Note

A combinatorial interpretation of the recurrence

\[ f_{n+1} = 6f_n - f_{n-1} \]

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Received 1 May 1997; accepted 23 February 1998

Abstract

Bonin et al. (1993) recalled an open problem related to the recurrence relation verified by NSW numbers. The recurrence relation is the following: \( f_{n+1} = 6f_n - f_{n-1} \), with \( f_1 = 1 \) and \( f_2 = 7 \), and no combinatorial interpretation seems to be known. In this note, we define a regular language \( \mathcal{L} \) whose number of words having length \( n \) is equal to \( f_{n+1} \). Then, by using \( \mathcal{L} \) we give a direct combinatorial proof of the recurrence. © 1998 Elsevier Science B.V. All rights reserved

Keywords: NSW numbers; Recurrence relation; Combinatorial interpretation; Regular language

1. Introduction

In [3] Newman et al. studied a sequence of numbers related to the order of simple groups. These numbers are called NSW numbers, in honour of the authors, and they are defined by the following formula:

\[ f_n = \frac{(1 + \sqrt{2})^{2n-1} + (1 - \sqrt{2})^{2n-1}}{2}. \]

The first values of \( f_n \) appear in Fig. 1. This sequence of integers has some nice properties which are similar to Mersenne numbers [4].

NSW numbers are also connected to the solutions of the following diophantine equation:

\[ x^2 + (x + 1)^2 = y^2. \quad (1.1) \]
In fact,
\[(\frac{f_n - 1}{2})^2 + (\frac{f_n - 1}{2} + 1)^2 = y_n^2,\]
where \(y_n\) is an integer number. Therefore, the sequence \(x_n\) of diophantine equation's solutions is \((f_n - 1)/2\) (see [2]).

Bonin et al. [1] take NSW numbers into consideration, because they arise from the area statistic's over Schröder paths. It is easy to prove that these numbers verify the following recurrence relation:

\[f_{n+1} = 6f_n - f_{n-1}, \quad n \geq 2,\]  \hspace{1cm} (1.2)

where \(f_1 = 1\) and \(f_2 = 7\). These authors stated that:

the recurrence (1.2) cries out for a combinatorial interpretation. Finding this interpretation is an open problem.

The purpose of this note is to fill this gap. We start out by introducing a regular language \(\mathcal{L}'\). We prove that the cardinality of the set \(\mathcal{L}_n\) (i.e. the set of words of \(\mathcal{L}\) having length \(n\)) is equal to \(f_{n+1}\). Then, by means of an operator \(\vartheta\) over \(\mathcal{L}_{n-1}\), we construct the set \(\vartheta(\mathcal{L}_{n-1})\) whose elements have length \(n\) and we show that its cardinality is \(6f_n\). Finally, we determine a bijection between \(\vartheta(\mathcal{L}_{n-1})\) and the set \(\mathcal{L}_n \cup \mathcal{L}_{n-2}\). Consequently, we obtain a direct combinatorial proof of the recurrence (1.2).

2. The regular language \(\mathcal{L}'\)

The language \(\mathcal{L}'\) is defined by the following grammar \(G = (\{S,A\}, \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}, P, S)\):

\[
S \rightarrow x_1Sx_2Sx_3Sx_4A|x_5A|x_6A|x_7A|\varepsilon, \\
A \rightarrow x_1S|x_2S|x_3A|x_4A|x_5A|\varepsilon
\]

and the DFA that accepts the language \(\mathcal{L}'\) is shown in Fig. 2. By setting \(A\) (instead of \(S\)) as the start symbol of the grammar \(G\), we obtain another language \(\mathcal{L}'\) different from \(\mathcal{L}'\). We denote:

\[
\mathcal{L}'_n = \{w \mid S \Rightarrow^* w \text{ and } |w| = n\} \quad \mathcal{L}'_n = \{w \mid A \Rightarrow^* w \text{ and } |w| = n\}
\]

\[
\begin{array}{c|cccccccc}
  n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
  \hline
  f_n & 1 & 7 & 41 & 239 & 1393 & 8119 & 47321 & 275807 \\
\end{array}
\]

Fig. 1. NSW numbers.
Fig. 2. The DFA that accepts the language $L$.

and

\[
\begin{align*}
\sigma_n &= |L_n| \quad \sigma_n = \sigma_n', \\
\alpha_n &= |L_n'| \\
\end{align*}
\]

From the definitions of $L$ and $L'$, we can deduce the following relations:

\[
\begin{align*}
\sigma_n &= 3\sigma_{n-1} + 4\alpha_{n-1}, \\
\alpha_n &= 2\sigma_{n-1} + 3\alpha_{n-1}, \\
\end{align*}
\]

so we obtain

\[
\begin{align*}
\sigma_{n+1} &= 6\sigma_n - \sigma_{n-1}, \quad \text{where } \sigma_0 = 1, \sigma_1 = 7 \\
\end{align*}
\]

and

\[
\begin{align*}
\alpha_{n+1} &= 6\alpha_n - \alpha_{n-1}, \quad \text{where } \alpha_0 = 1, \alpha_1 = 5. \\
\end{align*}
\]

Consequently,

**Proposition 2.1.** The number $\sigma_n$ of words of $L$ having length $n$ is equal to $\sigma_{n+1}$.

**Remark 2.2.** The sequence $\gamma_n$ of integer numbers which are solutions of the diophantine equation (1.1) verifies the following recurrence relation:

\[
\begin{align*}
\gamma_{n+1} &= 6\gamma_n - \gamma_{n-1}, \\
\end{align*}
\]

where $\gamma_1 = 1, \gamma_2 = 5$. Therefore, the number $\alpha_n$ of words belonging to $L'$ having length $n$ is equal to $\gamma_{n+1}$.

3. The combinatorial proof

By means of the language $L$, we will give a combinatorial interpretation of the recurrence relation (1.2). We rewrite the relation in the following way:

\[
6\sigma_n = \sigma_{n+1} + \sigma_{n-1}. 
\]
Since $f_n = |\mathcal{L}_{n-1}|$, the formula will be proved combinatorially, if we can construct a bijection $\varphi$ such that:

$$\{1,2,3,4,5,6\} \times \mathcal{L}_{n-1} \overset{\varphi}{\to} \mathcal{L}_n \cup \mathcal{L}_{n-2}.$$  

To construct the bijection $\varphi$ we define an operator $\theta$ which adds to each word $w \in \mathcal{L}_{n-1}$ a terminal symbol in order to obtain six different words having length $n$. We prove that the cardinality of $\theta(\mathcal{L}_{n-1})$ is equal to $6f_n$. Finally, we show that $\theta(\mathcal{L}_{n-1})$ is in bijection with $\mathcal{L}_n \cup \mathcal{L}_{n-2}$.

Let $w = w_1 w_2 \ldots w_{n-1} \in \mathcal{L}_{n-1}$. If we denote the left factor $w_1 \ldots w_{n-2}$ of $w$ by $u$, the operator $\theta$ over $w = uw_{n-1}$ gives the following set of words:

- if $w_{n-1} = x_1 \Rightarrow \theta(w) = \{wx_1, wx_2, wx_3, wx_4, wx_5, wx_6\}$
- if $w_{n-1} = x_2 \Rightarrow \theta(w) = \{wx_1, wx_2, wx_3, wx_4, wx_5, wx_6\}$
- if $w_{n-1} = x_3 \Rightarrow \theta(w) = \{wx_1, wx_2, wx_3, wx_4, wx_5, wx_6\}$
- if $w_{n-1} = x_4 \Rightarrow \theta(w) = \{wx_1, wx_2, wx_3, wx_4, wx_5, wx_6\}$
- if $w_{n-1} = x_5 \Rightarrow \theta(w) = \{wx_1, wx_2, wx_3, wx_4, wx_5, wx_6\}$
- if $w_{n-1} = x_6 \Rightarrow \theta(w) = \{wx_1, wx_2, wx_3, wx_4, wx_5, wx_6\}$
- if $w_{n-1} = x_7 \Rightarrow \theta(w) = \{wx_1, wx_2, wx_3, ux_1x_7, ux_2x_7, ux_3x_7\}$

We note that the operator $\theta$ produces six different words of length $n$ starting from a word of length $n - 1$, and:

**Proposition 3.1.** If $w$ and $w'$ belong to $\mathcal{L}_{n-1}$ and $w \neq w'$, then $\theta(w) \cap \theta(w') = \emptyset$.

**Proof.** We assume that $\theta(w) \cap \theta(w') \neq \emptyset$. The operator $\theta$ inserts a terminal symbol immediately before or after the last symbol of $w$ and $w'$. Therefore, if by performing $\theta$ over $w$ and $w'$ we obtain the same word $\tilde{w}$, then the first $n - 2$ symbols of $w$ and $w'$ are equal (i.e. $w = uw_{n-1}$ and $w' = uw'_{n-1}$). Moreover, since $w \neq w'$, we have $w_{n-1} \neq w'_{n-1}$. Consequently, to obtain the same word $\tilde{w}$ the operator $\theta$ has to add a terminal symbol $x$ to $w$ after $w_{n-1}$ and a terminal symbol $x'$ to $w'$ before $w'_{n-1}$, so that $\tilde{w} = u w_{n-1} x = u x' w'_{n-1}$. From the definition of $\theta$, it follows that the cases in which the operator inserts a terminal symbol $x'$ before $w'_{n-1}$ are:

- (i) $w' = ux_4 \Rightarrow \tilde{w} = ux_2 x_4$,
- (ii) $w' = ux_5 \Rightarrow \tilde{w} = ux_2 x_5$,
- (iii) $w' = ux_7 \Rightarrow \begin{cases} \tilde{w} = ux_1 x_7 & \text{or} \\ \tilde{w} = ux_2 x_7 & \text{or} \\ \tilde{w} = ux_3 x_7 \end{cases}$.
The first case gives \( w_{n-1}x = x'w_{n-1} = x_7x_4 \). Since \( \tilde{w} = uw_{n-1}x \), we have \( w = ux_7 \) and \( x = x_4 \). From the definition of \( \vartheta \), it follows that the operator cannot add \( x_4 \) after \( x_7 \) and so \( \tilde{w} \notin \vartheta(w) \). Therefore, \( w' \neq ux_4 \). By proceeding in the same way we prove that \( \tilde{w} \) cannot be produced by the cases (ii) and (iii). Consequently, the assumption is false and the proposition follows. □

From this proposition we deduce that \( |\vartheta(L_{n-1})| = 6f_n \). Let us now prove that there is a bijection between \( \vartheta(L_{n-1}) \) and \( L_n \cup L_{n-2} \). We start by examining the words of \( \vartheta(L_{n-1}) \). We denote the set of words of \( L_n \) ending in the state \( S(A) \) by \( S_n \) (\( A_n \)). For instance, \( x_1x_2x_4x_1 \in S_4 \) and \( x_1x_2x_4x_3 \in A_4 \). We have, \( L_n = S_n \cup A_n \). Let \( w = uw_{n-1} \in L_{n-1} \). We perform the operator \( \vartheta \) over \( w \) and from the definition of \( \vartheta \) we deduce the following properties:

- If \( w_{n-1} = x_i \), with \( i = 1,2 \), then \( \vartheta(w) = \{ wx_1, wx_2, wx_3, wx_4, wx_5, wx_6 \} \) and \( \vartheta(w) \subseteq L_n \).
- If \( w_{n-1} = x_3 \), then \( \vartheta(w) = \{ wx_1, wx_2, wx_3, wx_4, wx_5, wx_6 \} \). We have two cases:
  - if \( u \in S_{n-2} \), then \( \vartheta(w) \subseteq L_n \),
  - if \( u \in A_{n-2} \), then \( \{ wx_1, wx_2, wx_3, wx_4, wx_5 \} \subseteq L_n \), and \( wx_6 \notin L_n \).
- If \( w_{n-1} = x_4 \), then \( \vartheta(w) = \{ wx_1, wx_2, wx_3, wx_4, wx_5, ux_7x_4 \} \). We have two cases:
  - if \( u \in S_{n-2} \), then \( \vartheta(w) \subseteq L_n \),
  - if \( u \in A_{n-2} \), then \( \{ wx_1, wx_2, wx_3, wx_4, wx_5 \} \subseteq L_n \), and \( ux_7x_4 \notin L_n \).
- If \( w_{n-1} = x_5 \), then \( \vartheta(w) = \{ wx_1, wx_2, wx_3, wx_4, wx_5, ux_7x_5 \} \). We have two cases:
  - if \( u \in S_{n-2} \), then \( \vartheta(w) \subseteq L_n \),
  - if \( u \in A_{n-2} \), then \( \{ wx_1, wx_2, wx_3, wx_4, wx_5 \} \subseteq L_n \), and \( ux_7x_5 \notin L_n \).
- If \( w_{n-1} = x_6 \), then \( u \in S_{n-2} \) and \( \vartheta(w) = \{ wx_1, wx_2, wx_3, wx_4, wx_5, wx_6 \} \). We have, \( \{ wx_1, wx_2, wx_3, wx_4, wx_5 \} \subseteq L_n \), and \( wx_6 \notin L_n \).
- If \( w_{n-1} = x_7 \), then \( u \in S_{n-2} \) and \( \vartheta(w) = \{ wx_1, wx_2, wx_3, ux_1x_7, ux_2x_7, ux_3x_7 \} \). We have \( \vartheta(w) \subseteq L_n \).

Therefore, the set \( \vartheta(L_{n-1}) \) is made up of two subsets: \( \mathcal{F} \) whose words belong to \( L_n \), and \( \mathcal{G} \) whose words do not belong to \( L_n \). Moreover, the set \( \mathcal{E} \) is such that

\[
\mathcal{E} = A_{n-2}x_7x_4 \cup A_{n-2}x_7x_5 \cup A_{n-2}x_3x_6 \cup L_{n-2}x_6x_6.
\]

We call the words of \( \mathcal{E} \) exceeding. From the definition of \( \vartheta \) and the previous assertions over \( \vartheta(L_{n-1}) \), we deduce that there is a set \( \mathcal{M} \) of the words of \( L_n \) that do not belong to \( \vartheta(L_{n-1}) \). We call the words of \( \mathcal{M} \) missing and it is easy to check that:

\[
\mathcal{M} = S_{n-2}x_7x_1 \cup A_{n-2}x_3x_7.
\]

Table 1 summarizes the state. We wish to point out that the following subset of exceeding words

\[
\mathcal{E}_1 = A_{n-2}x_7x_4 \cup A_{n-2}x_7x_5
\]

is in bijection with the set \( \mathcal{M} \) of missing words. We define the application \( \psi \) as follows:

- if \( w = ux_7x_4 \) with \( u \in A_{n-2} \) then \( \psi(w) = ux_1x_7 \),
Table 1

<table>
<thead>
<tr>
<th>Exceeding words</th>
<th>Missing words</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{A}_{n-2} x_2 x_4$</td>
<td>$\mathcal{A}_{n-2} x_1 x_7$</td>
</tr>
<tr>
<td>$\mathcal{A}_{n-2} x_2 x_5$</td>
<td>$\mathcal{A}_{n-2} x_2 x_7$</td>
</tr>
<tr>
<td>$\mathcal{A}_{n-2} x_3 x_6$</td>
<td>$\mathcal{A}_{n-2} x_2 x_7$</td>
</tr>
<tr>
<td>$\mathcal{A}_{n-2}$</td>
<td>$\mathcal{A}_{n-2} x_2 x_7$</td>
</tr>
<tr>
<td>$\mathcal{L}_{n-2} x_6 x_6$</td>
<td>$\mathcal{L}_{n-2} x_2 x_7$</td>
</tr>
</tbody>
</table>

— if $w = u x_2 x_7$ with $u \in A_{n-2}$ then $\psi(w) = u x_2 x_7$,  
— if $w = u x_3 x_6$ with $u \in A_{n-2}$ then $\psi(w) = u x_6 x_6$,  
otherwise $\psi(w) = w$. By applying $\psi$ to $\vartheta(\mathcal{L}_{n-1})$, we get  
\[
\psi(\vartheta(\mathcal{L}_{n-1})) = \mathcal{L}_n \cup \mathcal{E}_2,
\]

where  
\[
\mathcal{E}_2 = \mathcal{A}_{n-2} x_6 x_6 \cup \mathcal{L}_{n-2} x_6 x_6 = \mathcal{L}_{n-2} x_6 x_6.
\]

It is easy to invert $\psi$, and so $\psi$ is a bijection between $\vartheta(\mathcal{L}_{n-1})$ and $\mathcal{L}_n \cup \mathcal{E}_2$, where $\mathcal{E}_2$ is trivially in bijection with $\mathcal{L}_{n-2}$.

Consequently, a combinatorial proof of the recurrence (1.2) follows.

4. Conclusions

By using a regular language $\mathcal{L}$ we give a direct combinatorial proof of the recurrence (1.2) verified by NSW numbers.

Two other interesting problems related to NSW numbers are the following:

- **do there exist infinitely many $f_n$ prime?**
- **do there exist infinitely many $f_n$ composite?**

To the authors’ knowledge, these problems are open.

References