Lie Ideals and Derivations of Prime Rings

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In this paper, we shall consider the relationship between the derivations and Lie ideals of a prime ring. Some of the results we obtain have been obtained earlier, even for rings more general than prime rings, in the case of inner derivations. We shall also look at the action of derivations on Lie ideals; the results we obtain extend some that had been proved earlier only for the action of derivations on the ring itself.

Let $R$ be a ring and $d \neq 0$ a derivation of $R$. If $U$ is a Lie ideal of $R$, we shall be concerned about the size of $d(U)$. How does one measure this size? One way is to look at the centralizer of $d(U)$ in $R$; the bigger $d(U)$, the smaller this centralizer should be. This explains our interest in the centralizer of $d(U)$. The result we obtain generalizes the principal theorem of [1].

We may also measure the size of $d(U)$ by looking at how large $d(U)$, the subring generated by $d(U)$, turns out to be. We view $d(U)$ as large if it contains a non-zero ideal of $R$. For our special setting, we will obtain a result which generalizes one in [2].

Finally, a well-known and often used result states that if $d$ is a derivation of $R$, which is semi-prime and 2-torsion-free, such that $d^2 = 0$ then $d = 0$ (see the proof of Lemma 1.1.9 in [3]). If $R$ is prime, of characteristic not 2, and $d^2(I) = 0$ for a non-zero ideal, $I$, of $R$, it also follows that $d = 0$. What can one say if $d^2(U) = 0$ for some non-central Lie ideal of $R$? For inner derivations this was studied and answered in [4]. For prime rings and for any derivation $d \neq 0$ we answer the question of when $d^2(U) = 0$ completely in our Theorem 1.

We shall be working in the context of prime rings of characteristic not 2 in all that we do here. However, many of the results have some suitable analog for semi-prime, 2-torsion-free rings. In the presence of 2-torsion most of our results are not valid as stated, but something non-trivial can be said in

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even that situation. We do not go into a study of our results when there is 2-torsion present, here.

In all that follows $R$ will be a prime ring of characteristic not 2. The center of $R$ will be denoted by $Z$ throughout. If $A$ is a subset of $R$, by $C_R(A)$ we shall mean the centralizer of $A$, defined by $C_R(A) = \{ x \in R \mid xa = ax \ \text{all} \ \dot{a} \in A \}$. We shall also use the notation $[a, b]$ for the commutator $ab - ba$ of $a$ and $b$.

We begin with a special case of a far more general result (Theorem 5 in [4]), which we include for the sake of completeness. Since the result for prime rings is implicitly contained in Lemma 1.3 of [5], we state the result and give only an indication of its proof.

**Lemma 1.** If $U \subseteq Z$ is a Lie ideal of $R$, then there exists an ideal, $M$, of $R$ such that $[M, R] \subseteq U$, but $[M, R] \not\subseteq Z$.

**Proof:** Since $\text{char } R \neq 2$ and $U \subseteq Z$, it follows from results in [5] that $[U, U] \neq 0$ and that $[M, R] \subseteq U$ where $M = R[U, U]R \neq 0$ is the ideal of $R$ generated by $[U, U]$. That $[M, R] \subseteq Z$ follows easily; for, if $[M, R] \subseteq Z$ then $[M, [M, R]] = 0$, which would force $M \subseteq Z$ and, since $M \not= 0$ is an ideal of $R$, so $R = Z$.

**Lemma 2.** If $U \subseteq Z$ is a Lie ideal of $R$, then $C_R(U) = Z$.

**Proof:** $C_R(U)$ is both a subring and a Lie ideal of $R$. Since $C_R(U)$ cannot contain a non-zero ideal of $R$—otherwise $U$ centralizes a non-zero ideal of $R$, so is in $Z$—by Lemma 1.3 of [5] we conclude that $C_R(U) \subseteq Z$. Hence $C_R(U) = Z$.

The next result is a special case of Lemma 2 in [4].

**Lemma 3.** If $U$ is a Lie ideal of $R$ and $a \in R$ centralizes $[U, U]$ then $a$ centralizes $U$. That is, $C_R([U, U]) = C_R(U)$.

**Proof:** If $[U, U] \subseteq Z$ then, by Lemma 2, $a \in Z$, so certainly a centralizes $U$. On the other hand, if $[U, U] \subseteq Z$ and $u \in U$, $x \in R$ then $a = [u, [u, x]] \in Z$ and $au = [u, [u, ux]] \in Z$. If $a \not= 0$ we get $u \in Z$, which leads to $a = 0$. So $a = 0$; thus $[u, [u, x]] = 0$ for all $x \in R$. But then, by the Sublemma on p. 5 of [5], $u \in Z$; hence $U \subseteq Z$. In both cases we see that $a \in C_R(U)$. This gives that $C_R([U, U]) = C_R(U)$.

Primeness of a ring $R$ is defined by: $aRb = 0$ implies $a = 0$ or $b = 0$. The next lemma shows that primeness could also be defined by an analogous property for Lie ideals.

**Lemma 4.** If $U \subseteq Z$ is a Lie ideal of $R$ and if $aUb = 0$ then $a = 0$ or $b = 0$. 
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Proof. By Lemma 1, there exists an ideal \( M \) of \( R \) such that \([M, R] \subset Z\) but \([M, R] \not\subset U\). If \( u \in U, \ m \in M, \) and \( y \in R \) then \([mau, y] \in [M, R] \subset U\), thus \( 0 = a[mau, y]b = a[ma, y]ub + ama[u, y]b = a(may - yma)ub = amayub, \) since \( a[u, y]b \in aU = 0 \). Thus \( aMaRUb = 0 \). If \( a \neq 0 \), since \( R \) is prime we obtain \( Ub = 0 \); so, if \( x \in R, \ u \in U \) then \((ux - xu) \in U, \) whence \((ux - xu)b = 0, \) and so \( uxb = 0 \). In other words, \( uRb = 0 \); since \( U \neq 0, \) we get \( b = 0 \).

We now bring the derivation into play for the first time.

Lemma 5. If \( d \neq 0 \) is a derivation of \( R \), and \( U \) a Lie ideal of \( R \) such that \( d(U) = 0 \), then \( U \subset Z \).

Proof. Let \( u \in U, \ x \in R; \) since \( d(u) = 0 \) and \( d(ux - xu) = 0 \) we get \( ud(x) - d(x)u = d(ux - xu) = 0 \). Therefore \( u \) centralizes \( d(R) \). By [1] we have \( u \in Z; \) hence \( U \subset Z \).

We sharpen Lemma 5 considerably to

Lemma 6. If \( d \neq 0 \) is a derivation of \( R \), and \( U \) is a Lie ideal of \( R \) such that \( d(U) \subset Z \) then \( U \subset Z \).

Proof. If \( U \subset Z \), by Lemma 3 \( V = [U, U] \subset Z \). But, if \( u, w \in U \) then \( d(uw - wu) = (d(u)w - wd(u)) + (ud(w) - d(w)u) = 0 \) since \( d(u), d(w) \in Z \). Thus \( d(V) = 0; \) by Lemma 5 we get the contradiction \( V \subset Z \).

We look at some other property of \( d(U) \).

Lemma 7. If \( d \neq 0 \) is a derivation of \( R \), and \( U \subset Z \) is a Lie ideal of \( R \), then, if \( td(U) = 0 \) (or \( d(U)t = 0 \)), we must have \( t = 0 \).

Proof. Let \( u \in U, \ x \in R; \) then \((ux - xu)u = u(xu) - (xu)u \in U \). Thus \( td((ux - xu)u) = 0; \) that is \( t(d(ux - xu))u + t(ux - xu)d(u) = 0 \). Since \( ux - xu \in U, \) \( td(ux - xu) = 0, \) so the above relation gives us \( t(ux - xu)d(u) = 0 \) for all \( u \in U, \ x \in R \). Let \( x = d(v)y \) where \( v \in U, \ y \in R; \) we get, since \( tx = 0, \) that \( tud(U)Rd(u) = 0 \). Since \( d(U) \neq 0 \) we easily get from this last relation that \( tUd(U) = 0 \). By Lemma 4, since \( d(U) \neq 0, \) we conclude that \( t = 0 \).

We are ready for our first theorem.

Theorem 1. If \( d \neq 0 \) is a derivation of \( R \) and if \( U \) is a Lie ideal of \( R \) such that \( d(U) = 0 \) then \( U \subset Z \).

Proof. Suppose that \( U \subset Z; \) by Lemma 3, \( V = [U, U] \subset Z. \) So, to prove the theorem, it is enough to show that \( V \subset Z \).

By Lemma 1, \( U \supset [M, R] \) where \( M \) is an ideal of \( R \) such that \([M, R] \subset Z \). Let \( m \in [M, R] \subset U \cap M \) and \( u \in V; \) then \( w = d(u) \in d[U, U] \subset U, \) hence \( d(w) = 0 \) since \( d^2(u) = 0. \)
If \( y \in R \) then, since \( mw, y \in [M, R] \subset U \). Hence 
\[ 0 = d^2([mw, y]) = d^2\{m, y|w + m|w, y]\} = 2d(m)d([w, y]), \]
since \( 0 = d(w) = d^2[m, y] = d^2[m] = d^2[w, y] \). Thus \( d([M, R])d([d(v), R]) = 0 \). But \([M, R]\)
is a non-central Lie ideal of \( R \), therefore, by Lemma 7, 
\( d([d(V), R]) = 0 \). Hence, if \( u \in V \), \( v \in R \) then 
\[ 0 = d(d(u)v - xd(u)) = d(u)d(x) - d(x)d(u), \]
since \( d^2(u) = 0 \). Therefore \( d(V) \) centralizes \( d(R) \). By \([1]\), \( d(V) \subset Z \), hence by 
Lemma 6, \( V \subset Z \). This proves the Theorem.

The special cases, when \( U = R \) or \( U \) is an ideal of \( R \), where the conclusion is that 
\( d = 0 \) if \( d^2(U) = 0 \), are immediate consequences of Theorem 1.

If \( d \) is the inner derivation defined by \( a \in R \), that is, if \( d(x) = ax - xa \) for all \( x \in R \), and if \( d^2(U) = 0 \) (that is, \( [a, [a, U]] = 0 \)) we conclude from the 
thorem that if \( U \subset Z \) then \( [a, U] = 0 \) and so \( a \in Z \) by Lemma 2. Since the 
ideals (and so, the prime ideals) of \( R \) are invariant with respect to inner 
derivations, we easily derive one of the results of [4] as a corollary to 
Theorem 1, namely,

**Corollary.** If \( R \) is a semi-prime, 2-torsion-free ring and \( U \) a Lie ideal of \( R \) then, if \( [a, [a, U]] = 0 \) for some \( a \in R \), we must have \( [a, U] = 0 \).

We now examine the largeness of \( d(U) \) by proving the smallness of it centralizer.

**Theorem 2.** If \( U \subset Z \) is a Lie ideal of \( R \) and \( d \neq 0 \) is a non-zero 
derivation of \( R \), then \( C_R(d(U)) = Z \).

**Proof.** Let \( a \in C_R(d(U)) \), and suppose that \( a \notin Z \). Since \( U \subset Z \), \( V = 
[U, U] \subset Z \) by Lemma 3; moreover, \( d(V) \subset U \). Thus \( ad^2(u) = d^2(u)a \) for all 
\( u \in V \). But \( ad(u) = d(u)a \); applying \( d \) to this and using the above, we get 
\( d(a)d(u) = d(u)d(a) \). So both \( a \) and \( d(a) \) centralize \( d(V) \). But \( d(au - ua) = 
(d(a)u - ad(a) \in d(V) \), hence \( [d(a), d(V)] = 0 \). By Theorem 1 we have that 
\( [d(a), V] = 0 \), and since \( V \subset Z \) we have, by Lemma 2, that \( d(a) \in Z \).

By the same token, since \( a^2 \in C_R(d(U)) \), \( 2ad(a) = ad(a^2) \in Z \); because 
\( a \notin Z \) and \( d(a) \in Z \), the fact that \( ad(a) \in Z \) forces \( d(a) = 0 \). Hence \( d(a) = 0 \) for all \( a \in C_R(d(U)) \) which are not in \( Z \). If \( d(b) \neq 0 \) for some \( b \in C_R(d(U)) \), 
then, by the above, \( b \in Z \). Furthermore, if \( a \in C_R(d(U)) \), \( a \notin Z \) then 
\( d(a) = 0 \), hence \( d(a + b) = d(b) \neq 0 \). Consequently, \( a + b \in Z \); together with 
\( b \in Z \) we conclude from this that \( a \in Z \), a contradiction. Hence, if we 
suppose that \( C_R(d(U)) \subset Z \) then we are forced to \( d(a) = 0 \) for all 
\( a \in C_R(d(U)) \).

Let \( W = \{x \in R \mid d(x) = 0\} \); by what we have just done, \( C_R(d(U)) \subset W \).
Moreover, if \( a \in C_R(d(U)) \) and \( u \in U \) then 
\( d(au - ua) = ad(u) - d(u)a = 0 \) since \( d(a) = 0 \). Thus \( [a, U] \subset W \).

Now, since \( U \subset Z \), by Lemma 1 \( U \supset [M, R] \) where \( M \) is an ideal of \( R \) 
such that \( [M, R] \subset Z \). If \( m \in [M, R] \subset U \cap M \) then \( ma \in M \) hence, for
u \in U, [ma, u] \in U, that is \([m, u][a + m[a, u] \in U. Therefore a centralizes \(d([m, u]a + m[a, u]) = (d([m, u]))a + (d(m))[a, u]\) since \(d(a) = d[a, u] = 0\) because \(a, [a, u] \in W\). Since a centralizes \(d([m, u])\) and \(d(m)\) we get \(d(m)[a, [a, u]] = 0\) for all \(m \in [M, R], u \in U. Thus (d([M, R])) [a, [a, U] = 0. Since \([M, R]\) is a non-central Lie ideal of \(R\), by Lemma 7 we have that \([a, [a, U]] = 0. Therefore, by Theorem 1 (or its Corollary), since \(U \subseteq Z\), we get that \(a \in Z\). With this Theorem 2 is proved.

We now want to consider \(\overline{d(U)}\), the subring generated by \(d(U)\), where \(U \subseteq Z\) is a Lie ideal of \(R\) and \(d \neq 0\) is a derivation of \(R\). In [2] it was shown, for any ring \(R\), that \(\overline{d(R)}\) contains a non-zero ideal of \(R\), provided \(d^3 \neq 0\). We shall show here that, in the prime case, if \(d^3 \neq 0\) then \(\overline{d(U)}\) contains a non-zero ideal of \(R\).

In what follows we assume—as we have up to now—that \(R\) is a prime ring, \(\text{char } R \neq 2\), and that \(U \subseteq Z\) is a Lie ideal of \(R\) and that \(d \neq 0\) is a derivation of \(R\). We shall make frequent use of the Lie ideals \(V = [U, U]\) and \(W = [V, V]\) which are closely related to \(U\).

Our result will follow as a consequence of several lemmas.

**Lemma 8.** If \(d^3 \neq 0\) and if \(\overline{d(V)}\) contains a non-zero left ideal \(\lambda\) of \(R\) and a non-zero right ideal \(\rho\) of \(R\) then \(\overline{d(U)}\) contains a non-zero ideal of \(R\).

**Proof.** Since \(V = [U, U]\) and \(d(V) \subseteq U\) we know that \(d(\overline{d(V)}) \subseteq \overline{d(U)}\). Let \(a \in \lambda \subseteq \overline{d(V)}\) and \(x \in R\); then \(d(xa) \in d(\lambda) \subseteq d(\overline{d(V)}) \subseteq d(U)\), hence \(d(x)a + xd(a) \in \overline{d(U)}\). Since \(d(x)a\) is in \(\overline{d(U)}\), and so, in \(\overline{d(V)} \subseteq d(U)\), we get that \(xd(a) \in \overline{d(U)}\). Thus \(Rd(\lambda) \subseteq \overline{d(U)}\). Similarly, \(d(\rho) \subseteq \overline{d(U)}\).

If \(a \in \lambda, u \in V\) then \(d(ua - au) \in \overline{d(V)}\), hence, \(d(a)u - ud(a) + ad(u) - d(u)a \in \overline{d(V)}\). But \(d(u)a \in \lambda \subseteq \overline{d(V)}, ud(a) \in \overline{d(U)}\) by the above, and \(ad(u) \in \lambda d(V) \subseteq \overline{d(V)}\). The upshot of this is that \(d(\lambda)V \subseteq \overline{d(U)}\). Similarly, \(Vd(\rho) \subseteq \overline{d(U)}\).

Let \(I = \lambda V \rho; I\) is an ideal of \(R\) and, by Lemma 4, \(I \neq 0\). Moreover, \(d(I) = d(\lambda V \rho) \subseteq d(\lambda) V \rho + \lambda d(V) \rho + \lambda V d(\rho)\) lies in \(\overline{d(U)}\) since \(d(\lambda)V, Vd(\rho), \lambda, \rho\) are all in \(\overline{d(U)}\). Thus \(\overline{d(I)} \subseteq \overline{d(U)}\). But, if \(d^3 \neq 0\), it is easy to see, as in [2], that because \(I\) is an ideal of the prime ring \(R\), \(\overline{d(I)}\) contains a non-zero ideal of \(R\). Therefore \(\overline{d(U)}\) contains a non-zero ideal of \(R\).

**Lemma 9.** If \(I \neq 0\) is an ideal of \(R\) and if \(\overline{d(U)}\) does not contain both a non-zero left-ideal and a non-zero right-ideal of \(R\) then, if \([c, I] \subseteq \overline{d(U)}, c\) must be in \(Z\).

**Proof.** Let \(t \in d(U)\) and \(i \in I\); then \([c, ti] = [c, t]i + t[c, i] \in \overline{d(U)}\).

Because \(t \in \overline{d(U)}, [c, I] \subseteq \overline{d(U)}\) we have \(t[c, I] \subseteq \overline{d(U)}\). Hence \([c, t]i \in \overline{d(U)}\); that is, the right ideal of \(R\), \([c, d(U)] \subseteq \overline{d(U)}\). Similarly \(I[c, d(U)] \subseteq \overline{d(U)}\) is a left-ideal of \(R\) lying in \(\overline{d(U)}\). By our hypothesis one of
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\[ I[c, d(U)] = 0 \text{ or } [c, d(U)]I = 0, \text{ therefore } [c, d(U)] = 0. \] By Theorem 2 we conclude that \( c \in Z \).

We prove a highly special and somewhat messy

**Lemma 10.** If \( d^2(U)^2 = 0 \) then \( d^3(W) = 0 \).

**Proof.** Since \( U \subseteq Z \), by Lemma 3 none of \( U, V = [U, U], W = [V, V] \) is in \( Z \). Also \( d(V) \subseteq U, d(W) \subseteq V, d^2(W) \subseteq U \). If \( u \in U, v \in V, w \in W \) then, for any \( t \in U \), since \( d^2(U)^2 = 0 \), we have

\[ d^2(u) d^2(d(v) d^2(w) t - d^2(w) t d(v)) = 0. \] (1)

Expanding this explicitly and making use of \( d(v) \in U, d^2(w) \in U \) and \( d^2(U)^2 = 0 \), (1) reduces to

\[ d^2(u) d(v) d^4(w) t + 2d^2(w) d(t)) = 0. \] (2)

If in (2) we choose \( t \in d(V) \subseteq U \), because \( d^3(w) d(t) = 0 \) for such a \( t \), we get from (2) that \( d^2(u) d(v) d^4(w) d(V) = 0 \). By Lemma 7 we conclude that \( d^2(u) d(v) d^4(w) = 0 \). But then (2) further reduces to

\[ d^2(u) d(v) d^3(w) d(U) = 0 \quad \text{for } u \in U, v \in V, w \in W, \] (3)

hence, by Lemma 7 we get

\[ d^2(u) d(v) d^3(w) = 0 \quad \text{for all } u \in U, v \in V, w \in W. \] (4)

Similarly, reversing sides, we get

\[ d^3(w) d(v) d^2(u) = 0 \quad \text{for all } u \in U, v \in V, w \in W. \] (5)

Consider \( d^2(d(t)) d^2(vd(w) - d(w)v) = 0 \) where \( t, w \in W \) and \( v \in V \). Expanding this and making use of (5) gives us that \( d^2(t) V d^3(w) = 0 \) for all \( t, w \in W \). By Lemma 4 we obtain \( d^3(w) = 0 \) for all \( w \in W \), hence \( d^3(W) = 0 \), as claimed.

We still need one more lemma before we can prove our next Theorem.

**Lemma 11.** If \( d^3(U) = 0 \) then \( d^3 = 0 \).

**Proof.** Let \( u \in U \) and \( r \in R \); then

\[ 0 = d^3[u, r] = 3[d^2(u), d(r)] + 3[d(u), d^3(r)] + [u, d^3(r)]. \] (1)

In this replace \( u \) by \( d^2(w) \) where \( w \in W \), to obtain

\[ [d^2(w), d^3(r)] = 0 \quad \text{for } w \in W, r \in R. \] (2)
We now replace \( u \) by \( d(w) \), \( r \) by \( d(r) \) where \( w \in W \), in (1); we get, using (2), that \([d(w), d^4(r)] = 0\) for all \( w \in W \), \( r \in R \). Since \( W \triangleq Z \), by Theorem 2 we get that \( d^4(R) \subset Z \).

Since \( d^4(r) \in Z \) for all \( r \in R \), if \( u \in U \), \( r \in R \) then \( 0 = d^4[u, r] = 6[d^2(u), d^3(r)] + 4[d(u), d^3(r)] \). But we also have that \( 0 = d^3[u, d(r)] = 3[d^2(u), d^3(r)] + 3[d(u), d^3(r)] \). Playing these last two relations off against each other leads us to \( 2[d(u), d^3(r)] = 0 \), and so \([d(u), d^3(r)] = 0\) for all \( u \in U \), \( r \in R \). By Theorem 2, \( d^4(R) \subset Z \).

Thus, if \( r \in R \), \( u \in U \) then \( Z \ni d^3(rd^*(u)) = d^3(r)d^2(u) \). However, \( d^3(R) \subset Z \), so since \( d^3(R)d^2(U) \subset Z \) if \( d^3(R) \neq 0 \) we are forced to \( d^2(U) \subset Z \).

Suppose, then, that \( d^3(R) \neq 0 \); as we have seen, we must have \( d^2(U) \subset Z \). If \( r \in R \), \( u \in U \) then \( Z \ni d^3(rd(u)) = d^3(r)d^2(u) \), and since \( d^3(r) \in Z \), \( d^2(u) \in Z \) we see that \( d^3(r)d^2(u) \in Z \); that is, \( d^3(R)d^2(U) \subset Z \). By Lemma 6 we know that \( d(U) \triangleq Z \), by the above we know that \( d^3(R) \subset Z \); these, combined with \( d^4(R)d(U) \subset Z \) force \( d^4(R) = 0 \).

Again, if \( r \in R \), \( u \in U \) then \( 0 = d^4(rd(u)) = 4d^3(r)d^2(u) \), so that \( d^3(R)d^2(U) = 0 \). But \( d^3(U) \neq 0 \subset Z \) (by Theorem 1) so we conclude that \( d^3(R) = 0 \). This proves the lemma.

We have all the ingredients to prove

**Theorem 3.** If \( U \triangleq Z \) is a Lie ideal of \( R \) and \( d \) a derivation of \( R \) such that \( d^3 \neq 0 \), then \( d(U) \) contains a non-zero ideal of \( R \).

**Proof.** If \( V = [U, U] \) and \( W = [V, V] \), in view of Lemma 8 it is enough to show that \( d(V) \) contains a non-zero left, and a non-zero right, ideal of \( R \). So, suppose not. We shall show that this leads to \( d^2([W, W]) = 0 \); by Lemmas 10 and 11 we shall reach the contradiction \( d^3 = 0 \).

Let \( a = d(w) \), where \( w \in [W, W] \); thus, for \( x \in R \), \( a(ax - xa) = a(ax) - (ax)a \in W \), hence \( d(a(ax - xa)) = d(a)(ax - xa) + ad(ax - xa) \) is in \( d(W) \). But \( a \in d(W) \subset d(V) \) and \( d(ax - xa) \in d(V) \), whence we get

\[
d(a)(ax - xa) \in d(V) \quad \text{for all} \quad a \in d[W, W], \ x \in R. \tag{1}
\]

On the other hand, if \( u \in V \) then \( d[a, u] = [d(a), u] + [a, d(u)] \in d(V) \), and since \( a \in d(W) \subset d(V) \) we have that \([a, d(u)] \in d(V)\); hence;

\[
[d(a), V] \subset d(V) \quad \text{for all} \quad a \in d[W, W]. \tag{2}
\]

We also have \( d(V) \ni d(a) \, d(ar - ra) = d(a) \, [d(a), r] + d(a) \, [a, d(r)] \); by (1), \( d(a) \, [a, d(r)] \in d(V) \). The net result of the above becomes

\[
d(a) \, [d(a), r] \in d(V) \quad \text{for all} \quad a \in d[W, W], \ r \in R. \tag{3}
\]
We linearize (3) on $a$ to get
\[ s = d(a) [d(b), r] + d(b) [d(a), r] \in \overline{d(V)} \quad \text{for all} \quad a, b \in d[W, W], r \in R. \]

(4)

If $t = [d(a)d(b), r] = d(a)[d(b), r] + [d(a), r]d(b)$ then $s - t = d(b) [d(a), r] - [d(a), r]d(b) \in \overline{d(V)}$ by (2). Thus we conclude that $t \in \overline{d(V)}$, that is, $[d(a)d(b), R] \subset \overline{d(V)}$.

Because $\overline{d(V)}$ does not contain both a non-zero left-ideal and a non-zero right-ideal of $R$, by Lemma 9 we have that $d(a)d(b) \in Z$ for all $a, b \in d[W, W]$. Let $a = d(a)d(b)$, by (1), $d(b)(bx - xb) \in \overline{d(V)}$ and since $d(a) \in \overline{d(V)}$, we get that $a(bx - xb) = d(a)d(b)(bx - xb) \in \overline{d(V)}$. Because $a \in Z$ this says that $[b, I] \subset \overline{d(V)}$ where $I = aR$ is an ideal of $R$. By our hypothesis on $\overline{d(V)}$, if $I \neq 0$, we would conclude by Lemma 9 that $b \in Z$ for all $b \in d[W, W]$; by Lemma 6 we would be led to $[W, W] \subset Z$, and so $U \subset Z$, a contradiction. Thus $I = aR = 0$, hence $a = 0$. In other words, $d(a)d(b) = 0$ for all $a, b \in d[W, W]$, that is, $(d[W, W])^2 = 0$. By Lemmas 10 and 11 we reach the contradiction $d^3 = 0$. With this the proof of Theorem 3 is complete.

We conclude the paper with a result which simultaneously implies those of Theorems 1 and 2. This is

**Theorem 4.** Let $R$ be a prime ring, char $R \neq 2$, and let $U \subset Z$ be a Lie ideal of $R$. Suppose that $\delta$ and $d$ are derivations of $R$ such that $\delta d(U) = 0$. Then either $d = 0$ or $\delta = 0$.

**Proof.** Suppose that $d \neq 0$ and $\delta \neq 0$. Let $V = [U, U]$. Then, for $v \in V$, $d(v) \in U$ hence $\delta[d[u, d(v)]] = 0$ for all $u \in U$. Thus $\delta[d(u), d(v)] + [u, d^2(v)] = 0$, which gives us, since $\delta d(U) = 0$ and $\delta$ is a derivation of $R$, that $[\delta(u), d^2(v)] = 0$ for all $u \in U, v \in V$. Thus $d^2(v) \in C_R(\delta(U)) \subset Z$, by Theorem 2.

If $v \in V$ and $r \in R$ then $0 = \delta[d(d(v), r)] = \delta([d(v), d(r)])$ since $d^2(v) \in Z$. Thus, expanding, we get $[\delta d(v), d(r)] + [d(v), \delta d(r)] = 0$ and so $[d(v), \delta d(r)] = 0$ for all $v \in V, r \in R$; that is $[d(V), \delta d(R)] = 0$. Since $d \neq 0$, by Theorem 2, $\delta d(R) \subset Z$.

Now, for $v \in V, u \in U$ we have $Z \ni \delta d(d(v)u) = \delta(d^2(v)u + d(v)d(u)) = d^2(v)\delta(u)$ since $\delta d^2(v) = \delta d(v) = 0$ because $d(v) \in U$, and $\delta(d(v)d(u)) = 0$. Therefore $d^2(V) \delta(U) \subset Z$. But, since $U \subset Z$, $\delta(U) \subset Z$ by Lemma 6; in consequence, $d^2(V) = 0$, since we know that $d^2(V) \subset Z$ and $d^2(V) \delta(U) \subset Z$. Since $V \subset Z$ and $d^2(V) = 0$, by Theorem 1 we obtain $d = 0$.

To see that Theorem 4 implies Theorem 1, merely choose $d = \delta$. As for Theorem 2, if $d \neq 0$ is a derivation of $R$ and if $a \in C_R(d(U))$, let $\delta$ be defined by $\delta(x) = ax - xa$; we see that $\delta d(U) = 0$, hence, by Theorem 4 since $d \neq 0$, $\delta$ must be 0. Therefore $ax = xa$ for all $x \in R$, that is to say, $a \in Z$. 


REFERENCES