$$

where

$$h_{\mu\nu} = \frac{1}{2} \left(\theta_{\mu\alpha} F_{\alpha\nu} + \theta_{\mu\alpha} F_{\alpha\mu} + \frac{1}{2} \eta_{\mu\nu} \theta_{\alpha\beta} F_{\alpha\beta} \right). \tag{12}$$

The second term in the action above is the new θ-dependant interaction term introduced by the non-commutative nature of the space–time. The partition function for this theory is

$$Z = \int D\alpha\, D\eta\, DA\, D\Phi_a^*\, D\Phi_a\, e^{-S}, \tag{13}$$

where

$$S=\int d^3x\Big[|\partial_\mu\Phi_a|^2-4g^2A_\mu^2+h_{\mu\nu}\big(\Phi_a^*\partial_\mu\partial_\nu\Phi_a+4g^2A_\mu A_\mu\big)-\frac{i\lambda}{4\pi^2}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda$$
$$-\alpha_\mu\big(4g^2A_\mu+i\Phi_a^*\partial_\mu\Phi_a\big)+\eta^2-2i\eta\sqrt{\rho}\big(|\Phi_1|^2+|\Phi_2|^2-4g^2\big)\Big]. \tag{14}$$

Here, notice that constraint on CP^1 fields in Eq. (2) is implemented in the path integral through $\rho(|\Phi_1|^2+|\Phi_2|^2-4g^2)^2$, with the parameter[1] $\rho\to\infty$ and this term is then linearised using an auxiliary field η. The constraint given in Eq. (3) is introduced using the multiplier fields α_μ.[2] Now introducing the fields b_μ and a_μ we linearise the quadratic term in A_μ and the Chern–Simons term (Hopf term), respectively, to write the partition function of this theory as

$$Z=\int DC\,D\alpha\,D\eta\,DB\,Da\,DA\,Db\,D\Phi_a^*\,D\Phi_a\,e^{-\mathcal{S}}, \tag{15}$$

where the action

$$\mathcal{S}=\int d^3x\,\Phi_a^*[-D_\mu D_\mu+V]\Phi_a-C_{\mu\nu}\big(\Phi_a^*\partial_\mu\partial_\nu\Phi_a+4g^2A_\mu A_\nu\big)+(C_{\mu\nu}+h_{\mu\nu})B_{\mu\nu}$$
$$-4g^2\alpha_\mu A_\mu+\eta^2-8ig^2\eta\sqrt{\rho}+4g^2\left[\frac{\alpha_\mu^2}{4}+\alpha_\mu(b_\mu+ik\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda)\right]$$
$$-4g^2\big[2ik\epsilon_{\mu\nu\lambda}b_\mu\partial_\nu a_\lambda-k^2(\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda)^2+ik\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda\big]. \tag{16}$$

Using the auxiliary fields $C_{\mu\nu}$ and $B_{\mu\nu}$ we have conveniently re-expressed the above action where there is no direct coupling between θ-dependant terms and the CP^1 fields Φ_α. Here the covariant derivative is defined as $D_\mu=\partial_\mu+iW_\mu$ where the gauge field is given by

$$W_\mu=b_\mu+\frac{1}{2}\alpha_\mu+ik\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda, \tag{17}$$

with

$$k=-\frac{\lambda}{(4\pi^2)(4g^2)}\quad\text{and}\quad V=2i\eta\sqrt{\rho}. \tag{18}$$

3. Duality equivalence

We now carry out the integrations over Φ_a^* and Φ_a in the partition function in Eq. (15) after re-writing the action in Eq. (16) as

$$\mathcal{S}=\int d^3x\,\Phi_a^*\mathcal{O}\Phi_a+S_0, \tag{19}$$

where

[1] The constraints are treated as functional delta function following our earlier work [17] and also that of Mitter and Ramdas [18].

[2] All the θ-dependant terms coming from the constraint in Eq. (3) when SW map is applied cancel when plugged back into SW mapped CP^1 action. This justifies the use of the commutative constraint in the SW mapped action. See [25] for a detailed discussion on this aspect.

$$S_0 = \int d^3x\, C_{\mu\nu}\big[i\partial_\mu W_\nu - 2\cdot 4g^2 W_\mu A_\nu - 4g^2 W_\mu W_\nu\big] - 4g^2 C_{\mu\nu} A_\mu A_\nu + (C_{\mu\nu} + h_{\mu\nu})B_{\mu\nu}$$
$$- 4g^2\alpha_\mu A_\mu + \eta^2 - 8ig^2\eta\sqrt{\rho} - 4g^2\left[\frac{\alpha_\mu^2}{4} + \alpha_\mu(b_\mu + ik\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda)\right]$$
$$- 4g^2\big[2ik\epsilon_{\mu\nu\lambda} b_\mu \partial_\nu a_\lambda - k^2(\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda)^2 + ik\epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda\big], \tag{20}$$

and the operator $\mathcal{O}$ is given by

$$\mathcal{O} = -(\delta_{\mu\nu} + C_{\mu\nu}) D_\mu D_\nu + V. \tag{21}$$

Thus the partition function reduces to

$$Z = \int DC\, D\alpha\, D\eta\, DB\, Da\, DA\, Db\, e^{-S_0 - 2\ln\det\mathcal{O}}. \tag{22}$$

Using the well-known proper time representation of determinant for the operator $\mathcal{O}$ defined in Eq. (21), we get

$$-2\ln\det\mathcal{O} = 2\int_{\Lambda^{-2}}^{\infty} \frac{d\beta}{\beta} \int Dq_\mu(\tau)\, e^{-\int_0^\beta d\tau\,[\frac{1}{4}(\delta_{\mu\nu} - C_{\mu\nu})\dot{q}_\mu\dot{q}_\nu + V] - i\oint_C W_\mu\, dx^\mu}. \tag{23}$$

Notice that the $\det\mathcal{O}$ depends on the gauge field W_μ through the Wilson loop. Also the auxiliary field $C_{\mu\nu}$ appears in the det where as there is no explicit θ dependence.

Substituting this in Eq. (22) and expanding $e^{-2\ln\det\mathcal{O}}$, we get the partition function as

$$Z = \int DC\, D\alpha\, D\eta\, DB\, Da\, DA\, Db \left(1 + \sum_{i=1}^{\infty} \frac{Z_n}{n!}\right) e^{-S_0}, \tag{24}$$

where

$$Z_n = \prod_{i=1}^{\infty} 2^n \int \frac{d\beta_i}{\beta_i} \int Dq_\mu^i(\tau)\, e^{-\int_0^\beta d\tau\,[\frac{1}{4}(\delta_{\mu\nu} - C_{\mu\nu})\dot{q}_\mu^i\dot{q}_\nu^i + V] - i\oint_{C_i} W_\mu\, dx^\mu}. \tag{25}$$

Here we notice that all the dependence of the partition function on the NC parameter θ comes through S_0.

Notice that the term $(1 + \sum_{i=1}^{\infty} \frac{Z_n}{n!})$ in Eq. (24) above contains *all* the terms in the series expansion of $e^{-2\ln\det\mathcal{O}}$. We do not neglect any terms here and thus we are evaluating the partition function *exactly*. Z_n in the above can be taken as the defining the paths of Φ_a particles [17].

We consider the first term in Eq. (24)

$$Z_0 = \int DC\, D\alpha\, D\eta\, DB\, Da\, DA\, Db\, e^{-S_0}$$

which after the integrations over b_μ, A_μ and a_μ becomes

$$Z_0 = \int DC\, D\alpha\, D\eta\, DB\, Dv_\mu\, e^{-S_{\text{eff}}}, \tag{26}$$

where the effective action is

$$S_{\text{eff}} = \int d^3x\, 4g^2\left[\frac{\alpha_\mu^2}{4} + \frac{\alpha_\mu F_\mu(\theta)}{2\cdot 4g^2} + \frac{1}{4}\left(\alpha_\mu + \frac{F_\mu(\theta)}{4g^2}\right) C_{\mu\nu}^{-1}\left(\alpha_\nu + \frac{F_\nu(\theta)}{4g^2}\right)\right]$$
$$+ \frac{1}{4\cdot 4g^2} F_\mu^2(\theta) + \frac{i}{4\cdot 4g^2 k} F_\mu(\theta) d_{\mu\nu}^{-1} F_\nu(\theta) + C_{\mu\nu} B_{\mu\nu} + \eta^2 + i8g^2\eta\sqrt{\rho}$$

$$+\frac{i4g^2}{2}C_{\mu\nu}\partial_\mu\alpha_\nu+g^2(\partial_\mu C_{\mu\nu})^2-\frac{1}{k}\partial_\nu C_{\mu\nu}d^{-1}_{\mu\alpha}\left[ig^2\partial_\beta C_{\beta\alpha}-\frac{1}{2}F_\alpha(\theta)\right]+\frac{1}{3}v_\mu^2. \tag{27}$$

In the above we have used the definition

$$F^\mu(\theta)=\left[\theta^{\rho\alpha}\partial^\alpha B^{\rho\mu}-\theta^{\rho\mu}\partial^\alpha B^{\rho\alpha}+\frac{1}{2}\theta^{\alpha\mu}\partial^\alpha B_{\sigma\sigma}\right], \tag{28}$$

and $d_{\mu\nu}=-\epsilon_{\mu\nu\lambda}\partial_\lambda$. Also we use $C^{-1}_{\mu\nu}$ where $C_{\mu\nu}C^{-1}_{\nu\lambda}=\delta_{\mu\lambda}$. In Eq. (26) we have introduced a new field v_μ in the measure and a Gaussian factor in the action (see Eq. (27)). This is done for later convenience (see Eq. (40) below). Thus the Z_0 in Eq. (26) contains the contribution from the first term in the series expansion of $e^{-2\ln\det\mathcal{O}}$. Next we evaluate the contribution to the partition function from the remaining terms of this series. From Eqs. (24) and (25), we see that these terms contain expectation value of the products of Wilson loops (for every i in Eq. (25) we have a Wilson loop to be averaged with weight factor S_0). Here we use the fact that the averaging over the products of Wilson loops is factorisable and hence it is equal to the product of the averaging over the Wilson loops when the coefficient of the Hopf term $\lambda=\frac{\pi}{2s}$. That is, we use the property of the expectation value of Wilson loop $W(C_i)$,

$$\big\langle W(C_1)\cdots W(C_n)\big\rangle=\prod_{i=1}^{n}\big\langle W(C_i)\big\rangle \tag{29}$$

when $\lambda=\frac{\pi}{2s}$, which can be easily verified in a straightforward manner in the present case by considering that the Wilson loops are non-intersecting [17]. Also notice that the product of Wilson loop is nothing but the union of the Wilson loops. Using these results we get the second term in Eq. (24) to be

$$Z'=\int D\Omega\left[\prod_{i=1}^{\infty}2^n\int\frac{d\beta_i}{\beta_i}\int Dq^i_\mu(\tau)\,e^{-\int_0^\beta d\tau\,(\frac{1}{4}(\delta_{\mu\nu}-C_{\mu\nu})\dot q^i_\mu\dot q^i_\nu+V)}\right]e^{-i\oint_C W_\mu\,dx^\mu-S_0}, \tag{30}$$

where the measure $D\Omega=DC\,D\alpha\,D\eta\,DB\,Da\,DA\,Db$.

Now we carry out the integrations over the fields b and A. Here the terms coming from the Wilson loops also contribute to these integrations unlike in the case of Z_0 in Eq. (26). The partition function becomes

$$Z'_1=\int D\tilde\Omega\left[\prod_{i=1}^{\infty}2^n\int\frac{d\beta_i}{\beta_i}\int Dq^i_\mu(\tau)\,e^{-\int_0^\beta d\tau\,(\frac{1}{4}(\delta_{\mu\nu}-C_{\mu\nu})\dot q^i_\mu\dot q^i_\nu+V)}\right]\delta(\chi)e^{-S_1}, \tag{31}$$

where the measure is $D\tilde\Omega=DC\,D\alpha\,D\eta\,DB\,Da$. The integration over the vector potential A_μ gives the delta function in Eq. (31). The explicit form of this delta function is

$$\delta(\chi)\equiv\delta\big(F_\mu(\theta)+iJ_\mu-4g^2i\partial_\alpha C_{\alpha\mu}-2i\cdot 4g^2k\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda\big) \tag{32}$$

with $F_\mu(\theta)$ as given in Eq. (28). The action S_1 in Eq. (31) is given as

$$S_1=S'_{\rm eff}-\int d^3x\,4g^2\left[(k\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda)^2-\frac{2k}{4g^2}\epsilon_{\mu\nu\lambda}J_\mu\partial_\nu a_\lambda+ik\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda+\frac{i}{8g^2}\alpha_\mu J_\mu\right]. \tag{33}$$

The $S'_{\rm eff}$ here is same as $S_{\rm eff}|_{v_\mu=0}$. The J_μ that appears in Eqs. (32) and (33) is the current associated with the particles moving along the Wilson loops C_i and is given by

$$J_\mu=\sum_i^n\int\frac{\partial q^i_\mu}{\partial\tau}\delta^3\big(q-q^{C_i}_\mu\big). \tag{34}$$

From Eq. (32), we note that even when the coefficient of the Hopf term λ is set to zero (i.e., $k=0$) the current J_μ do *not* vanish because of the θ-dependant terms. Thus the non-commutativity of the space–time which gave rise to new interaction terms also results a *non-vanishing* current even when $\lambda=0$. This has to be contrasted with the commutative case where the current vanishes when $\lambda = 0$ signalling the confinement of particles and anti-particles and a non-vanishing λ leads to deconfinement [17]. Here, in our case, we see that when the NC parameter is non-vanishing, there is no confinement of particles and anti-particles even when $\lambda = 0$.

Now we integrate over the field a_μ in the partition function in Eq. (31). With the delta function in Eq. (32) this is done trivially, leading to

$$Z_1 = \int D\bar{\Omega}\left[\prod_{i=1}^{\infty} 2^n \int \frac{d\beta_i}{\beta_i}\int Dq_\mu^i(\tau)\, e^{-\int_0^\beta d\tau\,(\frac{1}{4}(\delta_{\mu\nu}-C_{\mu\nu})\dot{q}_\mu^i\dot{q}_\nu^i+V)}\right]e^{-(S_{\mathrm{eff}}+S_j)}, \tag{35}$$

where

$$\begin{aligned} S_j = &-\int d^3x\left[\frac{i}{4\cdot 4g^2k}J_\mu d_{\mu\nu}^{-1}J_\nu + \frac{1}{2\cdot 4g^2}J_\mu\left(i\left(F_\mu(\theta)+4g^2\alpha_\mu\right)+\frac{1}{k}d_{\mu\nu}^{-1}F_\nu(\theta)\right)\right]\\ &+ J_\mu\left[\frac{i}{2k}d_{\mu\nu}^{-1}+\delta_{\mu\nu}\right]\partial_\alpha C_{\alpha\nu} - \int d^3x\left[\frac{1}{3}v_\mu^2+\frac{1}{2g}J_\mu v_\mu\right] \end{aligned} \tag{36}$$

and $D\bar{\Omega} = DC\, D\alpha\, D\eta\, DB\, Dv$. Here the field v_μ is introduced to linearise the quadratic term in the current J_μ. With J_μ as given in Eq. (34) and $d_{\mu\nu} = \epsilon_{\mu\nu\lambda}\partial_\lambda$, the contribution from the first term of S_j is well known:

$$e^{-\frac{i\pi^2}{\lambda}\int d^3x\, J_\mu d_{\mu\nu}^{-1}J_\nu} = e^{\frac{i\pi^2}{\lambda}(\sum_{i=1}^n \mathcal{W}(C_i)+\sum_{i\neq j}2n_{ij})}. \tag{37}$$

In the above $\mathcal{W}(C_i)$ is the writhe of the curve which in terms of the solid angle subtended by the tangent to the C_i on a sphere traced out by it and an *odd* integer as $\mathcal{W}(C_i) = \frac{1}{2\pi}\Omega(C_i) + (2k+1)$. n_{ij} is the linking number of the curves C_i and C_j and its contribution to partition function is *unity* when $\lambda = \frac{\pi}{2s}$. Using these results in Eq. (35), we get

$$Z_1 = \int D\bar{\Omega}\left[\prod_{i=1}^{\infty} 2^n \int \frac{d\beta_i}{\beta_i}\int_{q_\mu} e^{-\int_0^\beta d\tau\,(\frac{1}{4}(\delta_{\mu\nu}-C_{\mu\nu})\dot{q}_\mu^i\dot{q}_\nu^i+V)+(-)^{2s}is\Omega-iV_\mu J_\mu}\right]e^{-S_{\mathrm{eff}}}. \tag{38}$$

Here

$$V_\mu = \frac{i}{2\cdot 4g^2}\left(i\left(F_\mu(\theta)+4g^2\alpha_\mu\right)+\frac{1}{k}d_{\mu\nu}^{-1}F_\nu(\theta)\right)+\frac{i}{2g}v_\mu+\left(\frac{1}{2k}d_{\mu\nu}^{-1}-i\delta_{\mu\nu}\right)\partial_\alpha C_{\alpha\nu}. \tag{39}$$

Notice the $(-)^{2s}$ factor in Eq. (38) above. This factor is due to the odd integer $2k+1$ appearing in the expression of writhe $\mathcal{W}(C_i)$.

We now use the Z_0 and Z_1 given above in Eq. (24) to get

$$Z = \int D\bar{\Omega}\,\exp\left\{-\left(S_{\mathrm{eff}} - 2\int\frac{d\beta}{\beta}\int_{q_\mu} e^{-\int_0^\beta d\tau\,(\frac{1}{4}(\delta_{\mu\nu}-C_{\mu\nu})\dot{q}_\mu^i\dot{q}_\nu^i+V)+(-)^{2s}is\Omega-iV_\mu J_\mu}\right)\right\}. \tag{40}$$

Here we notice that the θ dependence of the partition function comes from S_{eff} and also through the potential V_μ.

The effect of adding the Polyakov phase factor to the path integral of spinless particle for free as well as in presence of background fields has been studied and it is well known to give path integral corresponding to particles with spin s [16,17]. This has been shown using the $SU(2)$ coherent state path integral which gives

$$\int_{\hat{U}(0)=\hat{U}(\lambda)} \mathcal{D}\hat{U}\, e^{is\int_0^\lambda d\tau\,(H(\hat{U})+\Omega)} = \mathrm{Tr}\langle\hat{U}|e^{isH(\tau_\mu)}|\hat{U}\rangle, \tag{41}$$

where $\hat{U}$ are the $SU(2)$ coherent states and τ_μ are the generators of spin-s representation of $SU(2)$ [26]. We adapt these results to our present case and obtain

$$\int_{\Lambda^{-2}}^{\infty} \frac{d\beta}{\beta} \int_{q_\mu} e^{-\int_0^\beta d\tau (\frac{1}{4}(\delta_{\mu\nu}-C_{\mu\nu})\dot{q}^i_\mu \dot{q}^i_\nu + V) + (-)^{2s} i s \Omega - i\oint V_\mu dx_\mu} = (-)^{2s} \int_{\Lambda^{-2}}^{\infty} \frac{d\beta}{\beta} \operatorname{Tr} e^{-\beta[\frac{D}{s\mathcal{A}} + \frac{\sqrt{\pi}\epsilon}{4} V + M]}, \tag{42}$$

where Λ is the cut-off and $D = \operatorname{sgn}(\lambda)(i\partial_\mu - V_\mu)\tau^\mu$. Here τ_μ are the generators of spin-s representation of $SU(2)$, $M = \Lambda \frac{\sqrt{\pi}\ln(2s+1)}{4}$, $\mathcal{A} = \sqrt{\det(\delta_{\mu\nu} - C_{\mu\nu})}$ and V is defined in Eq. (18).

Using this in Eq. (40) we get

$$Z = \int D\bar{\Omega}\, e^{-S_{\text{eff}}(-1)^{2s+1}} \det\left[\frac{D}{s\mathcal{A}} + \frac{\sqrt{\pi}\Lambda^{-1}}{4} V + M\right]. \tag{43}$$

The above determinant can be expressed as a functional integral over $\bar{\Psi}$ and Ψ which are complex doublet fields or fermionic fields depending whether $2s+1$ is odd or even integer. Here we see that the factor $(-)^{2s}$ appeared in Eq. (38) (coming from the writhe of the Wilson loop calculated in Eq. (37)) is the important factor deciding the statistics of the dual theory. This factor of $(-)^{2s}$ in the exponential in Eq. (38), in turn, is obtained by choosing the coefficient of the Hopf term in Eq. (7). Since the Hopf term do not change under the SW map, we see that the NC parameter do not affect the statistics of the dual fields.

Thus exponentiating the determinant in the above, we get the partition function as

$$Z = \int D\bar{\Omega}\, D\bar{\Psi}\, D\Psi\, e^{-S_{\text{eff}}} e^{-\int d^3x\, \bar{\Psi}[\frac{D}{s\mathcal{A}} + \frac{\sqrt{\pi}}{4} V + M]\Psi}. \tag{44}$$

Thus we see that all the dependence on the NC parameter θ comes through terms linear and quadratic $F_\mu(\theta)$ appearing in S_{eff} and also from the $\bar{\Psi} V_\mu \tau_\mu \Psi$ where it is coupled linearly. Since we have kept only terms of order θ in SW map while writing the NC action in terms of commutative fields and θ in Eq. (11), in the action $-S'_{\text{eff}}$ appearing in Eq. (44) also we keep only linear terms in θ and carry out integrations over α_μ, v_μ and η. Thus we get the partition function of the dual theory as

$$Z = \int DC\, DB\, D\bar{\Psi}\, D\Psi\, e^{-\mathcal{S}}, \tag{45}$$

where the dual action is

$$\begin{aligned}\mathcal{S} = \int d^3x\, & C_{\mu\nu} B_{\mu\nu} - \frac{1}{k}\partial_\nu C_{\mu\nu} d^{-1}_{\mu\alpha}\left[ig^2\partial_\beta C_{\beta\alpha} - \frac{1}{2}F_\alpha(\theta)\right] - \frac{i}{2}F_\mu(\theta)\partial_\alpha C_{\alpha\mu} + g^2(\partial_\nu C_{\mu\nu})^2 \\ & + g^2\partial_\alpha C_{\mu\alpha}\big(\delta_{\mu\nu} + C^{-1}_{\mu\nu}\big)\partial_\beta C_{\beta\nu} + \bar{\Psi}\left[\frac{\operatorname{sgn}(\lambda)}{s\mathcal{A}}(i\partial_\mu - \tilde{V}_\mu)\tau^\mu + \big(M + 2g^2\rho\sqrt{\pi}\Lambda^{-1}\big)\right]\Psi \\ & + \frac{\pi\rho}{16\Lambda}(\bar{\Psi}\Psi)^2 + \frac{3}{16g^2s^2\mathcal{A}^2}(\bar{\Psi}\tau_\mu\Psi)^2 + \frac{1}{16g^2s^2\mathcal{A}^2}(\bar{\Psi}\tau_\mu\Psi)\big(\delta_{\mu\nu} + C^{-1}_{\mu\nu}\big)^{-1}(\bar{\Psi}\tau_\nu\Psi).\end{aligned} \tag{46}$$

Here

$$\tilde{V}_\mu = \frac{i}{8g^2k} d^{-1}_{\mu\nu} F_\nu(\theta) + \frac{1}{2}\left(\frac{1}{k}d^{-1}_{\mu\nu} - i\delta_{\mu\nu} + C^{-1}_{\mu\nu}\right)\partial_\alpha C_{\alpha\nu}. \tag{47}$$

Notice here that through $\tilde{V}_\mu$, the θ-dependant terms directly get coupled to $\bar{\Psi}$ and Ψ. The dual action has further θ-dependant terms which are coupled to the auxiliary field $C_{\mu\nu}$. In the commutative limit all these later terms vanish and the integrations over the fields $B_{\mu\nu}$ and $C_{\mu\nu}$ become trivial giving the action in the commutative plane

$$S = \int d^3x\, \bar{\Psi}\left[\frac{\operatorname{sgn}(\lambda)}{s} i\tau^\mu\partial_\mu + \big(M + 2g^2\rho\sqrt{\pi}\Lambda^{-1}\big)\right]\Psi + \frac{1}{4g^2s^2}(\bar{\Psi}\tau_\mu\Psi)^2 + \frac{\pi\rho}{16\Lambda^2}(\bar{\Psi}\Psi)^2 \tag{48}$$

obtained in [17]. The four-Fermi interaction term in the dual theory comes when we integrate over the auxiliary field η. This field was introduced in the action (see Eq. (14)) to incorporate the condition in Eq. (2) which is the same in the commutative case also. Thus it is not surprising to see that the four-Fermi interaction term in both NC case and commutative model are the same. In contrast, the Thirring term gets a θ dependence (through $\mathcal{A}$). Notice that the duality shown here is exact, to all orders in fermion mass and coupling constants.

Our dual theory in terms of spin-s fields is non-local as expected for a field theory on a non-commutative space. Our first aim is to see what is the dual theory for a CP^1 model in NC space–time by starting from a SW mapped CP^1 model. Our results clearly point to the fact that the dual action obtained here (Eq. (46)) is not the naïve NC generalisation of the commutative action obtained in [17] (but in the limit $\theta \to 0$, we recover the action obtained in [17]). Similar feature was also noticed in the context of the duality between Maxwell–Chern–Simons theory and self-dual model in the NC settings [24]. Also it has been shown that the effect of NC is same as that of a field-dependant gravitational background [27] and thus the proper time determinant in Eq. (23) can be thought of as evaluated in a non-trivial background. It is this background dependence coming because of the non-commutativity which leads to the appearance of $\mathcal{A}^{-1}$ and $(\delta_{\mu\nu} + C^{-1}_{\mu\nu})^{-1}$ in the dual action. In spite of these non-local and non-polynomial nature of the dual theory one would be able to show various relations between these theories by formally taking functional derivatives.

From the equivalence of the partition functions in Eq. (15) and Eq. (45) obtained here we can derive the mappings between various n-point correlators of CP^1 model and the dual spin-s theory in the NC plane by introducing appropriate source terms. The form of the SW mapped Hopf term and the SW mapped field strength of the vector field is suggestive to couple a topologically invariant current of the form

$$J^{\text{top}}_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda \tag{49}$$

using a source (a vector field here) to the partition function of the SW mapped CP^1 model with Hopf term in Eq. (15). Repeating the steps leading to Eq. (45), we get the dual partition function where the source filed-dependant terms are present. Now by taking functional derivatives we get

$$\langle J^{\text{top}}_\mu\rangle_{\text{NC}\,CP^1} = 2si\left\langle J^N_\mu + 2F_\mu(\theta) - i4g^2\partial_\nu C_{\mu\nu} + \frac{1}{8s\pi}\epsilon_{\mu\nu\lambda}\partial_\lambda\partial_\alpha C_{\alpha\nu}\right\rangle_{\text{NC spin-}s}, \tag{50}$$

where $J^N_\mu = \operatorname{sgn}\lambda\bar{\Psi}\frac{\tau_\mu}{s\mathcal{A}}\Psi$. The over all factor i in the above will be removed when we do a Wick rotation from Euclidean space. From Eq. (46) it is clear that the current J^N_μ gets the θ dependence through $\mathcal{A}$. Thus the above map between correlators shows the interesting feature that the Noether charge is θ dependant where as the corresponding soliton charge is not. We also notice that a spin-s particle in the NC dual theory corresponds to a soliton of index $2s$ as in the commutative case.

4. Conclusion

We have studied the duality equivalence in the NC plane and showed that the NC CP^1 model coupled with Hopf term is equivalent to an interacting NC spin-s theory with $s = \frac{\pi}{2\lambda}$ where λ is the coupling strength of the Hopf term. We have shown this equivalence after re-expressing the NC CP^1 model with Hopf term using SW map, keeping terms to order θ. We recover the dual interacting spin-s theory constructed in the commutative plane in the limit $\theta \to 0$ from the NC dual theory obtained here. There are couple of points worth mentioning here. Ours is among the first to study the NC dual equivalence using path integral approach. Secondly dual of CP^1 model in NC space is different from NC version of the dual of CP^1 model in commutative space. We have also shown here that the statistics of the dual theory do not get affected by the non-commutativity of the space–time. The mapping between the correlators of topological and Noether currents shows that while the topological index is unaffected by NC parameter θ the Noether charge of the NC dual theory depends on θ.

It will be of interest to see what are the new solutions in NC CP^1 model obtained in [28] correspond to in the dual spin-s theory obtained here. CP^1 model coupled with Hopf term has been constructed and studied in the non-commutative sphere also [29]. It will be interesting to study whether the equivalence obtained here can be generalised to fuzzy sphere and to analyse the various limits of the dual theory on fuzzy sphere.

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