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Polyominoes Which Tile Rectangles

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1. INTRODUCTION

In [1], D. Klarner defined the *order n* of a polyomino *P* as the minimum number of congruent copies of *P* which can be assembled (allowing translation, rotation, and reflection) to form a rectangle. For those

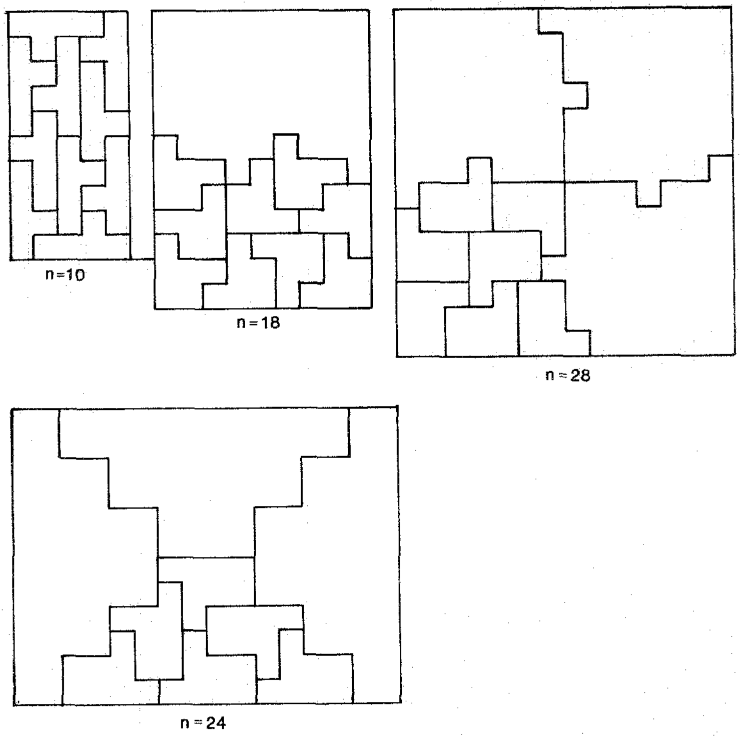

FIG. 1. Klarner's four "sporadic" polyominoes, of orders 10, 18, 24, and 28, respectively.

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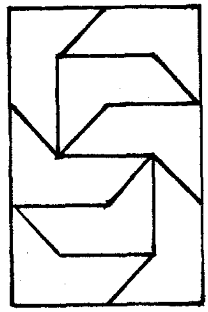


FIG. 2. A rectangle formed from eight congruent pieces.

polyominoes which will not tile any rectangle, the order is undefined. A polyomino has order $n = 1$ if and only if it is itself a rectangle. Conditions for order 2 and for order 4 are decribed in [1], and there are infinitely many dissimilar examples for each of the orders 1, 2, and 4. One example each is given for orders 10, 18, 24, and 28 (see Fig. 1), and these are "the only known examples" beyond order 4, according to [1].

In this paper, we show that there are infinitely many dissimilar polyomino examples for every order n which is a multiple of 4.

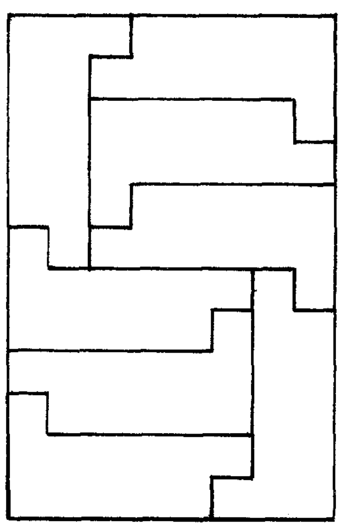


FIG. 3. A polyomino of order 8.

2. The Basic Construction for Order 8

In Fig. 2, we see a novel way in which a 4×6 rectangle can be dissected into eight congruent pieces. While the pieces in this dissection are not polyominoes (they are actually *triaboloes* in the terminology of [2]), it requires a minimum of eight of them to form a rectangle, and this theme is easily transferred to polyominoes, as shown in Fig. 3. This is the smallest polyomino of order eight obtained in this manner, but it is easy to form larger ones. A general scheme for this is illustrated in Fig. 4. For any two

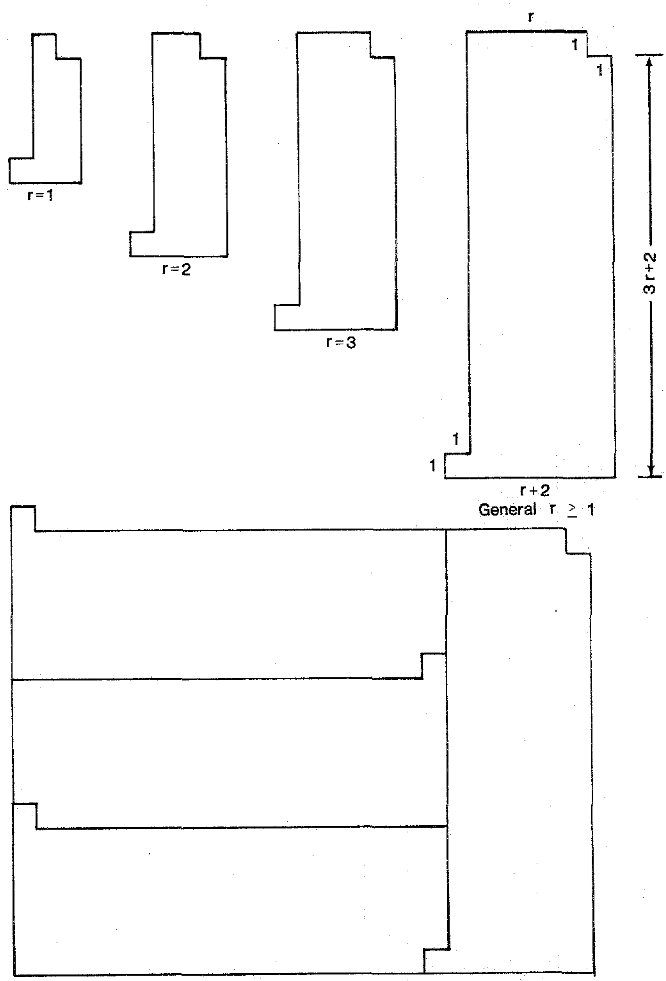


Fig. 4. Dissimilar polyominoes of order 8, and how to stack them.

distinct values of the positive integer r, two dissimilar polyominoes of order 8 are obtained.

It is easy to show that none of these figures can have order less than 8. The proof begins by observing that only the "heel of the boot" can be in the *corner* of a rectangle which is tiled. Then the "toe" must be mated with the notch at the top-back of another boot. The quickest way to finish off the rectangle then requires eight copies of the figure.

3. Polyominoes of Order 4s

In Fig. 5, we see a construction for a polyomino of order $n = 4s$ for every $s = 1, 2, 3, \ldots$. Starting with a rectangle which is $2 \times (4s - 2)$, we remove a

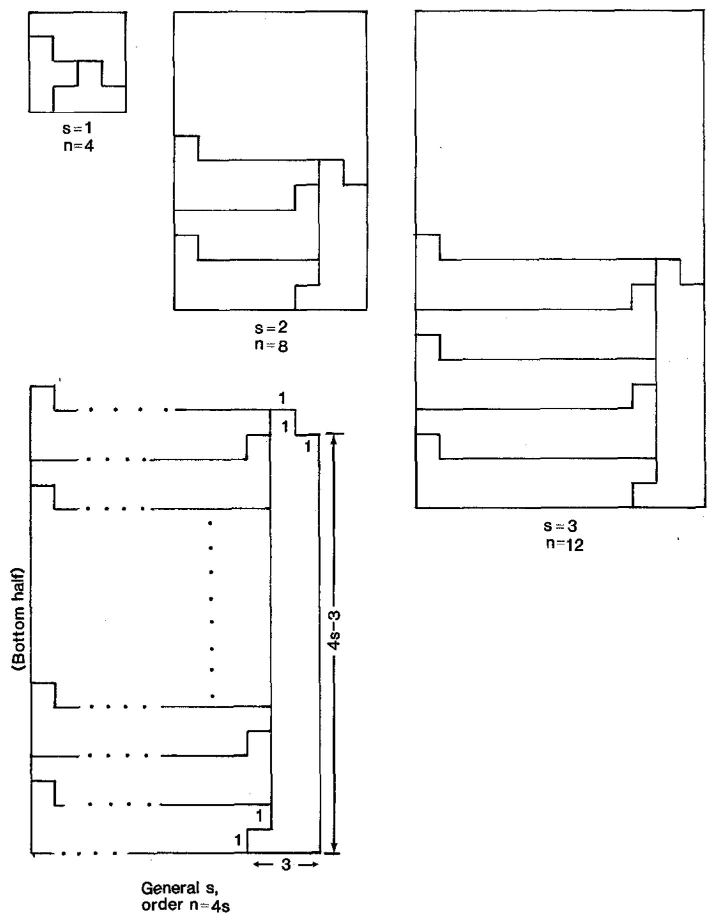
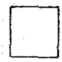
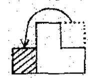
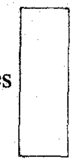
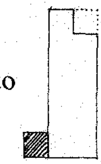


FIG. 5. Polyominoes of order $n = 4s$, for every positive integer s.

single square from one corner and attach it as a "toe" at the opposite corner. For $s=1$, this changes  to  to give the familiar simplest example of order 4. For $s=2$, it changes  to  which is the polyomino of order 8 shown in Fig. 3; etc. The idea shown in Fig. 4 can be applied not only to order $n=8$, but to any order $n=4s$ to obtain

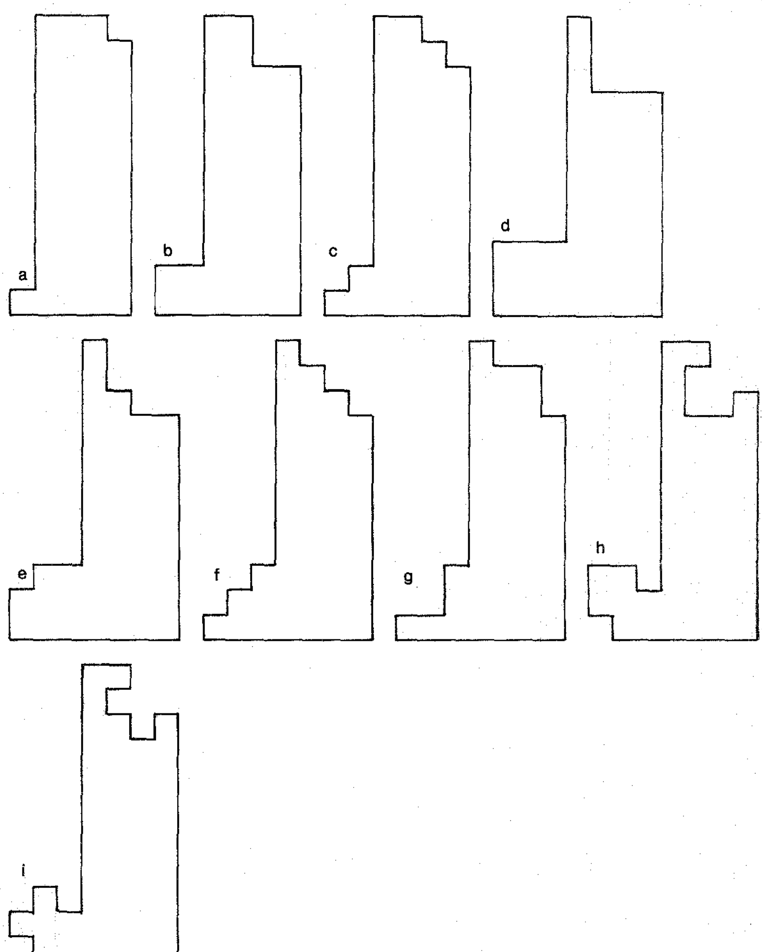


FIG. 6. Variations on a theme: nine different 48-ominoes, each of order 8. (Example b is similar to the 12-omino of order 8 in Fig. 3.)

infinitely many dissimilar polyominoes of order n whenever n is a multiple of 4 (Fig. 5). The general construction involving both r and s begins with a rectangle which is $(r+1) \times (2s-1)(r+1)$, and moves a single 1×1 square from the top-back of the "boot" to become a "toe" at the opposite corner. The proof that the resulting figure truly has order $n = 4s$ is analogous to the proof for $n = 8$.

Actually, there are many different shapes which can be removed from the top-back of the "boot" and then affixed to form the "toe." Some of these are illustrated in Fig. 6, for the case $s = 2$, $r = 3$. The necessary and sufficient condition for the "toe" to work is that it be symmetric around an axis of slope -1, and that it be removable from the top-back of the "boot" without disconnecting the figure. (Without the symmetry property, the underlying tiling idea shown in Fig. 2 will fail.) For the tiling concept to work, it is not necessary that the figure be a polyomino (cf. Fig. 2), nor even that all its edges be straight lines. For example, the "toe" could be a quadrant of a circle.

4. Unsolved Problems

We now have infinitely many examples of polyominoes of each order $n = 4s$ for all $s \geqslant 1$. From the examples with $n = 24$ and $n = 28$ in Fig. 1, we know that not all polyominoes of order $n = 4s$ are based on the theme in Fig. 2, even ignoring the special case $s = 1$, i.e., order $n = 4$, for which many kinds of examples exist (see [1]). Hence there is the unsolved problem:

(1) Characterize all the polyominoes of order n for each $n = 8, 12, 16, 20, 24, 28, 32, \ldots$.

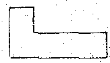
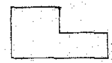
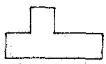
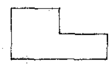
There are infinitely many polyominoes of order 2 (see [1]), and one example each known for orders 10 and 18 (see Fig. 1). Hence:

(2) Is there a polyomino of order n for every *even* value of n?

We need only supply examples when n is twice an odd number, since the multiples of 4 are already taken care of. The sequence "2, 10, 18, ..." suggests that examples with $n \equiv 2 \pmod 8$ may be easier to find than the more general case of $n \equiv 2 \pmod 4$.

(3) For $n > 1$, is there any polyomino of *odd* order n?

Small odd numbers (e.g., $n = 3$ and $n = 5$) seem particularly unlikely, but there is no obvious reason why there should be no polyominoes of order 15 (say), or other larger and preferably composite odd number orders.

Klarner [1] calls a polyomino P *odd* if it is *possible* to use an odd number of copies of P to form a rectangle (not necessarily the minimum rectangle). He shows that , , , and  are odd, being able to form rectangles made of 15, 21, 27, and 11 copies, respectively. He also shows that fifteen copies of any polyomino which is three quadrants of a rectangle () can be used to pack a rectangle. Note that all of these odd polyominoes have order 2. Klarner states that it is not known whether or not the order-10 pentomino  is odd; and that  is the only (non-rectangular) polyomino known for which as few as *eleven* copies suffice to make a rectangle containing an odd number of copies. It would be reasonable to call the smallest odd number of copies of P which can be

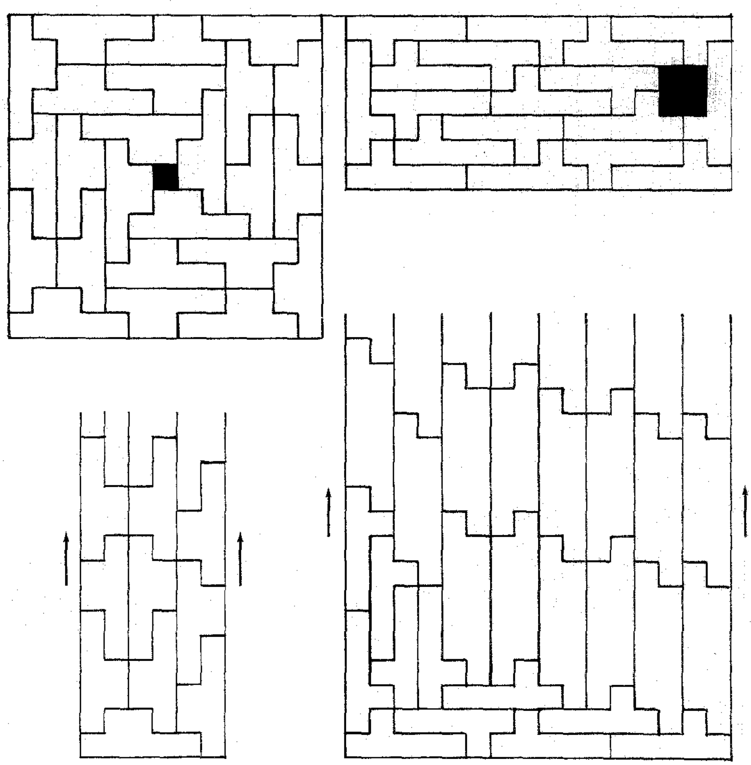
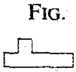
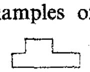
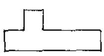


FIG. 7. Examples of infinite half-strips, and rectangles with square holes, made from  and 

assembled to form a rectangle the *odd-order of P*. Problem (3) can then be restated: "Is there any non-rectangular polyomino P whose *order* is equal to its *odd-order*?" We can also list Klarner's implied question:

(4) Is there any non-rectangular polyomino with an odd-order less than 11?

Finally,

(5) Is there *any* (finite) order associated with the polyominoes  and/or  ?

Each of these will tile a semi-infinite strip, and each will fill a finite rectangle except for a small square hole somewhere inside it. (These four constructions are illustrated in Fig. 7.)

It is known [3] that the general problem of whether a given, arbitrary polyomino can tile a rectangle is computationally undecidable. This implies that there is no computable function $f(n)$ which bounds the area of the minimum rectangle that a given n-omino might tile, for otherwise we would have a decision procedure: "Try all arrangements of the given n-omino in all rectangles of area $\leqslant f(n)$." (While this is no doubt computationally "hard," it is nonetheless much easier than "undecidable.") This result for *arbitrary* n-ominoes tells us little or nothing about the *specific* hexomino and heptomino in problem (5). (Both parts of problem (5) have been answered in the affirmative by Karl A. Dahlke. See the following articles.)

For tiling more general regions than rectangles with polynominoes, see [4]. If sets of polyominoes are allowed as the tiles, see [5].

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