The median procedure in the semilattice of orders

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Abstract

Let $X$ be a finite set; we are concerned with the problem of finding a consensus order $P$ that summarizes an $m$-tuple (profile) $P^*$ of (partial) orders on $X$. A classical approach is to consider a distance function $d$ on the set $O$ of all the orders of $X$ and to search to minimize the remoteness $\sum_{1 \leq i \leq m} d(P, P_i)$. We study some properties of this median procedure, and compare it with some other consensus approaches. Besides the classical symmetric difference metric, other distances are considered, and we particularly address the consequences for the consensus problem of the existence of a semilattice structure on the set $O$.

Résumé

On considère un ensemble $X$ fini, et le problème de trouver un ordre consensus $P$ résumant un $m$-uplet (profil) $P^*$ d'ordres (partiels) sur $X$. Une approche classique est de munir l'ensemble $O$ de tous les ordres sur $X$ d'une métrique $d$, puis de chercher à minimiser l'éloignement $\sum_{1 \leq i \leq m} d(P, P_i)$. Nous étudions certaines propriétés de cette procédure médiane, et nous la confrontons à d'autres approches du problème. Plusieurs métriques sont envisagées, à côté de la classique distance de la différence symétrique, et nous abordons particulièrement les apports au problème du consensus de la prise en considération de la structure de demi-treillis de l'ensemble $O$.

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1. Introduction

In the domain of social choice, important literature is devoted to the aggregation (or consensus) of binary relations. These relations are used as preference models, and the
purpose is to obtain a collective preference relation. Many types of binary relations have been used in preference modelling. Most of them, e.g. linear orders, weak orders, semiorders, have the common feature that their asymmetric part (the strict preference) is transitive, i.e. it is an (strict, partial) order (see for instance [29]). So, in the scope of the celebrated Arrow Theorem [1], there were several axiomatic works on aggregation functions admitting the set $O_X$ of the orders on a fixed finite set $X$ as their range, sometimes also as their domain ([5,11,22]; see Section 3.3).

Indeed, except these axiomatic studies, the consensus of strict orders does not seem to have been extensively studied in the literature; for instance, contrary to the case of linear orders (see for instance [13] for a survey), the metric aggregation of orders, that is the research of median, not necessarily linear, orders remains to be studied. In this paper, we start from the general results about metric aggregation in semilattices [18,19] and particularize them to the semilattice of orders. Our purpose is to give properties of median orders that can be useful to decide, in the instance of the aggregation of orders problem, whether the metric approach is a good one or not (a similar work was done in the case of partitions by Barthélémy and Leclerc [7], in the domain of classification). Another use is to facilitate the effective research of median orders.

We mainly emphasize the fact that semilattices of orders are of the so-called lower locally distributive (LLD) type, see [27]. So, we begin in Section 2 by recalling some properties of these semilattices, together with a new metric characterization (Proposition 2.1). We also consider the specific case of orders. In Section 3, we survey some recent results concerning the consensus problem in abstract lattices; we essentially retain properties that are useful for the case of orders, for instance a property of the so-called quota rules in the case of LLD semilattices (Proposition 3.1). Section 4 deals with the specific study of orders. The main results are: for a wide set of metrics defined on $O_X$, the covering graph of a median order includes only majority pairs (Theorem 4.2); this is a generalization of a well-known property of median linear orders. With the classical symmetric difference metric, the median procedure on orders has the Pareto unanimity property (Theorem 4.4). Previously known and new counter-examples show that this property does not generalize to other metrics or LLD semilattices.

2. The semilattice of the orders on $X$

2.1. Meet-semilattices and the lower local distributivity case

Several recent works give evidence for the appropriateness of lattice theory in the abstract study of consensus problems [6,18,20,28]. Almost all the obtained results apply in fact to semilattices. Here, we recall some definitions and properties that will be particularly useful in the sequel; for general information on lattice theory, see [10].

Let $L$ be a finite ordered set; a lower bound of a subset $S$ of $L$ is an element $t \leq L$ such that $t \leq s$ for all $s \in S$. Dually, $t$ is an upper bound of $S$ if $s \leq t$ for all $s \in S$. When any pair $s, t$ of the elements of $L$ has a greatest lower bound (g.l.b), denoted as $s \wedge t$ (and called the meet of $s$ and $t$), $L$ is endowed with a (meet) semilattice algebraic
structure. The binary operation $\wedge$ is associative, commutative and idempotent. Then, any subset $S$ of $L$ has a meet, denoted as $\wedge S$, and also, if it is bounded above, a least upper bound (denoted as $\vee S$ and called the join of $S$). The join of two elements $s$ and $t$ is denoted as $s \vee t$. Every principal ideal $(\langle t \rangle) = \{s \in S: s \leq t\}$ of $L$ is a lattice, where the join of two elements always exists. Of course, a lattice is a semilattice, but we are concerned here with meet semilattices that are not lattices.

An element $j$ of $L$ is \textit{join irreducible} if $S \subseteq L$ and $j = \vee S$ imply $j \in S$; an equivalent property is that $j$ covers exactly one element, denoted by $p(j)$, of $L$; as usual, $s$ covers $t$ (denoted $t \prec s$) means that $t \leq s' < s$ implies $t = s'$. The set of all the join irreducibles is denoted by $J$; we set $J(s) = \{j \in J: j \leq s\}$ for any $s \in L$. Since $s = \vee J(s)$ for all $s \in L$, the set $J$ is the unique minimal subset generating the entire lattice by the join operation. A join irreducible is an \textit{atom} if it covers the minimum element $0_L = \wedge L$ of $L$. The semilattice $L$ is \textit{atomistic} if all its join irreducibles are atoms, that is, $p(j) = 0_L$ for any $j \in J$.

The representation of $s$ by the set $J(s)$ is one-to-one, with the properties: $J(s \wedge t) = J(s) \cap J(t)$ (but only $J(s) \cup J(t) \subseteq J(s \vee t)$). This representation is generally redundant; a subset $K$ of $J$ is an \textit{irredundant representation} of an element $s$ of $L$ if $s = \vee K$, and $\forall (K - \{j\}) < s$ for any $j \in K$.

Let us briefly describe several classes of lattices. A lattice $L$ is \textit{distributive} if it satisfies the \textit{distributivity laws}: for all $s, s', t \in L$, $(s \vee s') \wedge t = (s \wedge t) \vee (s' \wedge t)$, or, equivalently, $(s \wedge s') \vee t = (s \vee t) \wedge (s' \vee t)$. With a finite set $L$, this property has the following useful characterization (D) bearing on the join-irreducible elements:

(D) For $j \in J$ and $S \subseteq L$, the inequality $j \leq \vee S$ implies that there exists $s \in S$ such that $j \leq s$.

A finite lattice $L$ is \textit{lower semimodular} if, for every $s, t \in L$, $s \prec s \vee t$ and $t \prec s \vee t$ imply $s \wedge t \prec s$ and $s \wedge t \prec t$. Distributive lattices are lower semimodular. The lattice $L$ is \textit{ranked} if it admits a numerical \textit{rank function} $r$ such that $r(0_L) = 0$ and $s \prec t$ implies $r(t) = r(s) + 1$. Lower semimodular lattices are ranked.

A finite lattice $L$ is \textit{lower locally distributive} (LLD) if it satisfies the following equivalent conditions (among many others; see the survey of Monjardet [27]); for $s \in L$, we set $s^- = \big\{s' \in L: s' \prec s\}$:

(1)\textit{ every element $s$ of $L$ admits a unique irredundant representation,}
(2)\textit{ for any $s \in L$, the interval $[s^-, s]$ is a boolean lattice,}
(3)\textit{ $L$ is ranked with the rank function given by $r(s) = |J(s)|$, for any $s \in L$,}
(4)\textit{ $L$ is lower semimodular with the rank function as in (LLD3) above,}
(5)\textit{ for any $j, j' \in J$ and $s \in L$, $j \not\leq s$, $j' \not\leq s$ and $j \leq s \vee j'$ imply $j' \not\leq S \vee j$.}

Distributive lattices are LLD. Property (LLD5) is the \textit{anti-exchange} one. The unique irredundant representation of $s$ assumed in (LLD1) will be denoted as $D(s)$; so, $D(s) \subseteq J(s)$ and, for $K \subseteq J$, $s = \vee K \iff D(s) \subseteq K \subseteq J(s)$. Moreover, from $s \vee t = (\vee D(s)) \vee (\vee D(t))$, it follows $D(s \vee t) \subseteq D(s) \cup D(t)$, for all $s, t \in L$. This type of lattices is frequently associated with convexity considerations. For instance, the set of intervals
of a finite linearly ordered set is ordered by inclusion as an LLD lattice (the same for
the subtrees of a finite tree). Another fundamental example of an LLD lattice is the
lattice of Moore families on a given finite set [12].

All these definitions extend to semilattices according to the following principle: a
meet semilattice is said to be of a given type if all its principal ideals are lattices
of this type: distributive, LLD and lower semimodular semilattices are defined in this
way.

2.2. Metrics on semilattices

In this section, we recall some classes of metrics on posets studied, for instance, in
[19,26]; see also [7].

Let \( L \) be a finite meet semilattice and \( v \) a real function on \( L \). Assume that \( v \)
is strictly isotone: for any \( s, t \in L \), \( s < t \) implies \( v(s) < v(t) \). Such a function \( v \)
is said to be a lower valuation if it satisfies one the following two equivalent
properties:

(LV1) For all \( s, t \in L \) such that \( s \lor t \) exists, \( v(s) + v(t) \leq v(s \lor t) + v(s \land t) \);

(LV2) The real function \( d_v \) defined on \( L^2 \) by the following formula (1) is a metric
on \( L \):

\[
d_v(s, t) = v(s) + v(t) - 2v(s \land t).
\]

The equivalence of conditions (LV1) and (LV2) in lattices goes back to [10]; see [3,26]
for the more general cases of semilattices and other ordered sets. A characteristic prop-
erty of lower semimodular semilattices is that their rank functions are lower valuations.

When Condition (LV1) is satisfied, the metric \( d_v \) is a minimum path length metric
in the valued undirected covering graph \( C(L) \) of the semilattice \( L \). The vertices
of \( C(L) \) are the elements of \( L \), and an unordered pair \( st \) of elements of \( L \) is an edge of
\( C(L) \) if \( s \prec t \) or \( t \prec s \); the length of this edge is \( |v(s) - v(t)| \). The metric \( d_v \) associated
in this way to the rank function \( r \) of \( L \) is denoted as \( \partial \) and called the lattice metric
on \( L \). Then, this metric corresponds to minimum path lengths in the unvalued graph
\( C(L) \).

A simple way to define a lower valuation \( v \) on \( L \) is to consider a real strictly positive
mapping \( w \) on \( J \) and to set \( v(0_L) = 0 \) and, for any \( s \in L, s \neq 0_L, v(s) = \sum_{j \in J(s)} w(j) \).
Such a function \( v \) is a lower valuation on \( L \), and will be said a weight valuation. The
metric \( d_v \) will be said to be a weight metric. It is given by (1), or, more precisely, by
formula (2), where \( \Delta \) is the symmetric difference of subsets:

\[
d_v(s, t) = \sum_{j \in J(s) \Delta J(t)} w(j).
\]

With constant, unit, weights, one gets the symmetric difference metric, denoted as \( \delta \):
for all \( s, t \in L \), \( \delta(s, t) = |J(s) \Delta J(t)| \).

LLD semilattices have a very simple metric characterization by the equality of met-
rics \( \partial \) and \( \delta \), already observed for distributive semilattices [19].
Proposition 2.1. Let $L$ be a finite semilattice. Then, $L$ is lower locally distributive if and only if the equality $\hat{c} = \delta$ holds.

Proof. Assume $L$ is LLD. Then, it satisfies Conditions (LLD3) and (LLD4) above. From the latter, its rank function $r$ is a lower valuation and, from the former together with property (LV2), the expression of the metric $\hat{c} = d_r$ is, for all $s, t \in L$, $\hat{c}(s, t) = |J(s)| + |J(t)| - 2|J(s \wedge t)| = |J(s)| + |J(t)| - 2|J(s) \cap J(t)| = |J(s) \Delta J(t)| = \delta(s, t)$.

Conversely, assume $\hat{c} = \delta$ and set, for any $s \in L$, $r(s) = \hat{c}(0_L, s) = \delta(0_L, s) = |J(s)|$. We show that $r$ is the rank function of $L$. Consider two elements $s, t$ of $L$ with $s \not< t$. Then, $r(t)$ is the minimum length of a path between $0_L$ and $t$ in the unvalued graph $C(L)$. If the pair $st$ is an edge of this path, then $r(t) = r(s) + 1$; otherwise, we have $r(t) \leq r(s)$, which implies $|J(t)| \leq |J(s)|$, a contradiction with $J(s) \subset J(t)$. So, $r$ is a rank function and Condition (LLD3) is satisfied.

2.3. The lower locally distributive semilattice of orders

Let $X$ be a finite set of $n$ alternatives. A (strict) order on $X$ is a binary relation $P \subseteq X^2$ of ordered pairs of elements of $X$, satisfying the following three properties:

- antisymmetry: for any $x, y \in X$, $(x, y) \notin P$;
- asymmetry: for any $x, y \in X$, $(x, y) \in P$ implies $(y, x) \notin P$;
- transitivity: for any $x, y, z \in X$, $(x, y) \in P$ and $(y, z) \in P$ imply $(x, z) \in P$.

The assertions $(x, y) \in P$ and $(x, y) \notin P$ are also denoted here $xPy$ and $xP^c y$, respectively. The order $P$ is linear if, for any $x, y \in X$, $xP^c y$ implies $yPx$.

Let $O_X$ (or simply $O$) be the set of all the orders on $X$, defined as above. Since the intersection of two orders is still an order, this set, ordered with inclusion, is a meet semilattice. Every linear order is maximal in $O$, which is not a lattice. If two orders $P$ and $P'$ are both included in some order $Q$, they have a join $P \vee P'$, which is in fact the transitive closure of the binary relation $P \cup P'$. A first study of the semilattices of orders is found in [2]. The following properties are obvious: the minimum of $O$ is the empty relation; an atom (covering the minimum) is an order, denoted $A_{xy}$ containing a unique pair $(x, y)$. Since, obviously, every order $P$ satisfies $P = \bigvee\{A_{xy} : (x, y) \in P\}$, the semilattice $O$ is atomistic. The set of all the atoms of $O$ is denoted as $\mathcal{A}$.

We denote $\prec$ the covering relation on $O$, and $\prec_P$ the covering relation on $X$ corresponding to a given order $P$, element of $O$. We recall a characterization of the relation $\prec$.

Proposition 2.2. Let $P, Q$ be two elements of $O$. Then, $Q$ covers $P$ if and only if there exist $x, y \in X$ such that $x \prec_Q y$ and $P = Q - \{(x, y)\}$.

Proof. Let $Q \in O$ and $x, y \in X$ such that $x \prec_Q y$. We show that $P = Q - \{(x, y)\}$ is still an element of $O$. Obviously, $P$ is antireflexive and asymmetric. It is also transitive, unless there exists $z \in X$ such that $xQz$ and $zQy$. But this would contradict the
hypothesis that $y$ covers $x$ in $Q$. So, $P \in O$, $P \subseteq Q$ and $P \prec Q$ since these two orders differ by only one pair.

Conversely, assume $P, Q \in O$ and $P \prec Q$. For any $(x, y) \in Q - P$, one may find $x_1, \ldots, x_k$ such that $x \prec_Q x_1, x_1 \prec_Q x_2, \ldots, x_{k-1} \prec_Q x_k, x_k \prec_Q y$. If none of these covering pairs belongs to $Q - P$, then $(x, y) \in P$, a contradiction. So, there is at least one covering pair $(x, y) \in Q - P$. Then, from the first part of the proof, $P = Q - \{(x, y)\}$. 

As a consequence of this result, the semilattice $O$ satisfies Condition (LLD3) above and is LLD. A family of binary relations having, ordered by inclusion, the cardinality as rank function is called well-graded by Doignon and Falmagne [14]. Any well-graded family which is closed under intersection defines an LLD semilattice.

The pairs of elements of $X$ appearing in the unique irredundant representation $D(P)$ of an order $P$ are those of the diagram of $P$. The interval $[P^-, P]$ is isomorphic to the lattice $\mathcal{P}(D(P))$ by the correspondence $R \subseteq D(P) \iff P - R \in [P^-, P]$.

The symmetric difference metric on $O$ is given by $\hat{d}(P, Q) = \hat{d}(P, Q) = |P \Delta Q|$. This metric is widely used on sets of binary relations, since its introduction by Kemeny [16] to define median linear orders. An axiomatic characterization of this distance on orders, among many sets of binary relations, is given in [4]. More generally, a weight metric $d_v$ on $O$ has the form $d_v(P, Q) = \sum_{(x, y) \in P \Delta Q} w(x, y)$, where $w$ is a strictly positive weighting of the ordered pairs of distinct elements of $X$. Such a weighting may come, for instance, from some previous knowledge on the pairs of alternatives in a specific instance of the aggregation problem.

Remark. In any semilattice, another class of metrics correspond to weightings of the meet irreducible elements (the dual notion of join-irreducibles); these metrics are called co-weight metrics in [7]. In the case of the semilattice of orders, the meet-irreducible elements are the linear orders on $X$. Due to the factorial increase of their number with the cardinality of $X$, the use of co-weight metrics on $O$ cannot be computationally efficient and presents also theoretical drawbacks. They do not seem adequate for the research of consensus orders.

3. Consensus elements in semilattices

3.1. Approaches for consensus

Let $L$ be a finite semilattice. A profile of length $m$ of $L$ is a $m$-tuple $s^* = (s_1, \ldots, s_m)$ of elements of $L$. A consensus element of $s^*$ is an element $s \in L$ that summarizes $s^*$ in some useful sense. The so-called consensus problem consists of making clear what is meant by “useful sense” and determining explicitly the related consensus elements. In the case of orders, it arises, for instance, when considering $m$ judges or $m$ criteria, each of them providing a preference order $P_i$, $1 \leq i \leq m$ on the elements of $X$. Then, a consensus order $P$ is a good candidate for a collective preference or multi-criterion order.

We denote by $L^*$ the set $\bigcup_m L^m$ of all profiles of $L$. The concatenation $s^*s'^*$ of two profiles $s^*$ and $s'^*$ is defined as usual. Three ways arise to tackle the consensus
problem, respectively, referred to as \( m \)-procedures, complete procedures, or complete multiprocedures:

- An \( m \)-procedure is a map \( f : L^m \to L \).
- A complete procedure is a map \( L^* \to L \).
- A complete multiprocedure is a map \( L^* \to 2^L \).

We say that a complete multiprocedure is definite if it is a map \( L^* \to 2^L - \{\emptyset\} \).

Three overlapping approaches have been used to tackle the consensus problem:

1. The axiomatic approach (in the case of orders, see [5,11,17,20,28]) is an extension of the classical Arrowian approach in the framework of social choice theory [1] and weak orders. In such an approach one retains, as a consensus order, an order satisfying some conditions that arise from experimental evidence or from ethical considerations. This approach leads to problems of existence (with possibility/impossibility theorems) and uniqueness of orders satisfying these conditions.

2. The constructive approach, where a way to construct a consensus is explicitly given. The most obvious constructive approach is surely the unanimity (or Pareto) rule, called also the strict consensus rule, which is described below for finite semilattices.

3. The combinatorial optimization approach. Here we have at our disposal some criterion measuring the remoteness \( \rho(s,s^*) \) of any element \( s \) of \( L \) to the given profile \( s^* \); we search for the solutions (or one of them) of \( \min \rho(s,s^*) \) in the semilattice \( L \).

3.2. A class of algebraic procedures: the quota rules

A first type of aggregation rules is provided by the algebraic structure of \( L \). Given a real number \( q \in [0,1] \), the \( q \)-quota rule consensus rule \( c_q \) is defined as

\[
c_q(s^*) = \bigvee \{ j \in J : \gamma(j) > q \},
\]

provided such a join exist for the given profile \( s^* \) (recall \( \bigvee \emptyset = 0_L \)). The index \( \gamma(j) = \gamma(j,s^*) \) is the number \( \vert I(j,s^*)/m \vert \), with \( I = \{1,\ldots,m\} \) and \( I(j,s^*) = \{i \in I : j \leq s_i\} \).

Similarly, the weak \( q \)-quota rule consensus function \( b_q \) is defined as

\[
b_q(s^*) = \bigvee \{ j \in J : \gamma(j) \geq q \}.
\]

In a semilattice, these consensus rules may be thought of as complete multiprocedure returning at most one element (that is, not definite); they become complete procedures in a lattice. From the definitions, \( q \leq q' \) and \( c_q(s^*) \) exists imply that \( c_{q'}(s^*) \) exists and \( c_{q'}(s^*) \leq c_q(s^*) \). Certain values of \( q \) correspond to the algebraic formalizations of classical consensus rules; the following notations and terminology will be used in the sequel:

\[
q = 0.5: \quad b(s^*) = \bigvee \{ j \in J : \gamma(j) \geq 0.5 \} \quad \text{(weak majority rule)},
\]

\[
c(s^*) = \bigvee \{ j \in J : \gamma(j) > 0.5 \} \quad \text{(majority rule)};
\]

\[
q = 1: \quad u(s^*) = \bigvee \{ j \in J : \gamma(j) = 1 \} \quad \text{(unanimity rule)}.
\]
Note that \( u(s^*) = \bigwedge_{1 \leq i \leq m} s_i \) always exists. The properties are recalled or proved in the sequel under condition, not systematically recalled, of existence of \( c_\theta(s^*) \) or \( b_\theta(s^*) \).

For instance, \( u(s^*) \leq c(s^*) \leq b(s^*) \) is always true when \( b(s^*) \) exists; if \( m \) is odd, one has \( c(s^*) = b(s^*) \).

In the following, we state a property of quota rules in LLD semilattices, not at all general for quota rules; for instance it is not true in the partition lattice [7].

**Proposition 3.1.** Let \( s^* \) and \( s'^* \) be two profiles of an LLD semilattice \( L \), and \( s \in L \) such that \( c_\theta(s^*) = c_\theta(s'^*) = s \). Then \( c_\theta(s^*s'^*) = s \).

**Proof.** Set \( K_\theta(s^*) = \{ j \in J : \gamma(j) > q \} \). We have \( D(s) \subseteq K_\theta(s^*) \subseteq J(s) \) and \( D(s) \subseteq K_\theta(s'^*) \subseteq J(s') \). We use the standard property of frequencies: for any \( j \in J \), \( \min(\gamma(j,s^*), \gamma(j,s'^*)) \leq \gamma(j,s^*s'^*) \leq \max(\gamma(j,s^*), \gamma(j,s'^*)) \).

So, consider a join irreducible \( j \in J \); if \( j \in D(s) \), then \( q \leq \min(\gamma(j,s^*), \gamma(j,s'^*)) \) and \( j \in K_\theta(s^*s'^*) \); if \( j \notin J(s) \), then \( \max(\gamma(j,s^*), \gamma(j,s'^*)) \leq q \) and \( j \notin K_\theta(s^*s'^*) \). So, \( D(s) \subseteq K_\theta(s^*s'^*) \subseteq J(s) \) and the result holds. \( \square \)

### 3.3. An axiomatic characterization of a class of m-procedures

Here, we consider \( m \)-procedures, that is consensus functions \( f \) which associate a unique element \( s \) of \( L \) to each profile \( s^* \) of length \( m \) of \( L \):

- The function \( f \) is **decisive** if for all \( j \in J \), \( s^*, s'^* \in L^m \),
  \[
  [I(j, s^*) = I(j, s'^*)] \implies [j \leq f(s^*) \iff j \leq f(s'^*)].
  \]

- The function \( f \) is **Paretian** if for every \( s^* \in L^m \), \( \bigwedge_{1 \leq i \leq m} s_i \leq f(s^*) \).

A **federation** on \( I = \{1, \ldots, m\} \) is a family \( \mathcal{F} \) of subsets of \( I \) satisfying the monotonicity property: \( [I \in \mathcal{F}, I' \supseteq I] \Rightarrow [I' \in \mathcal{F}] \). A federation consensus function \( f_\mathcal{F} \) on \( L \) is associated with any federation \( \mathcal{F} \) by \( f_\mathcal{F}(s^*) = \bigvee_{I \in \mathcal{F}} \left( \bigwedge_{i \in I} s_i \right) \). Especially, if \( \mathcal{F} = \{ I' \subseteq I : I' \supseteq I_0 \} \), for a given subset \( I_0 \) of \( I \), then \( f_\mathcal{F}(s^*) = \bigwedge_{i \in I_0} s_i \) is an **oligarchic** consensus function. The unanimity rule \( u \) is the oligarchic function with \( I_0 = I \), while an oligarchic procedure \( f \) reduces to the unanimity rule if and only if it is **symmetrical**, i.e. \( f(s_1, \ldots, s_m) = f(s_{\sigma(1)}, \ldots, s_{\sigma(m)}) \), for any permutation \( \sigma \) of \( \{1, \ldots, m\} \). More generally, it may be shown that the quota rules defined above have a lattice polynomial expression and constitute a special class of federation consensus functions.

In the semilattice \( L \), a dependence relation \( \beta \) is defined on the set \( J \) by: \( j\beta j' \) if \( j \neq j' \) and there exists \( t \in L \) such that \( j, j' \not< t \) and \( j < t \lor j' \). The following theorem is one of those established in [28]; see also [20].

**Theorem 3.2.** If the graph \((J, \beta)\) is strongly connected, then a federation consensus function \( f \) on \( L \) is oligarchic if and only if it is decisive and Paretian.

This is the case of the semilattice \( O \). This “Arrowian” characterization of oligarchic functions on orders (which include the unanimity rule \( u \)) has been given by Brown [11] and Barthélémy [5]; a similar result on equivalence relations is due to [24]. A common
proof of Brown and Mirkin results is given in [17], the unifying theory being provided in [28]. This oligarchic result admits extensions to other domains. A well-known one, where the domain consists of profiles of linear orders, was given by Mas-Collel and Sonnenschein [22].

3.4. Medians for metrics related with the semilattice structure

Let \( s^* = (s_1, \ldots, s_m) \) be a profile of \( L \), and \( d \) a metric on \( L \). We consider the multiprocedure \( \mathcal{M}_d \) which consists of searching for the medians of \( s^* \) for the metric \( d \), that is the elements \( \mu \) of \( L \) such that the remoteness \( \rho(\mu, s^*) = \sum_{i=1}^m d(\mu, s_i) \) is minimum [8]. As a complete and definite multiprocedure, this median procedure has the nice property of Young consistency, introduced by Young [32]: if the profiles \( s^* \) and \( s'^* \) are such that \( \mathcal{M}_d(s^*) \cap \mathcal{M}_d(s'^*) \neq \emptyset \), then \( \mathcal{M}_d(s^*s'^*) = \mathcal{M}_d(s^*) \cap \mathcal{M}_d(s'^*) \).

Indeed, according to Proposition 3.1, this property is also fulfilled, under definiteness, by quota rules in LLD semilattices.

With a weight metric, as defined above in Section 2.2, in a semilattice structure, medians are related with majority rules. In this case, the remoteness function \( \rho(s, s^*) \) may be written as

\[
\rho(s, s^*) = m \left( \Sigma - \sum_{j \in J(s)} (2\gamma(j) - 1)w(j) \right),
\]

where the constant \( \Sigma = \sum_{j \in J} \gamma(j)w(j) \) depends only on the profile \( s^* \).

Set, for each \( s \in L \), \( s_c = \sqrt[j \in J(s): \gamma(j) > 0.5]{j} \) and \( s_b = \sqrt[j \in J(s): \gamma(j) \geq 0.5]{j} \); then, \( s_c \) and \( c(s^*) \) whenever \( c(s^*) \) exists and \( s_b \) whenever \( b(s^*) \) exists.

From the results in [18,19] about medians for weight metrics on semilattices, we have:

**Proposition 3.3.** Let \( d \) be a weight metric on the semilattice \( L \). Then:

- for any profile \( s^* \) and median \( \mu \in \mathcal{M}_d(s^*) \), the equality \( \mu = \mu_b \) holds,
- there exists at least one median \( \mu_0 \in \mathcal{M}_d(s^*) \) satisfying \( \mu = \mu_c \).

In other terms, any median is the join of weak majority join irreducibles (that is, such that \( \gamma(j) \geq 0.5 \)) and at least one median is the join of majority join irreducibles (that is, such that \( \gamma(j) > 0.5 \)). This fact will be extensively used in Section 4. In the cases where \( c(s^*) \) or \( b(s^*) \) exist, we have:

**Theorem 3.4.** Let \( d \) be a weight metric on the semilattice \( L \). Then:

- for any profile \( s^* \) such that \( b(s^*) \) exists and for any median \( \mu \in \mathcal{M}_d(s^*) \), the inequality \( \mu \leq b(s^*) \) holds,
- for any profile \( s^* \) such that \( c(s^*) \) exists and for any median \( \mu \in \mathcal{M}_d(s^*) \), there exists a median \( \mu_0 \) of \( s^* \) such that:
(i) \( \mu_0 \leq c(s^*) \),
(ii) \( \mu_0 \leq \mu \),
(iii) every element of the interval \([\mu_0, \mu]\) is a median.

4. Median orders

4.1. Consensus rules in the semilattice of orders

Let \( P^* = (P_1, \ldots, P_m) \) be a profile of \( O \). For a join irreducible (atom) \( A_{xy} \) of \( O \), we use the notations \( I(xy, s^*), \gamma(xy, P^*) \) or \( \gamma(xy) \) instead of \( I(A_{xy}, s^*), \gamma(A_{xy}, P^*) \) or \( \gamma(A_{xy}) \).

For \( q \in [0, 1[, \) set \( E_q(P^*) = \{(x, y) \in X^2 : \gamma(xy, P^*) > q\} \); the consensus \( c_q(P^*) \) exists and is equal to the order \( P \) if and only if the inclusions \( D(P) \subseteq E_q(P^*) \subseteq P \) hold. Such an order \( P \) exists if and only if the oriented graph \( G_q(X, E_q(P^*)) \) has no circuit. The possibility of circuits for \( q < 1 \) is easy to recognize, and well-known in the case of the majority rule for a profile of linear orders, where it is called Condorcet effect.

For any profile \( P^* \), there exists a value \( \alpha(P^*) \in [0, 1] \) such that \( G_q \) has no circuit if and only if \( q \geq \alpha(P^*) \); formally, \( \alpha(P^*) = \min_H \text{ circuit on } x \max_{(x, y) \in H} \gamma(xy) \).

As noted above, the unanimity rule defined by \( u(P^*) = \bigcap_{1 \leq i \leq m} P_i \) is definite. It states that \( xPy \) holds in the consensus order \( P \) if and only if \( xPy \) holds in all the orders \( P_i \) of the profile. Moreover, it was recalled in Section 3.3 that this rule satisfies the symmetry and decisiveness conditions, the latter being now stated as

\[
[I(xy, P^*) = I(xy, P'^*)] \Rightarrow [xf(P^*)y \Leftrightarrow xf(P'^*)y].
\]

As recalled in [28], such an axiom is close to Arrow’s “independence of irrelevant alternatives”, where it is assumed, in the case of weak orders, that the restriction of \( f(P^*) \) to the subset \( \{x, y\} \) of \( X \) depends only on the restrictions of the \( P_i \)'s to that subset. Indeed, applied to orders, the decisiveness axiom is stronger, because it deals separately with each pair \( (x, y) \) and \( (y, x) \). In the case of orders, the unanimity rule is generally not adequate, since an order of low cardinality is obtained in most cases. Thus, despite the good properties of this rule, other consensus methods are needed. The consensus rule \( c_{\alpha(P^*)} \) may be of practical interest; the following counter-example and Part (ii) of Proposition 4.1 show that it does not satisfy consistency, but a weakened form of this property.

Counter-example. Let \( X = \{a, b, c, d, e, f\} \) and the profiles \( P^* = (A, B, C) \) and \( P'^* = (A', B', C') \) of \( O_X \) according to Fig. 1. We have \( \alpha(P^*) = \alpha(P'^*) = 1/3 \), and \( c_{\alpha(P^*)}(P^*) = c_{\alpha(P'^*)}(P'^*) = P \), while \( \alpha(PP'^*) = 1/6 \) and \( c_{\alpha(PP'^*)}(P^*P'^*) = Q \).

Proposition 4.1. Let \( P^* \) and \( P'^* \) be two profiles of \( O \), of lengths \( m \) and \( m' \), respectively, and \( P \in O \) such that \( c_{\alpha(P^*)}(P^*) = c_{\alpha(P'^*)}(P'^*) = P \). Then:

(i) \( \max(m_2(P^*)m_2(P'^*)) \leq \alpha(P^*P'^*) \leq \frac{m_2(P^*)m_2(P'^*)}{m+m'} \),
(ii) \( P \subseteq c_{\alpha(P^*P'^*)}(P^*P'^*) \).
Theorem 4.2.

Proof. We use the absolute frequencies instead of the relative ones. So, we set, for \( q \in [0, 1] \), \( F_q(P^*) = E_q(P^*) \), \( F_{m+q}^{q}(P^*) = E_q(P^{*}) \), and \( F_{(m+m')q}(P^*P^{*}) = E_q(P^{*}P^{*}) \). In the same way, we set \( \beta(P^*) = m \gamma(P^*) \), \( \beta(P^{*}) = m \gamma(P^{*}) \), and \( \beta(P^{*}P^{*}) = (m + m') \gamma(P^{*}P^{*}) \). Assume \( 0 < \beta(P^*) \leq \beta(P^{*}) \); there is a circuit in \( F_{[\beta(P^*)]}(P^*) \), and this circuit still exists in \( F_{[\beta(P^*)]}(P^{*}P^{*}) \). Thus, \( \max(\beta(P^*), \beta(P^{*})) \leq \beta(P^{*}P^{*}) \), which leads to the first inequality of (i).

By the hypotheses, \( D(P) \subseteq F_{(\beta(P^{*})+\beta(P^{*}))(P^{*}P^{*})} \subseteq P \) and \( D(P) \subseteq F_{(\beta(P^{*}))(P^{*}P^{*})} \subseteq P \). Then, for \((x, y) \in D(P)\), one gets \((x, y) \in F_{(\beta(P^{*})+\beta(P^{*}))(P^{*}P^{*})} \) and, for \((x, y) \notin P \), \((x, y) \notin F_{(\beta(P^{*})+\beta(P^{*}))(P^{*}P^{*})} \). From the latter, \( F_{(\beta(P^{*}))(P^{*}P^{*})} \subseteq P \), which has no circuit; so, \( \beta(P^{*}P^{*}) < \beta(P^{*}) + \beta(P^{*}) \), which leads to the second inequality of (i). Moreover, \( c_{[\beta(P^{*})+\beta(P^{*})](m+m')}(P^{*}P^{*}) = P \subseteq c_{[\beta(P^{*}P^{*})](m+m')}(P^{*}P^{*}) \). The part (ii) follows. \( \Box \)

Now we come to the median procedure for a weight metric \( d \). For \( q \in [0, 1] \), set \( E_{q}'(P^*) = \{(x, y) \in X^2; \gamma(x, y) \geq q\} \). For the remoteness, Formula (3) and Proposition 3.3 of Section 3.4 lead, respectively, to Formula (4) and Theorem 4.2 in the case of orders.

\[
\rho(P, P^*) = \frac{m}{2} \left( \Sigma - \sum_{(x, y) \in P} (2 \gamma(x, y) - 1)w(x, y) \right), \tag{4}
\]

where \( \Sigma = \sum_{(x, y) \in X^2} \gamma(x, y)w(x, y) \).

Theorem 4.2. Let \( d \) be a weight metric on the semilattice \( O \), and \( P^* \) a profile of \( O \). Then:

- for any median order \( M \in M_d(P^*) \), \( D(M) \subseteq E_{0.5}(P^*) \),
- there exists at least one median order \( M_0 \in M_d(P^*) \) satisfying \( D(M_0) \subseteq E_{0.5}(P^*) \).

Proof. By Proposition 3.3, we know that there is a subset \( E \) of \( E_{0.5}(P^*) \) such that \( M = \bigvee \{A_{xy}: (x, y) \in E\} \). By condition (LLD1) of Section 2.1, the unique
Theorem 4.2 generalizes the fact that, in the aggregation of linear orders, any median linear order corresponds to an hamiltonian path of the majority tournament of the given profile (see [13]). Another proof of this result could be obtained from the observation that, if $D(P) \subseteq E_{0.5}(P^*)$, then the remoteness is an antitone function on the boolean lattice $[P^*, P]$. Indeed, any median order $M$ has the form $M_0 \cup A$, where $D(M_0) \subseteq E_0$ and $A \subseteq \{(x, y) \in X^2; \gamma(xy) = 0.5\}$. In practice, when the medians whose diagrams are constituted of majority pairs only are known, it is not difficult to obtain all the medians by additions of pairs $(x, y)$ with $\gamma(xy) = 0.5$. Especially, when $m$ is odd, all the covering pairs in a median order are majority ones.

Example. Let $X = \{a, b, c, d\}$. Consider the $(2k + 1)$-profile $P^*$ of orders on $X$ where each order $A$ and $B$ of Fig. 2 appears $k$ times, the last element of $P^*$ being the linear order $T$. For this profile, $E_{0.5}(P^*) = \{(b, c), (c, d), (d, a)\}$. According to Theorem 4.2, the medians of $P^*$ take their covering pairs in $E_0$; so, we find them by checking, according to Formula (4), only eight orders among the 219 possible ones on $X$.

For $k = 1$, the unique median for the metric $\delta$ is the order $C$ in Fig. 2. For $k \geq 2$, there are four medians: $A, B, A_{cd}$, and the linear order $T'$.

Let $Y_1, \ldots, Y_p$ be the connected components of the relation $E_{0.5}(P^*)$, and, for $h = 1, \ldots, p$, set $R^h = Y_h^2$ and $R_0 = X^2 - (\bigcup_{1 \leq h \leq p} R^h)$. Let $P$ be an order included in $\bigcup_{1 \leq h \leq p} R^h$; from expression (4), the remoteness of $P$ admits an additive decomposition according to the $Y_h$’s as

$$\rho(P, P^*) = m \left( \sum_0 + \sum_{1 \leq h \leq p} \left( \sum_{h} - \sum_{(x, y) \in R_h} \bigcap_P (2 \gamma(xy) - 1)w(xy) \right) \right),$$

with the constants $\Sigma_h = \sum_{(x, y) \in R_h} \gamma(xy)w(xy)$, for $h = 0, \ldots, p$. Let $P|Y_h$ be the restriction of $P$ to $Y_h$ and $P^*|Y_h$ the profile of $O_{Y_h}$ whose components are the restrictions of the orders $P_i$ to $Y_h$.

Proposition 4.3. Let $d$ be a weight metric on the semilattice $O$, and $P^*$ a profile of $O$. Then, an order $M$ is a median if and only if $P \subseteq \bigcup_{1 \leq h \leq p} R^h$ and, for all $h = 1, \ldots, p$, $P|Y_h \in \mathcal{I}_d(P^*|Y_h)$.
**Proof.** By Theorem 4.2 above, we know that there is no pair of a median order \( M \) in the set \( R^0 \). The result is then an immediate consequence of Formula (5).

So, as soon as \( E_{0.5}(P^*) \) has several connected components, the research of medians may be solved separately on each of them. For instance, if \( E_{0.5}(P^*) = \emptyset \), then the only median is the empty order. It follows from Proposition 4.3 that the median procedure for orders has the following property, which is called here stability for objects by analogy with the case of partitions, see [7]:

(i) Let \( P \) be a consensus element of the profile \( P^* \) and let \( P_{\bar{Y}} \) be the restriction of \( P \) to \( Y \) and \( P^*_{\bar{Y}} \) the profile of \( O_{\bar{Y}} \), whose components are the restrictions of the orders \( P_i \) to \( Y \). Then \( P_{\bar{Y}} \) is a consensus element of \( P^*_{\bar{Y}} \).

(ii) Moreover, if \( P^* \) is a profile of \( O \) and \( P \) an order on \( X \), then \( P \) is a consensus element of \( P^* \) if and only if, for each union \( Y \) of connected classes of \( P \), the order \( P_{\bar{Y}} \) is a consensus element of \( P^*_{\bar{Y}} \).

### 4.2. Pareto property with the lattice metric on \( O \)

As the last example of Section 4.1 shows, a ordered pair \((b, a)\) with a null index may belong to a median order. Concerning the unanimity pairs (with the maximum index 1), we establish a specific property of the lattice metric \( \delta \) (or \( \delta \)): the corresponding median procedure has the so-called Pareto property: any median contains any unanimity pair.

**Theorem 4.4** (Birfet and Leclerc; see Birfet [9]). Let \( P^* \) be a profile of \( O \) and two distinct elements \( x \) and \( y \) of \( X \) such that \( \gamma(xy) = 1 \). Then, \((x, y) \in M \) for any median order \( M \) of \( P^* \) for the metric \( \delta \).

**Proof.** Assume that there exists a median \( M \) and a unanimity pair \((x_0, y_0)\) (that is, \( x_0 P_i y_0 \) for all \( i \)) with \((x_0, y_0) \notin M \). Two cases may arise: \((y_0, x_0) \in M \) or \((y_0, x_0) \notin M \).

If \((y_0, x_0) \in M \), consider the relation \( M' \) obtained by the exchange of \( y_0 \) and \( x_0 \). So, \((x_0, y_0) \in M' \); for \( z \in X \), \( z \neq x_0, y_0 \), \( x_0 M' z \) \( \iff \) \( y_0 M z \), \( z M' x_0 \) \( \iff \) \( z M y_0 \), \( y_0 M' z \) \( \iff \) \( x_0 M z \) and \( z M' y_0 \) \( \iff \) \( z M x_0 \); for \( z, z' \in X \), \( z, z' \neq x_0, y_0 \), \( z M' z' \) \( \iff \) \( z M z' \). Obviously, \( M' \) is an order on \( X \), isomorphic to \( M \). We compare the quantities \( \sum_{(x,y) \in M} \gamma(xy) \) and \( \sum_{(x,y) \in M'} \gamma(xy) \). Observe that \( x_0 M z \) and \( y_0 M x_0 \) imply \( y_0 M z \) and, so, \( x_0 M' z \); similarly, \( z M y_0 \) implies \( z M' y_0 \). The orders \( M \) and \( M' \) differ only on the pairs of the \( y_0 M z \) and \( z M x_0 \) types, respectively changed into \( x_0 M' z \) and \( z M' y_0 \), and on the replacement of the pair \((y_0, x_0)\) by \((x_0, y_0)\). For \( z \neq x_0, y_0, x_0 P_i y_0 \) and \( y_0 P_i z \) imply \( x_0 P_i z \) and, so, \( \gamma(x_0 z) \geq \gamma(y_0 z) \) and, similarly, \( z P_i x_0 \) and \( x P_i y_0 \) imply \( z P_i y_0 \) and, so, \( \gamma(z y_0) \geq \gamma(z x_0) \). Moreover, \( \gamma(x_0 y_0) = 1 \) and \( \gamma(y_0 x_0) = 0 \). Reporting these observations in the remoteness formula (1), with unit weights, and observing that the orders \( M \) and \( M' \) have the same cardinality, we obtain \( \rho(M', \pi) \leq \rho(M, \pi) - 1 \), a contradiction with the assumption that \( M \) is a median.
If \((y_0, x_0) \not\in M\) (that is, \(x_0\) and \(y_0\) are not comparable for the order \(M\)), consider the subsets \(Z = \{z \in X: x_0zMz\} \) and \(Z' = \{z \in X: y_0zMz\} \) and \(M' = (M - S) \cup T \) for a profile of a distributive (thus, LLD) semilattice endowed with the metric \(\rho\) that is an order. Assume \(\sum_{(x,y) \in E} (2\kappa(x,y) - 1) \leq \sum_{(x,y) \in T} (2\kappa(x,y) - 1)\), and consider the binary relation \(M' = (M - S) \cup T \cup \{(x_0, y_0)\}\). We first show that \(M'\) is an order.

- The deletion of the pairs of \(S\) preserves the transitivity property; otherwise, there exist \(z \in Z\) and \(z' \not\in Z\) such that \(zMz'\). Then, it follows from \(z'My_0\) that \(zMy_0\), a contradiction.
- The addition of the pairs of \(T\) preserves transitivity: let \(z, z' \in X\) with \(z(M - S)x_0\) and \((x_0, z') \in T\), that is \(y_0Mz'\). Since \(z \not\in S\), \(zMy_0\), \(zMz'\) and \(z(M - S)z'\) hold.
- The addition of the pairs of \(T\) preserves asymmetry: otherwise, there exists \(z \in Z\) such that \(z(M - S)x_0\), which implies \(zMx_0\) and, by transitivity of \(M\), \(y_0Mx_0\), a contradiction.
- The addition of the pair \((x_0, y_0)\) preserves transitivity: one has \(z((M - S) \cup T')x_0 \Rightarrow z(M - S)x_0 \Rightarrow zMy_0\).

Now we compare the remotenesses of \(M\) and \(M'\). We have \(\sum_{(x,y) \in M'} (2\kappa(x,y) - 1) = \sum_{(x,y) \in E} (2\kappa(x,y) - 1) - \sum_{(x,y) \in S} (2\kappa(x,y) - 1) + \sum_{(x,y) \in T} (2\kappa(x,y) - 1) + 1\), with, by hypothesis, \(\sum_{(x,y) \in E} (2\kappa(x,y) - 1) < \sum_{(x,y) \in T} (2\kappa(x,y) - 1)\). Moreover, for any \(z \in Z\), \(x_0P_1y_0\) and \(y_0P_2z\) imply \(x_0P_1z\) and, so, \(\gamma(x_0z) \geq \gamma(y_0z)\). Thus, since \(T\) and \(T'\) have the same cardinality, \(\sum_{(x,y) \in T} (2\kappa(x,y) - 1) \leq \sum_{(x,y) \in T'} (2\kappa(x,y) - 1)\) and \(\sum_{(x,y) \in M'} (2\kappa(x,y) - 1) \geq \sum_{(x,y) \in M} (2\kappa(x,y) - 1)\), which implies \(\rho(M', \pi) \leq \rho(M, \pi) - 1\), again a contradiction with the assumption that \(M\) is a median.

The case \(\sum_{(x,y) \in T} (2\kappa(x,y) - 1) < \sum_{(x,y) \in S} (2\kappa(x,y) - 1)\) is similar, with the order \(M'' = (M - T) \cup S \cup \{(x_0, y_0)\}\) instead of \(M'\).

The Pareto property is required in many consensus problems, where its absence may be thought of as a paradox. But Theorem 4.4 does not seem to have significant extensions to more general cases of weight metrics in LLD semilattices. A counter-example for a profile of a distributive (thus, LLD) semilattice endowed with the metric \(\hat{\kappa}\) is given in [19]. Two counter-examples about the Pareto property for medians for \(\hat{\kappa}\) in LLD lattices were provided by Li [21], in a dual presentation. The following counter-example shows that the Pareto property of median orders is no longer satisfied in \(O\) with a weight metric \(\hat{\kappa}\) different from \(\hat{\kappa}\).

**Counter-example.** Let \(X = \{a, b, c, d\}\) and the profile \(P^* = (T, A, B)\) of \(O_X\) according to Fig. 3. Let \(d\) be the weight metric associated with the weight function \(w\) defined by \(w(a, b) = w(c, d) = w(a, d) = 10; w(a, c) = w(b, d) = 4; w(x, y) = 1\) for all other ordered pairs \((x, y)\) of distinct elements of \(X\). The unique unanimity pair is \((b, c)\) and the other majority pairs are \((a, b)\) and \((c, d)\). According to Theorem 4.2, \(\mathcal{A}_d(P^*) \subseteq \{\emptyset, A_{ab}, A_{bc}, A_{cd}, A, B, C, T\}\). A straightforward investigation leads to the conclusion that \(C\) is the unique median, with \((b, c) \not\in C\). This example shows also that, contrary to the
Table 1

<table>
<thead>
<tr>
<th>Type of semilattice</th>
<th>Weight metrics</th>
<th>Metric $\delta$</th>
<th>Metric $\hat{\delta}$</th>
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<tbody>
<tr>
<td>Distributive lattice</td>
<td>Yes [25]</td>
<td>Yes$^{(\leftarrow)}$</td>
<td>Yes$^{(\leftarrow)}$</td>
</tr>
<tr>
<td>LLD lattice</td>
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<td>No$^{(\rightarrow)}$</td>
<td>No [21]</td>
</tr>
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<td>No$^{(\uparrow)}$</td>
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<tr>
<td>Distributive semilattice</td>
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<td>No [19]</td>
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<tr>
<td>LLD semilattice</td>
<td>No$^{(\uparrow)}$</td>
<td>No$^{(\uparrow)}$</td>
<td>No$^{(\uparrow)}$</td>
</tr>
<tr>
<td>Order semilattice</td>
<td>No$^{a}$</td>
<td>Yes$^{b}$</td>
<td>Yes$^{(\leftarrow)}$</td>
</tr>
</tbody>
</table>

$^{a}$This paper; see the counter-example above.
$^{b}$This paper, Theorem 4.4.
$^{(\leftarrow)}$ from the entry at left; $^{(\rightarrow)}$ from the entry at right; $^{(\uparrow)}$ from the entry above.

In the distributive case, the medians depend on the weights: by Theorem 4.4, $C$ cannot be a median for the metric $\hat{\delta}$.

In Table 1, we summarize the results about the Pareto property for the metrics and the semilattice structures considered in this paper (note that metrics $\delta$ and $\hat{\delta}$ generally differ in lower semimodular lattices, but are the same in all the other cases considered here).

4.3. Two open questions

A first problem is to devise efficient algorithmic procedures to find the median orders, besides the straightforward adaptation of an integer linear programming approach currently used in the case of linear orders, see [30]. It seems that the algorithmic complexity status of the decision problem associated to the research of medians in $O$ is not known for profiles of length at least three, even in the case where the elements of the profile are linear orders [15]. Two related problems are recognized as NP-hard:

- median linear orders of a profile of at least three linear orders (see [13]),
- median orders of a profile of at least two binary relations [31].
Table 2

<table>
<thead>
<tr>
<th></th>
<th>Definiteness</th>
<th>Pareto</th>
<th>Decisiveness</th>
<th>Symmetry</th>
<th>Consistency</th>
<th>Stability</th>
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<tr>
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</tr>
</tbody>
</table>

The median procedure has been axiomatically characterized as a consensus method in structures like distributive lattices or median semilattices [6, 23]; median semilattices are those distributive semilattices where $c(s^*)$ always exists. Besides consistency and others, the following Quasi-Condorcet condition (QC) has an important role in these characterizations; here this condition is given in the form corrected by Mc Morris et al. [23]:

(QC) For any profile $s^*$, for any $j \in J$ such that $\gamma(j) = 0.5$, and for any $s \in L$, $s \vee p(j)$ is a consensus element if and only if $s \vee j$ is a consensus element.

Such a condition is no longer satisfied by the median procedure in non-distributive semilattices, for instance in the semilattice $O$ (for the sake of brevity, we do not give here a relevant counter-example). So, the question remains open in such structures; specific characterizations of the median procedure (for the symmetric difference metric again) have been obtained in several cases, for instance for linear orders [33].

5. Conclusion

We summarize in Table 2 our knowledge on the properties fulfilled by the procedures mentioned in Sections 3 and 4.

Stability (for objects) of $c_q$ (when defined) comes from the fact that if $c_q(P^*) = P$, then the union $Y$ of some classes of $P$ is a union of connected classes of $E_q(P^*)$ and that $P_{|Y}$ is the set of these classes, while $E_q(P^*)$ admits as connected classes the connected classes of $E_q(P^*)$ that are included in $P$. In other words: $E_q(P^*)_{|Y} = E_q(P^*_{|Y})$. Since, for $Y \subseteq X$, only the inequality $\alpha(P^*_{|Y}) \leq \alpha(P^*)$ holds, the consensus rule $c_{(P^*)}$ is not stable; to construct a precise counter-example is left to the reader.
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References

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