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On the prime ideal structure of tensor products of algebras

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Abstract

This paper is concerned with the prime spectrum of a tensor product of algebras over a field. It seeks necessary and sufficient conditions for such a tensor product to have the S-property, strong S-property, and catenarity. Its main results lead to new examples of stably strong S-rings and universally catenarian rings. The work begins by investigating the minimal prime ideal structure. Throughout, several results on polynomial rings are recovered, and numerous examples are provided to illustrate the scope and sharpness of the results. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

All rings and algebras considered in this paper are commutative with identity element and, unless otherwise specified, are assumed to be non-zero. All ring homomorphisms are unital. Throughout, k denotes a field. We shall use t.d.$(A:k)$, or t.d.(A) when no confusion is likely, to denote the transcendence degree of a k-algebra A over k (for nondomains, t.d.$(A) = \sup\{\text{t.d.}(A/p)\colon p \in \operatorname{Spec}(A)\}$), and $k_A(p)$ to denote the quotient

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field of A/p, for each prime ideal p of A. Also, we use Spec(A), Max(A), and Min(A) to denote the sets of prime ideals, maximal ideals, and minimal prime ideals, respectively, of a ring A, and $\subset$ to denote proper inclusion. Recall that an integral domain A of finite Krull dimension n is a Jaffard domain if its valuative dimension, $\dim_v(A)$, is also n. A locally Jaffard domain is a finite-dimensional domain A such that A_p is a Jaffard domain for each $p \in \mathrm{Spec}(A)$. Finite-dimensional Prüfer domains and Noetherian domains are locally Jaffard domains. We assume familiarity with the above concepts, as in [1,15]. Any unreferenced material is standard, as in [12,18,20].

Since the EGA of Grothendieck [13], a few works in the literature have explored the prime ideal structure of tensor products of k-algebras (cf. [23,24,26,3,4]). These have mainly been concerned with dimension theory in specific contexts, such as tensor products of fields, AF-domains, or pullbacks. At present, the general situation remains unresolved. By analogy with known studies on polynomial rings, the investigation of some chain conditions may be expected to cast light on the spectrum of such constructions. Thus, we focus here on an in-depth study of central notions such as the S-property, strong S-property, and catenarity. In particular, our main result, Theorem 4.13, allows us to provide new families of stably strong S-rings and universally catenarian rings. Throughout, several results on polynomial rings are recovered and numerous examples are provided to illustrate the scope and sharpness of the main results.

In order to treat Noetherian domains and Prüfer domains in a unified manner, Kaplansky [18] introduced the concepts of S(eidenberg)-domain and strong S-ring. A domain A is called an S-domain if, for each height-one prime ideal p of A, the extension $pA[X]$ to the polynomial ring in one variable also has height 1. A commutative ring A is said to be a strong S-ring if A/p is an S-domain for each $p \in \mathrm{Spec}(A)$. It is noteworthy that while $A[X]$ is always an S-domain for any domain A [11], $A[X]$ need not be a strong S-ring even when A is a strong S-ring. Thus, as in [19], A is said to be a stably strong S-ring (also called a universally strong S-ring) if the polynomial ring $A[X_1,\ldots,X_n]$ is a strong S-ring for each positive integer n. The study of this class of rings was initiated by Malik and Mott [19] and further developed in [16,17]. An example of a strong S-domain which is not a stably strong S-domain was constructed in [8].

As in [6], we say that a domain A is catenarian if A is locally finite-dimensional (LFD for short) and, for each pair $P \subset Q$ of adjacent prime ideals of A, $\mathrm{ht}(Q) = 1 + \mathrm{ht}(P)$; equivalently, if for any prime ideals $P \subseteq Q$ of A, all the saturated chains in Spec(A) between P and Q have the same finite length. Note that catenarity is not stable under adjunction of indeterminates. Thus, as in [6], a domain A is said to be universally catenarian if $A[X_1,\ldots,X_n]$ is catenarian for each positive integer n. Cohen–Macaulay domains [20] or LFD Prüfer domains [7] are universally catenarian; and so are domains of valuative dimension 1 [6] and LFD domains of global dimension 2 [5]. Finally, recall that any universally catenarian domain is a stably strong S-domain [6, Theorem 2.4].

In Section 2, we extend the definitions of the S-property and catenarity to the context of arbitrary rings (i.e., not necessarily domains). Section 3 investigates the minimal prime ideal structure in tensor products of k-algebras. Vamos [25] proved that if K and L are field extensions of k, then the minimal prime ideals of $K \otimes_k L$ are pairwise comaximal. We give an example to show that this result fails for arbitrary domains A and B that are k-algebras, and then show that the minimal prime ideals of $A \otimes_k B$ are pairwise comaximal provided that A and B are integrally closed domains. As an application, we establish necessary and sufficient conditions for $A \otimes_k B$ to be an S-ring, and therefore extend (in Theorem 3.9) the known result that $A[X_1, \ldots, X_n]$ is an S-domain for any domain A and any integer $n \geqslant 1$ [11, Proposition 2.1]. Our purpose in Section 4 is to study conditions under which tensor product preserves the strong S-property and catenarity. We begin with a result of independent interest (Proposition 4.1) characterizing the LFD property for $A \otimes_k B$. Also noteworthy is Corollary 4.10 stating that the tensor product of two field extensions of k, at least one of which is of finite transcendence degree, is universally catenarian. Our main theorem (4.13) asserts that: given an LFD k-algebra A and an extension field K of k such that either $\text{t.d.}(A:k) < \infty$ or $\text{t.d.}(K:k) < \infty$, let B be a transcendence basis of K over k and L be the separable algebraic closure of $k(B)$ in K, and assume that $[L:k(B)] < \infty$; then if A is a stably strong S-ring (resp., universally catenarian and the minimal prime ideals of $K \otimes_k A$ are pairwise comaximal), $K \otimes_k A$ is a stably strong S-ring (resp., universally catenarian). This result leads to new families of stably strong S-rings and universally catenarian rings. Section 5 displays examples illustrating the limits of the results of earlier sections. The section closes with an example of a discrete rank-one valuation domain V (hence universally catenarian) such that $V \otimes_k V$ is not catenarian, illustrating the importance of assuming K is a field in Theorem 4.13.

2. Preliminaries

In this section, we extend the notions of S-domain and catenarian domain to the context of arbitrary rings (i.e., not necessarily domains). We then state some elementary results and recall certain basic facts about tensor products of k-algebras, providing a suitable background to the rest of the paper.

Consider the following four properties that a ring A may satisfy:

(P_1): A/P is an S-domain for each $P \in \text{Min}(A)$.

(P_2): $\text{ht}(P) = 1 \Rightarrow \text{ht}(P[X]) = 1$, for each $P \in \text{Spec}(A)$.

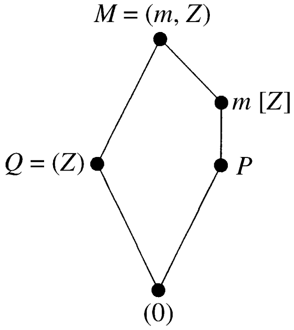
(Q_1): A is LFD and $\text{ht}(Q) = 1 + \text{ht}(P)$ for each pair $P \subset Q$ of adjacent prime ideals of A.

(Q_2): A/P is a catenarian domain for each $P \in \text{Min}(A)$.

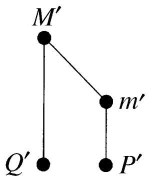
It is clear that a domain A satisfies (P_1) (resp., (P_2)) if and only if A is an S-domain; and that a domain A satisfies (Q_1) (resp., (Q_2)) if and only if A is catenarian. Some of these observations carry over to arbitrary rings. Using the basic facts from [18, p. 25], we verify easily that (P_1) $\Rightarrow$ (P_2); and that (Q_1) $\Rightarrow$ (Q_2). However, the inverse implications do not hold in general. The next example illustrates this fact.

Example 2.1. There exists a (locally) finite-dimensional ring A which satisfies both (P_2) and (Q_2) but neither (P_1) nor (Q_1).

Let $V := k(X)[Y]_{(Y)} = k(X) + m$, where $m := YV$. Let $R = k + m$. There exist two saturated chains in $\mathrm{Spec}(R[Z])$ of the form:



Indeed, let $I = PQ$ and $A = (R[Z]/I)_{M/I}$. Then A is a two-dimensional quasilocal ring, and hence trivially satisfies (Q_2). Further, part of $\mathrm{Spec}(A)$ displays as follows:



where $M' = (M/I)_{M/I}$, $Q' = (Q/I)_{M/I}$, $m' = (m[Z]/I)_{M/I}$, and $P' = (P/I)_{M/I}$. It is clear that m' is the unique prime ideal of A of height 1. By [8, Example 5], $\mathrm{ht}(m[Z]) = \mathrm{ht}(m[Z,T]) = 2$, so that $\mathrm{ht}(m'[T]) = 1$. Thus A satisfies (P_2). Now, $A/Q' \cong (R[Z]/Q)_{M/Q} \cong R$ is not an S-domain, since $\mathrm{ht}(m) = 1$ and $\mathrm{ht}(m[Z]) = 2$, whence A does not satisfy (P_1). Moreover, A fails to satisfy (Q_1), since $Q' \subset M'$ is a saturated chain in $\mathrm{Spec}(A)$ such that $\mathrm{ht}(M') = 2 \neq 1 + \mathrm{ht}(Q') = 1$.

By avoiding a feature of Example 2.1, we shall find a natural context in which (P_2) implies (P_1), and (Q_2) implies (Q_1). Let us say that a ring A satisfies MPC (for Minimal Primes Comaximality) if the minimal prime ideals in A are pairwise comaximal; i.e., if each maximal ideal of A contains only one minimal prime ideal. In the literature, MPC has also been termed "locally irreducible," presumably because any domain evidently satisfies MPC.

Remark 2.2. Let A be a ring satisfying MPC. Then:
(a) A satisfies (P_1) (resp., Q_1) if and only if A satisfies (P_2) (resp., Q_2).
(b) $S^{-1}A$ satisfies MPC for any multiplicative subset S of A.
(c) $A[X_1,\ldots,X_n]$ satisfies MPC for all integers $n \geqslant 1$.

Proof. The proof of (a) may be left to the reader. Now, (b) follows from basic facts about localization, while (c) is immediate since the minimal prime ideals of $A[X_1,\ldots,X_n]$ are of the form $p[X_1,\ldots,X_n]$, where $p \in \mathrm{Min}(A)$. $\square$

We now extend the domain-theoretic definitions of the S-property and catenarity to the MPC context. A ring A is called an S-ring if it satisfies MPC and (P_1); equivalently, MPC and (P_2). A ring A is said to be catenarian if A satisfies MPC and (Q_1); equivalently, MPC and (Q_2). It is useful to note that if A is an S-ring (resp., a catenarian ring), then so is A_S ($=S^{-1}A$), for each multiplicative subset S of A.

Next, we extend a domain-theoretic result of Malik and Mott [19, Theorem 4.6].

Proposition 2.3. *Let $A \subseteq T$ be an integral ring extension. If T is a strong* S-*ring* (*resp.*, *stably strong* S-*ring*), *then so is A.*

Proof. Let $p \in \mathrm{Spec}(A)$. Since T is an integral extension of A, the Lying-over theorem provides $P \in \mathrm{Spec}(T)$ such that $P \cap A = p$. Hence T/P is an integral extension of A/p, and T/P is a strong S-domain by hypothesis. Consequently, by [19, Theorem 4.6], A/p is a (strong) S-domain. The "stably strong S-ring" assertion follows from the "strong S-ring" assertion since $A[X_1,\ldots,X_n] \subseteq T[X_1,\ldots,X_n]$ inherits integrality from $A \subseteq T$. □

Proposition 2.5 generalizes the following domain-theoretic result.

Proposition 2.4 (Bouvier et al. [6, Corollary 6.3]). *Let A be a one-dimensional domain. Then the following conditions are equivalent*:
(i) *A is universally catenarian*;
(ii) *$A[X]$ is catenarian*;
(iii) *A is a stably strong* S-*domain*;
(iv) *A is a strong* S-*domain*;
(v) *A is an* S-*domain.*

Proposition 2.5. *Let A be a one-dimensional ring. Then*
(a) *The following three conditions are equivalent*:
  (i) *A is a stably strong* S-*ring*;
  (ii) *A is a strong* S-*ring*;
  (iii) *A satisfies* (P_2).
(b) *Suppose*, *in addition*, *that A satisfies MPC. Then* (i)–(iii) *are equivalent to each of* (iv)–(vi):
  (iv) *A is universally catenarian*;
  (v) *$A[X]$ is catenarian*;
  (vi) *A is an* S-*ring.*

Proof. (a) It is trivial that (i) ⇒ (ii) (even without one-dimensionality). Also, any field is an S-domain. As $\dim(A)=1$, (ii) is therefore equivalent to the requirement that A/Q is an S-domain for each $Q \in \mathrm{Min}(A)$. This requirement is obviously equivalent to (iii). Thus, (ii) ⇔ (iii).

(ii) ⇒ (i) Clearly, it suffices to prove that A/p is a stably strong S-domain for each $p \in \mathrm{Min}(A)$. By Proposition 2.4, this assertion holds, since for any $p \in \mathrm{Min}(A)$, A/p is either a field or a one-dimensional strong S-domain.

(b) (iii) $\Leftrightarrow$ (P_1) $\Leftrightarrow$ (vi), since A satisfies MPC.

(v) $\Rightarrow$ (vi) Let $p \in \mathrm{Min}(A)$. Then $(A/p)[X] \cong A[X]/p[X]$ is a catenarian domain, since $p[X] \in \mathrm{Min}(A[X])$. So, by Proposition 2.4, A/p is an S-domain. Hence (in view of the MPC condition), A is an S-ring.

(vi) $\Rightarrow$ (iv) It suffices to prove that A/p is universally catenarian, for each minimal prime ideal p of A. This holds by Proposition 2.4, since for any $p \in \mathrm{Min}(A)$, A/p is either a field or a one-dimensional S-domain. The proof is complete. □

For the convenience of the reader, we close this section by discussing some basic facts connected with the tensor product of k-algebras. These will be used frequently in the sequel without explicit mention.

Let A and B be two k-algebras. If A' is an integral extension of A, then $A' \otimes_k B$ is an integral extension of $A \otimes_k B$. If S_1 and S_2 are multiplicative subsets of A and B, respectively, then $S_1^{-1}A \otimes_k S_2^{-1}B \cong S^{-1}(A \otimes_k B)$, where $S := \{s_1 \otimes s_2\colon s_1 \in S_1 \text{ and } s_2 \in S_2\}$. Recall also that if A is an integral domain, then $\mathrm{ht}(p) + \mathrm{t.d.}(A/p) \leqslant \mathrm{t.d.}(A)$, for each $p \in \mathrm{Spec}(A)$ (cf. [21, p. 37] and [28, p. 10]). It follows that $\dim(A) \leqslant \mathrm{t.d.}(A)$ for any ring A. Moreover, we assume familiarity with the natural isomorphisms for tensor products. In particular, we identify A and B with their canonical images in $A \otimes_k B$. Also, $A \otimes_k B$ is a free (hence faithfully flat) extension of A and B. Here we recall that if $R \hookrightarrow S$ is a flat ring extension and $P \in \mathrm{Min}(S)$, then $P \cap R \in \mathrm{Min}(R)$ by going-down. Finally, we refer the reader to the useful result of Wadsworth [26, Proposition 2.3] which yields a classification of the prime ideals of $A \otimes_k B$ according to their contractions to A and B.

3. Minimal prime ideal structure and S-property

This section studies the transfer of the MPC property and S-property to tensor products of k-algebras. As a prelude to this, we first investigate the minimal prime ideal structure of such constructions. In [25, Corollary 4], Vamos proved that if K and L are field extensions of k, then $K \otimes_k L$ satisfies MPC. We first illustrate by an example the failure of this result for arbitrary k-algebras A and B, and then show that $A \otimes_k B$ satisfies MPC provided A and B are integrally closed domains. As an application, we establish necessary and sufficient conditions for $A \otimes_k B$ to be an S-ring, and therefore extend the known result that $A[X_1, \dots, X_n]$ is an S-domain, for any domain A and any integer $n \geqslant 1$ [11, Proposition 2.1]. Throughout Sections 3 and 4, LO (resp., GD) refers to the condition “Lying-over” (resp., “Going-down”), as in [18, p. 28].

We begin by providing a necessary condition for $A \otimes_k B$ to satisfy MPC.

Proposition 3.1. (a) *If $C \subseteq D$ is a ring extension satisfying LO and GD, and D satisfies MPC, then C satisfies MPC.*

(b) *If A and B are k-algebras such that $A \otimes_k B$ satisfies MPC, then A and B each satisfy MPC.*

Proof. (a) Let $p, q \in \mathrm{Min}(C)$ and $m \in \mathrm{Spec}(C)$ such that $p+q \subseteq m$. Since $C \subseteq D$ satisfies LO and GD, there exist $P, Q, M \in \mathrm{Spec}(D)$ with $P \cap C = p$, $Q \cap C = q$, $M \cap C = m$, and $P+Q\subseteq M$. Choose $P_0, Q_0 \in \mathrm{Min}(D)$ such that $P_0 \subseteq P$ and $Q_0 \subseteq Q$. Therefore $P_0+Q_0 \subseteq M$, with $P_0 \cap A = p$ and $Q_0 \cap A = q$. Since D satisfies MPC, we have $P_0 = Q_0$; consequently $p=q$, as desired.

(b) It suffices to treat A. Now, $A\otimes_k B$ is A-flat, and so $A \to A\otimes_k B$ satisfies GD. It also satisfies LO by [26, Proposition 2.3]. Apply (a), to complete the proof. □

The following example shows that Vamos' result (mentioned above) does not extend to arbitrary k-algebras. It also provides a counterexample to the converse of Proposition 3.1(b).

Example 3.2. There exists a separable algebraic field extension K of finite degree over k and a k-algebra A satisfying MPC such that $K\otimes_k A$ fails to satisfy MPC.

Let $k=\mathbb{R}$ and $K=\mathbf{C}$ be the fields of real numbers and complex numbers, respectively. Let $V := \mathbf{C}[X]_{(X)} = \mathbf{C} + X\mathbf{C}[X]_{(X)}$ and $A := \mathbb{R} + X\mathbf{C}[X]_{(X)}$. Clearly, A is a one-dimensional local domain with quotient field $L=\mathbf{C}(X)$ and maximal ideal $p = X\mathbf{C}[X]_{(X)}$, such that $A/p=\mathbb{R}$. We wish to show that $K\otimes_{\mathbb{R}} A$ does not satisfy MPC. Indeed, let $f(Z)=Z^2+1$ be the minimal polynomial of i over $\mathbb{R}$. We have $K\otimes_{\mathbb{R}} A \cong \mathbb{R}[Z]/(f(Z))\otimes_{\mathbb{R}} A \cong A[Z]/(f(Z))$. Therefore, the minimal prime ideals of $K\otimes_{\mathbb{R}} A$ are $\bar{I}=I/(f)$ and $\bar{J}=J/(f)$, where $I=(Z-i)L[Z]\cap A[Z]$ and $J=(Z+i)L[Z]\cap A[Z]$. Since $K\otimes_{\mathbb{R}} A$ is an integral extension of A, then so are $A[Z]/I \cong (A[z]/(f))/\bar{I}$ and $A[Z]/J \cong (A[z]/(f))/\bar{Y}$, whence $\dim(A[Z]/I)=\dim(A[Z]/J)=\dim(A)=1$. It follows that I and J are not maximal ideals in $A[Z]$. Then, there exist P_I and P_J in $\mathrm{Spec}(A[Z])$ such that $I\subset P_I$ and $J\subset P_J$. Clearly, $P_I\cap A = P_J\cap A = p$. Further, since $f\in I\cap J$ and $f\notin p[Z]$, then P_I and P_J are both uppers to p. As $A/p=\mathbb{R}$ and f is an irreducible monic polynomial over $\mathbb{R}$, it follows that $P_I = P_J = (p,f)$ (cf. [18, Theorem 28]). Therefore $I+J\subseteq P := (p,f)$, and hence $\bar{I}+\bar{J}\subseteq \bar{P} := P/(f(Z))$. Consequently, $A[z]/(f)\cong K\otimes_{\mathbb{R}} A$ does not satisfy, MPC, establishing the claim.

We next investigate various contexts for the tensor product to inherit the MPC property. The following result treats the case where the ground field k is algebraically closed.

Theorem 3.3. *Let k be an algebraically closed field. Let A and B be k-algebras. Then $A\otimes_k B$ satisfies MPC if and only if A and B each satisfy MPC.*

Proof. Proposition 3.1(b) handles the "only if" assertion. Next, assume that A and B each satisfy MPC. Let $P_0, Q_0 \in \mathrm{Min}(A\otimes_k B)$ and $P\in \mathrm{Spec}(A\otimes_k B)$ such that $P_0+Q_0\subseteq P$. Let $p_1 := P_0\cap A$, $q_1 := P_0\cap B$ and $p_2 := Q_0\cap A$, $q_2 := Q_0\cap B$. We have $p_1,p_2\in\mathrm{Min}(A)$ and $q_1,q_2\in\mathrm{Min}(B)$, since $A\subseteq A\otimes_k B$ and $B\subseteq A\otimes_k B$ each satisfy GD. Let $p := P\cap A$ and $q := P\cap B$. Then $p_1+p_2\subseteq p$ and $q_1+q_2\subseteq q$. As A and B each satisfy MPC, $p_1=p_2=: p_0$ and $q_1=q_2=: q_0$. Since k is algebraically closed, it follows from [27, Corollary 1, Chapter III, p. 198] and the lattice-isomorphism in [26, Proposition 2.3] that there is a unique prime Q of $A\otimes_k B$ that is minimal with respect to the properties $Q\cap A = p_0$, $Q\cap B = q_0$. Hence, $P_0 = Q = Q_0$, and the proof is complete. □

The next theorem generalizes the above-mentioned result of Vamos.

Theorem 3.4. *If A and B are integrally closed domains that are k-algebras, then $A \otimes_k B$ satisfies MPC.*

Proof. Let K (resp., L) denote the quotient field of A (resp., B). Let K_s (resp., L_s) denote the separable algebraic closure of k in K (resp., in L). Since A is integrally closed and $k \subseteq A \subseteq K$, the algebraic closure of k in K is contained in A. In particular, $K_s \subseteq A$; and, similarly, $L_s \subseteq B$. By [25, Theorem 3], $\mathrm{Min}(K \otimes_k L)$ and $\mathrm{Spec}(K_s \otimes_k L_s)$ are canonically homeomorphic, with the prime ideals of $K_s \otimes_k L_s$ being the contractions of the minimal prime ideals of $K \otimes_k L$. Observe that $K \otimes_k L$ is the localization of $A \otimes_k B$ at $\{a \otimes b\colon a \in A \setminus \{0\}, b \in B \setminus \{0\}\}$. If follows that there is a one-to-one correspondence between $\mathrm{Min}(K \otimes_k L)$ and $\mathrm{Min}(A \otimes_k B)$. Since $K_s \otimes_k L_s \subseteq A \otimes_k B \subseteq K \otimes_k L$, we obtain, via contraction, a bijection between $\mathrm{Min}(A \otimes_k B)$ and $\mathrm{Spec}(K_s \otimes_k L_s)$. Now, consider $P_0, Q_0 \in \mathrm{Min}(A \otimes_k B)$ and $P \in \mathrm{Spec}(A \otimes_k B)$ such that $P_0 + Q_0 \subseteq P$. Taking contractions to $K_s \otimes_k L_s$, we obtain $P_0^c = P^c$ and $Q_0^c = P^c$, since $\dim(K_s \otimes_k L_s) = 0$ [26]. In particular, $P_0^c = Q_0^c$. By the above bijection, $P_0 = Q_0$, as desired. □

The proof of Theorem 3.4 actually gives the following result. Let A and B be domains that are k-algebras. Let K_s (resp., L_s) be the separable algebraic closure of k in the quotient field K (resp., L) of A (resp., B). If $K_s \subseteq A$ and $L_s \subseteq B$, then $A \otimes_k B$ satisfies MPC.

Moving beyond the contexts of Theorems 3.3 and 3.4, we next show that $A \otimes_k B$ can satisfy MPC when k is not algebraically closed and when A, B are not integrally closed domains.

Example 3.5. Let $k := \mathbb{Q}$ be the filed of rational numbers and let $A := B := Q(i)[X^2, X^3]$. The quotient field of A (resp., B) is $K = L = Q(i)(X)$. We can easily check that A and B are not integrally closed (in fact, they are not seminormal), and $K_s = L_s = \mathbb{Q}(i)$, since $\mathbb{Q}(i)[X^2, X^3] \subseteq \mathbb{Q}(i)[X]$ which is integrally closed. Then $K_s \subseteq A$ and $L_s \subseteq B$. By the above remark, $A \otimes_{\mathbb{Q}} B$ satisfies MPC, although $k = \mathbb{Q}$ is not algebraically closed and A, B are not integrally closed.

In Example 3.2, we exhibited a separable algebraic extension field K of k and a k-algebra A satisfying MPC such that $K \otimes_k A$ fails to satisfy MPC. The following result studies the case where K is purely inseparable over k.

Proposition 3.6. *Let A be a k-algebra and K a purely inseparable field extension of k. Then $K \otimes_k A$ satisfies MPC if and only if A satisfies MPC.*

Proof. Proposition 3.1(b) handles the “only if” assertion. Conversely, assume that A satisfies MPC. Let P_0, Q_0 be minimal prime ideals of $K \otimes_k A$ and let $P \in \mathrm{Spec}(K \otimes_k A)$ such that $P_0 + Q_0 \subseteq P$. Put $p_0 := P_0 \cap A$, $q_0 := Q_0 \cap A$, and $p := P \cap A$. Hence $p_0 + q_0 \subseteq p$. Of course, p_0 and q_0 are in $\mathrm{Min}(A)$ since flatness ensures that $A \subseteq K \otimes_k A$ satisfies GD. Thus, since A satisfies MPC, we obtain $p_0 = q_0$. However, $\mathrm{Spec}(K \otimes_k A) \to \mathrm{Spec}(A)$ is an injection, since “radiciel” is a universal property [13]. Consequently, $P_0 = Q_0$, as desired. □

Theorem 3.9 is an extension to tensor products of k-algebras of the result [11, Proposition 2.1] that $A[X_1,\dots,X_n]$ is an S-domain, for any domain A and any integer $n \geqslant 1$. This latter result was generalized to infinite sets of indeterminates in [10, Corollary 2.13].

First we establish the following preparatory lemmas.

Lemma 3.7. *If A is a ring that satisfies MPC, then $A[X_1,\dots,X_n]$ is an* S*-ring, for every integer $n \geqslant 1$.*

Proof. By Remark 2.2(c), $A[X_1,\dots,X_n]$ satisfies MPC. Thus, it suffices to show the result when $n=1$. Let $Q \in \mathrm{Min}(A[X])$. Then there exists $q \in \mathrm{Min}(A)$ such that $Q=q[X]$. Hence, $A[X]/Q \cong (A/q)[X]$ is an S-domain, by the above remark, since A/q is an integral domain. Thus, $A[X]$ is an S-ring. □

Lemma 3.8. *Let A be a k-algebra and let K be a field extension of k such that $K \otimes_k A$ satisfies MPC. Then $K \otimes_k A$ is an* S*-ring if and only if either A is an* S*-ring or* $\mathrm{t.d.}(K:k) \geqslant 1$.

Proof. Suppose that $t := \mathrm{t.d.}(K:k) = 0$, i.e., that K is algebraic over k. Then $K \otimes_k A$ is an integral extension of A and thus satisfies LO. Furthermore $A \subseteq K \otimes_k A$ satisfies GD and so it follows easily that A inherits MPC from $K \otimes_k A$. It remains to show that if $P \in \mathrm{Min}(K \otimes_k A)$ and $p = P \cap A$, then $(K \otimes_k A)/P$ is an S-domain if and only if A/p is an S-domain. The "only if" statement follows from the proof of [19, Theorem 4.6], while the treatment of the "if statement is similar to that of proof of [19, Theorem 4.9].

In the remaining case, $t := \mathrm{t.d.}(K:k) \geqslant 1$. Let B be a transcendence basis of K over k. As $K \otimes_k A \cong K \otimes_{k(B)} (k(B) \otimes_k A)$, we see that $k(B) \otimes_k A$ satisfies MPC, by Proposition 3.1(b). Also, if $X \in B$, $B_1 := B \setminus \{X\}$ and $S := k(B_1)[X] \setminus \{0\}$, then $k(B) \otimes_k A = k(B_1)(X) \otimes_k A \cong S^{-1}((k(B_1) \otimes_k A)[X])$. As $k(B_1) \otimes_k A$ satisfies MPC, Lemma 3.7 yields that $(k(B_1) \otimes_k A)[X]$ is an S-ring. Hence, so is its ring of fractions $k(B) \otimes_k A$. Therefore, by the first case, so is $K \otimes_{k(B)} (k(B) \otimes_k A) \cong K \otimes_k A$, to complete the proof. □

Theorem 3.9. *Let A and B be k-algebras such that $A \otimes_k B$ satisfies MPC. Then $A \otimes_k B$ is an* S*-ring if and only if at least one of the following statements is satisfied*:
(1) *A and B are* S*-rings*;
(2) *A is an* S*-ring and* $\mathrm{t.d.}(A/p:k) \geqslant 1$ *for each* $p \in \mathrm{Min}(A)$;
(3) *B is an* S*-ring and* $\mathrm{t.d.}(B/q:k) \geqslant 1$ *for each* $q \in \mathrm{Min}(B)$;
(4) $\mathrm{t.d.}(A/p:k) \geqslant 1$ *and* $\mathrm{t.d.}(B/q:k) \geqslant 1$ *for each* $p \in \mathrm{Min}(A)$ *and* $q \in \mathrm{Min}(B)$.

Proof. We claim that $A \otimes_k B$ is an S-ring if and only if $k_A(p) \otimes_k B$ and $A \otimes_k k_B(q)$ are S-rings for each $p \in \mathrm{Min}(A)$ and $q \in \mathrm{Min}(B)$. Indeed, assume that $A \otimes_k B$ is an S-ring. Clearly, by [26, Proposition 2.3], for each minimal prime ideal p of A, $(A/p) \otimes_k B \cong (A \otimes_k B)/(p \otimes_k B)$ satisfies MPC, and thus so does its ring of fractions $k_A(p) \otimes_k B$. Similarly, so does $A \otimes_k k_B(q)$, for each minimal prime ideal q of B. In view of Remark

2.2(a), we may focus on (P_2). Let $p \in \mathrm{Min}(A)$ and $P \in \mathrm{Spec}(A \otimes_k B)$ such that $P \cap A = p$ and $\mathrm{ht}(P/(p \otimes_k B)) = 1$. Since $p \in \mathrm{Min}(A)$, we have $\mathrm{ht}(P) = \mathrm{ht}(P/(p \otimes_k B)) = 1$. By the hypothesis on $A \otimes_k B$, $1 = \mathrm{ht}(P[X]) = \mathrm{ht}((P/(p \otimes_k B))[X])$. Hence, $k_A(p) \otimes_k B$ is an S-ring for each $p \in \mathrm{Min}(A)$. Similarly, so is $A \otimes_k k_B(q)$ for each $q \in \mathrm{Min}(B)$.

Conversely, suppose that $k_A(p) \otimes_k B$ and $A \otimes_k k_B(q)$ are S-rings for each $p \in \mathrm{Min}(A)$ and $q \in \mathrm{Min}(B)$. Let $P \in \mathrm{Spec}(A \otimes_k B)$ such that $\mathrm{ht}(P) = 1$. By [26, Corollary 2.5], we have that either $p := P \cap A$ is a minimal prime ideal of A or $q := P \cap B$ is a minimal prime ideal of B. Without loss of generality, $p \in \mathrm{Min}(A)$. Then $\mathrm{ht}(P/(p \otimes_k B)) = \mathrm{ht}(P) = 1$. Since $k_A(p) \otimes_k B$ is an S-ring, we have $1 = \mathrm{ht}((P/(p \otimes_k B))[X]) = \mathrm{ht}(P[X])$. Consequently, $A \otimes_k B$ is an S-ring, and the claim has been proved. The theorem now follows from Lemma 3.8. □

It is clear from the above proof that the statement of Theorem 3.9 remains true without the MPC hypothesis if we substitute (P_2) for the S-ring property.

Corollary 3.10. *Let k be an algebraically closed field. Let A and B be domains that are k-algebras. Then $A \otimes_k B$ is an* S-*domain if and only if at least one of the following statements is satisfied.*
(1) *A and B are* S-*domains*;
(2) *A is an* S-*domain and* $\mathrm{t.d.}(A : k) \geqslant 1$;
(3) *B is an* S-*domain and* $\mathrm{t.d.}(B : k) \geqslant 1$;
(4) $\mathrm{t.d.}(A : k) \geqslant 1$ *and* $\mathrm{t.d.}(B : k) \geqslant 1$.

Proof. Apply Theorem 3.9, bearing in mind that $A \otimes_k B$ is an integral domain (hence satisfies MPC) since k is algebraically closed [27, Corollary 1, Chapter III, p. 198]. □

Corollary 3.11. *Let A and B be integrally closed domains that are k-algebras. Then $A \otimes_k B$ is an* S-*ring if and only if at least one of the following statements is satisfied*:
(1) *A and B are* S-*domains*;
(2) *A is an* S-*domain and* $\mathrm{t.d.}(A : k) \geqslant 1$;
(3) *B is an* S-*domain and* $\mathrm{t.d.}(B : k) \geqslant 1$;
(4) $\mathrm{t.d.}(A : k) \geqslant 1$ *and* $\mathrm{t.d.}(B : k) \geqslant 1$.

Proof. Combine Theorems 3.9 and 3.4. □

4. Strong S-property and catenarity

Our purpose in this section is to seek conditions for the tensor product of two k-algebras to inherit the (stably) strong S-property and (universal) catenarity. The main theorem of this section generates new families of stably strong S-rings and universally catenarian rings. Our interest is turned essentially to studying $A \otimes_k B$ in case at least one of A, B is a field extension of k. Beyond this context, the study of these properties becomes more intricate, as one may expect. In fact, a glance ahead to Example 5.5

reveals a non-catenarian ring of the form $A \otimes_k B$ in which A, B are each universally catenarian domains (in fact DVRs).

To determine when a tensor product of k-algebras is catenarian, we first need to know when it is LFD. That is handled by the first result of this section.

Proposition 4.1. *Let A and B be k-algebras. Then*:

(a) *If $A \otimes_k B$ is LFD, then so are A and B, and either* $\mathrm{t.d.}(A/p : k) < \infty$ *for each prime ideal p of A or* $\mathrm{t.d.}(B/q : k) < \infty$ *for each prime ideal q of B.*

(b) *If both A and B are LFD and either* $\mathrm{t.d.}(A : k) < \infty$ or $\mathrm{t.d.}(B : k) < \infty$, *then $A \otimes_k B$ is LFD. The converse holds provided A and B are domains.*

The proof of this proposition requires the following preparatory lemma.

Lemma 4.2. *Let K and L be field extensions of k. Then $K \otimes_k L$ is LFD if and only if either* $\mathrm{t.d.}(K : k) < \infty$ *or* $\mathrm{t.d.}(L : k) < \infty$.

Proof. ($\Leftarrow$) Straightforward, since $\dim(K \otimes_k L) = \min(\mathrm{t.d.}(K : k), \mathrm{t.d.}(L : k))$ (cf. [23, Theorem 3.1]).

($\Rightarrow$) Let B (resp., B') be a transcendence basis of K (resp., L) over k. As $K \otimes_k L \cong K \otimes_{k(B)} (k(B) \otimes_k k(B')) \otimes_{k(B')} L$, then $k(B) \otimes_k k(B') \subset\!\!\!\rightarrow K \otimes_k L$ is an integral extension that satisfies GD. Therefore, $K \otimes_k L$ is LFD if and only if $k(B) \otimes_k k(B')$ is LFD. Suppose that $\mathrm{t.d.}(K : k) = \mathrm{t.d.}(L : k) = \infty$. Let $T := k(x_1, x_2, \ldots) \otimes_k k(y_1, y_2, \ldots)$, where the $x_i \in B$ and the $y_i \in B'$. Since $T \subseteq k(B) \otimes_{k(x_1,x_2,\ldots)} T$ and $k(B) \otimes_{k(x_1,x_2,\ldots)} T \subseteq (k(B) \otimes_{k(x_1,x_2,\ldots)} T) \otimes_{k(y_1,y_2,\ldots)} k(B') \cong k(B) \otimes_k k(B')$ are ring extensions that satisfy GD and LO, then so does $T \subseteq k(B) \otimes_k k(B')$. Thus T is not LFD $\Rightarrow k(B) \otimes_k k(B')$ is not LFD $\Rightarrow K \otimes_k L$ is not LFD.

Let $K_n = k(x_1, \ldots, x_n)$ and $S_n = k[y_1, \ldots, y_n] \setminus \{0\}$, for each $n \geqslant 1$. Consider the following ring homomorphisms:

$$K_n[y_1, \ldots, y_n] \underset{i_n}{\subset\!\!\!\rightarrow} k(x_1, x_2, \ldots)[y_1, y_2, \ldots] \xrightarrow{\varphi} k(x_1, x_2, \ldots),$$

where $\varphi(y_i) = x_i$ for $i \geqslant 1$. Let $M = \mathrm{Ker}(\varphi)$ and $M_n = M \cap K_n[y_1, \ldots, y_n] = \mathrm{Ker}(\varphi_n)$, where $\varphi_n := \varphi \circ i_n$, for all $n \geqslant 1$. Since $x_1, \ldots, x_n$ are algebraically independent over k, $M_n \cap S_n = \emptyset$, for all $n \geqslant 1$. On the other hand, since $K_n[y_1, \ldots, y_n]$ is an AF-domain (we recall early in Section 5 the definition of an AF-domain), then, for every $n \geqslant 1$,

$$\mathrm{ht}(M_n) + \mathrm{t.d.}\left(\frac{K_n[y_1, \ldots, y_n]}{M_n} : K_n\right) = \mathrm{t.d.}(K_n[y_1, \ldots, y_n] : K_n) = n.$$

Hence $\mathrm{ht}(M_n) = n$, since $\frac{K_n[y_1, \ldots, y_n]}{M_n} \cong K_n$, for all $n \geqslant 1$. Therefore $M \cap S = \emptyset$, where $S := \bigcup_n S_n = k[y_1, y_2, \ldots] \setminus \{0\}$. We wish to show that $\mathrm{ht}(M) = \infty$. Indeed, observe that, for any integer $n \geqslant 1$,

$$\begin{aligned} M_n k(x_1, \ldots)[y_1, \ldots] &= M_n(k(x_1, \ldots) \otimes_{K_n} K_n[y_1, \ldots, y_n] \otimes_k k[y_{n+1}, \ldots]) \\ &= k(x_1, \ldots) \otimes_{K_n} M_n \otimes_k k[y_{n+1}, \ldots], \quad \text{and} \end{aligned}$$

$$\begin{aligned}
\frac{k(x_1,\dots)[y_1,\dots]}{M_nk(x_1,\dots)[y_1,\dots]} &\cong \frac{k(x_1,\dots)\otimes_{K_n}K_n[y_1,\dots,y_n]\otimes_k k[y_{n+1},\dots]}{k(x_1,\dots)\otimes_{K_n}M_n\otimes_k k[y_{n+1},\dots]}\\
&\cong k(x_1,\dots)\otimes_{K_n}\frac{K_n[y_1,\dots,y_n]}{M_n}\otimes_k k[y_{n+1},\dots] \quad (\text{cf.}[26])\\
&\cong k(x_1,\dots)\otimes_{K_n}K_n\otimes_k k[y_{n+1},\dots]\\
&\cong k(x_1,\dots)[y_{n+1},\dots], \quad \text{an integral domain.}
\end{aligned}$$

Thus $M_nk(x_1,\dots)[y_1,\dots]$ is a prime ideal in $k(x_1,\dots)[y_1,\dots]$, for all $n \geqslant 1$. Since $K_n[y_1,\dots,y_n] \to k(x_1,\dots)[y_1,\dots]$ is a faithfully flat homomorphism (and hence satisfies GD), we obtain $M_nk(x_1,\dots)[y_1,\dots]\cap K_n[y_1,\dots,y_n]=M_n$, and thus $\mathrm{ht}(M_nk(x_1,\dots)[y_1,\dots]) \geqslant \mathrm{ht}(M_n)=n$. By direct limits (cf. [10]), it follows that $\mathrm{ht}(M)=\infty$, as desired. Consequently, $S^{-1}M$ is a prime ideal of $T=k(x_1,\dots)\otimes_k k(y_1,\dots)$ with $\mathrm{ht}(S^{-1}M)=\infty$. Therefore T is not LFD, completing the proof. □

Proof of Proposition 4.1. (a) Assume that $A\otimes_k B$ is LFD. Let $p\in\mathrm{Spec}(A)$ and $q\in\mathrm{Spec}(B)$. As the extensions $A\subseteq A\otimes_k B$ and $B\subseteq A\otimes_k B$ satisfy LO, there exist prime ideals P and Q of $A\otimes_k B$ such that $P\cap A=p$ and $Q\cap B=q$. By [26, Corollary 2.5], $\mathrm{ht}(p)\leqslant\mathrm{ht}(P)<\infty$ and $\mathrm{ht}(q)\leqslant\mathrm{ht}(Q)<\infty$. It follows that A and B are LFD. Now, suppose that there exists a prime ideal q of B such that $\mathrm{t.d.}(B/q:k)=\infty$. Let p be any prime ideal of A. Then $A/p\otimes_k B/q\cong A\otimes_k B/(p\otimes_k B+A\otimes_k q)$ is LFD. Hence $k_A(p)\otimes_k k_B(q)$ is LFD, since it is a ring of fractions of $A/p\otimes_k B/q$. Therefore, by Lemma 4.2, $\mathrm{t.d.}(k_A(p):k)=\mathrm{t.d.}(A/p:k)<\infty$.

(b) Suppose that $\mathrm{t.d.}(A:k)<\infty$ and both A and B are LFD. Consider a chain $\Omega:=\{P_0\subset P_1\subset\cdots\subset P\}$ of prime ideals of $A\otimes_k B$ and let l be its length. We claim that l is finite, with an upper bound depending on P. Let $p_0\subset\cdots\subset p_r=p:=P\cap A$ and $q_0\subset\cdots\subset q_s=q:=P\cap B$ be the chains of intersections of Ω over A and B, respectively. We can partition Ω into subchains Ω_{ij} the prime ideals of which contract to p_i in $\mathrm{Spec}(A)$ and q_j in $\mathrm{Spec}(B)$. Thus each Ω_{ij} of length $l_{ij}\leqslant\dim(k_A(p_i)\otimes_k k_B(q_j))$, by [26, Proposition 2.3]. Therefore, we have

$$\begin{aligned}
l &\leqslant \sum_{i=0,j=0}^{r,s}(\dim(k_A(p_i)\otimes_k k_B(q_j))+1)\\
&\leqslant \sum_{i=0,j=0}^{r,s}\left(\min\left(\mathrm{t.d.}\left(\frac{A}{p_i}:k\right),\mathrm{t.d.}\left(\frac{B}{q_j}:k\right)\right)+1\right) \quad [23,\text{Theorem } 3.1]\\
&\leqslant \sum_{i=0,j=0}^{r,s}\left(\mathrm{t.d.}\left(\frac{A}{p_i}:k\right)+1\right)\\
&\leqslant (\mathrm{t.d.}(A:k)+1)(r+1)(s+1)\\
&\leqslant (\mathrm{t.d.}(A:k)+1)(\mathrm{ht}(p)+1)(\mathrm{ht}(q)+1)<\infty, \text{ as desired.}
\end{aligned}$$

Now, if A and B are domains, then the converse holds, by (a). □

Given an integer $n \geqslant 1$, Malik and Mott proved that $A[X_1,\dots,X_n]$ is a strong S-ring if and only if so is $A_p[X_1,\dots,X_n]$ for each prime ideal p of A [19, Theorem 3.2]. We next extend this result to tensor products of k-algebras.

Proposition 4.3. *Let A_1 and A_2 be k-algebras. Then the following statements are equivalent*:

(1) $A_1 \otimes_k A_2$ *is a strong* S-*ring* (*resp*., *catenarian*);

(2) $S_1^{-1}A_1 \otimes_k S_2^{-1}A_2$ *is a strong* S-*ring* (*resp*., *catenarian*) *for each multiplicative subset* S_i *of* A_i, *for* $i = 1,2$;

(3) $(A_1)_{p_1} \otimes_k A_2$ *is a strong* S-*ring* (*resp*., *catenarian*) *for each* $p_1 \in \mathrm{Spec}(A_1)$;

(4) $(A_1)m_1 \otimes_k A_2$ *is a strong* S-*ring* (*resp*., *catenarian*) *for each* $m_1 \in \mathrm{Max}(A_1)$;

(5) $A_1 \otimes_k (A_2)_{p_2}$ *is a strong* S-*ring* (*resp*., *catenarian*) *for each* $p_2 \in \mathrm{Spec}(A_2)$;

(6) $A_1 \otimes_k (A_2)_{m_2}$ *is a strong* S-*ring* (*resp*., *catenarian*) *for each* $m_2 \in \mathrm{Max}(A_2)$;

(7) $(A_1)_{m_1} \otimes_k (A_2)_{m_2}$ *is a strong* S-*ring* (*resp*., *catenarian*) *for each* $m_i \in \mathrm{Max}(A_i)$, *for* $i = 1,2$.

Proof. The class of strong S- (resp., catenarian) rings is stable under formation of rings of fractions. Thus (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (7), and (2) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7). Therefore, it suffices to prove that (7) $\Rightarrow$ (1). Note that if $(A_1)_{m_1} \otimes_k (A_2)_{m_2}$ satisfies MPC for each maximal ideal m_i of A_i, for $i=1,2$, then $A_1 \otimes_k A_2$ satisfies MPC. Indeed, let P_1 and P_2 be two minimal prime ideals contained in a common prime ideal P of $A_1 \otimes_k A_2$. Choose a maximal ideal m_i of A_i such that $P \cap A_i \subseteq m_i$, for $i = 1,2$. Then $P_i((A_1)_{m_1} \otimes_k (A_2)_{m_2}) \subseteq P((A_1)_{m_1} \otimes_k (A_2)_{m_2})$, for $i = 1,2$. Hence, by hypothesis, $P_1((A_1)_{m_1} \otimes_k (A_2)_{m_2}) = P_2((A_1)_{m_1} \otimes_k (A_2)_{m_2})$. Taking contractions to $A_1 \otimes_k A_2$, we obtain $P_1 = P_2$, since $(A_1)_{m_1} \otimes_k (A_2)_{m_2}$ is a ring of fractions of $A_1 \otimes_k A_2$. Then $A_1 \otimes_k A_2$ satisfies MPC. Also, if $(A_1)_{m_1} \otimes_k (A_2)_{m_2}$ is LFD for each maximal ideal m_i of A_i, for $i = 1,2$, then it is clear that $A_1 \otimes_k A_2$ is LFD.

Now suppose that (7) holds. Let $P \subset Q$ be a saturated chain in $\mathrm{Spec}(A_1 \otimes_k A_2)$, $p_i := P \cap A_i$ and $q_i := Q \cap A_i$, for $i = 1,2$. Choose $m_i \in \mathrm{Max}(A_i)$ such that $p_i \subseteq q_i \subseteq m_i$, for $i = 1,2$. Then $P((A_1)_{m_1} \otimes_k (A_2)_{m_2}) \subset Q((A_1)_{m_1} \otimes_k (A_2)_{m_2})$ is a saturated chain in $\mathrm{Spec}((A_1)_{m_1} \otimes_k (A_2)_{m_2})$. Since $(A_1)_{m_1} \otimes_k (A_2)_{m_2}$ is a strong S-ring (resp., catenarian), we have $P((A_1)_{m_1} \otimes_k (A_2)_{m_2})[X] \subset Q((A_1)_{m_1} \otimes_k (A_2)_{m_2})[X]$ is a saturated chain in $\mathrm{Spec}((A_1)_{m_1} \otimes_k (A_2)_{m_2}[X])$ (resp., $\mathrm{ht}(Q((A_1)_{m_1} \otimes_k (A_2)_{m_2})) = 1 + \mathrm{ht}(P((A_1)_{m_1} \otimes_k (A_2)_{m_2})))$. Therefore, $\mathrm{ht}Q[X]/P[X] = 1$ (resp., $\mathrm{ht}(Q) = 1 + \mathrm{ht}(P)$). Then (1) holds, completing the proof. □

It will follow from Theorem 4.9 (proved below) that if K and L are field extensions of k with $\mathrm{t.d.}(K) < \infty$, then $K \otimes_k L$ is a strong S-ring and catenarian. Applying Proposition 4.3, it follows that if A and B are von Neumann regular k-algebras with $\mathrm{t.d.}(A) < \infty$, then $A \otimes_k B$ is a strong S-ring and catenarian.

Proposition 4.4. *Let A be a k-algebra and K an algebraic field extension of k. If $K \otimes_k A$ is a strong* S-*ring* (*resp*., *catenarian*), *then A is a strong* S-*ring* (*resp*., *catenarian*).

Proof. The strong S-property is straightforward from Proposition 2.3. Assume that $K \otimes_k A$ is catenarian. Then $K \otimes_k A$ satisfies MPC, and thus, by Proposition 3.1(b), A satisfies MPC. Let $p \subset q$ be a saturated chain of prime ideals of A. Since $K \otimes_k A$ is an integral extension of A, there exists a saturated chain of prime ideals $P \subset Q$ of $K \otimes_k A$ such that $P \cap A = p$ and $Q \cap A = q$. Hence $\mathrm{ht}(Q) = 1 + \mathrm{ht}(P)$. As $A \subset_{\rightarrow} K \otimes_k A$ satisfies also GD, we obtain $\mathrm{ht}(q) = \mathrm{ht}(Q) = 1 + \mathrm{ht}(P) = 1 + \mathrm{ht}(p)$. Since, by Proposition 4.1, A is LFD, we conclude that A is catenarian. $\square$

Note that Proposition 4.4 fails, in general, when the extension field K is no longer algebraic over k, as it is shown by Examples 5.2 and 5.3.

Next, we investigate sufficient conditions, on a k-algebra A and a field extension K of k, for $K \otimes_k A$ to inherit the (stably) strong S-property and (universal) catenarity.

Proposition 4.5. *Let A be a k-algebra and K a purely inseparable field extension of k. Then $K \otimes_k A$ is a strong* S-*ring* (*resp.*, *stably strong* S-*ring*, *catenarian*, *universally catenarian*) *if and only if so is A.*

Proof. $k \hookrightarrow K$ is radiciel, hence a universal homeomorphism. In particular, both $A \hookrightarrow K \otimes_k A$ and (for each $n \geqslant 1$) $A[X_1,\dots,X_n] \hookrightarrow K \otimes_k A[X_1,\dots,X_n] \cong (K \otimes_k A)[X_1,\dots,X_n]$ induce order-isomorphisms on Specs. Moreover, by Proposition 3.6, $K \otimes_k A$ satisfies MPC if and only if A satisfies MPC. Hence, the "catenarian" and "universally catenarian" assertions now follow immediately. Also, by applying Spec to the commutative diagram

$$\begin{array}{ccc} A & \hookrightarrow & K \otimes_k A \\ \downarrow & & \downarrow \\ A[X] & \hookrightarrow & K \otimes_k A[X] \end{array}$$

we obtain the "strong S-ring" assertion and, hence, the "stably strong S-ring" assertion. $\square$

Proposition 4.6. *Let A be a domain that is a k-algebra and K an algebraic field extension of k. Assume that A contains a separable algebraic closure of k. Then $K \otimes_k A$ is a strong* S-*ring* (*resp.*, *stably strong* S-*ring*, *catenarian*, *universally catenarian*) *if and only if so is A.*

Proof. Proposition 4.4 handles the "only if" assertion. Conversely, let $\bar{k}$ be the separable algebraic closure of k contained in A. First, we claim that the contractions of any adjacent prime ideals of $K \otimes_k A[X_1,\dots,X_n]$ are adjacent in $A[X_1,\dots,X_n]$. Indeed, let n be a positive integer and $P \subset Q$ be a pair of adjacent prime ideals of $K \otimes_k A[X_1,\dots,X_n]$. Put $P' := P \cap A[X_1,\dots,X_n]$ and $Q' := Q \cap A[X_1,\dots,X_n]$. Since also $\bar{k} \subseteq A[X_1,\dots,X_n]/P'$, then $K \otimes_k (A[X_1,\dots,X_n]/P')$ satisfies MPC (see the remark following Theorem 3.4). Furthermore, since K is algebraic over k, P is the unique prime ideal of $K \otimes_k A[X_1,\dots,X_n]$ contained in Q and contracting to P' by [26, Proposition 2.3] and [23, Theorem 3.1]. Hence, $1 = \mathrm{ht}(Q/P) = \mathrm{ht}(Q/(K \otimes_k P')) = \mathrm{ht}(Q'/P')$, proving the claim. Now the "strong S-ring" and "stably strong S-ring" assertions follow easily. Moreover, since $K \otimes_k A[X_1,\dots,X_n]$ is an integral extension of $A[X_1,\dots,X_n]$ that satisfies GD, for any integer n, we have for any prime ideals $P \subseteq Q$ of $K \otimes_k A[X_1,\dots,X_n]$, $\mathrm{ht}(P)=\mathrm{ht}(P')$ and $\mathrm{ht}(Q)=\mathrm{ht}(Q')$, where $P' := P \cap A[X_1,\dots,X_n]$ and $Q' := Q \cap A[X_1,\dots,X_n]$. Then, in view of the above claim, the "catenarian" and "universally catenarian" statements follow, completing the proof. □

Theorem 4.7. *Let A be a domain that is a k-algebra and K an algebraic field extension of k. Assume that the integral closure A' of A is a Prüfer domain. Then $K \otimes_k A$ is a stably strong* S-*ring.*

Proof. We claim that $K \otimes_k A'$ is a stably strong S-ring. In fact, let P_0 be a minimal prime ideal of $K \otimes_k A'$. Then $P_0 \cap A' = (0)$, and thus $(K \otimes_k A')/P_0$ is an integral extension of A'. Since A' is a Prüfer domain, $(K \otimes_k A')/P_0$ is a stably strong S-domain by [19, Proposition 4.18]. It follows that $K \otimes_k A'$ is a stably strong S-ring, as desired. Proposition 2.3 completes the proof. □

Theorem 4.8. *Let A be an LFD Prüfer domain that is a k-algebra and K an algebraic field extension of k. Then $K \otimes_k A$ is catenarian.*

Proof. First, we have that $K \otimes_k (A/p)$ satisfies MPC, by Theorem 3.4, since A/p is integrally closed for any $p \in \mathrm{Spec}(A)$. An argument similar to the treatment of the claim in the proof of Proposition 4.6 allows us to see that the contractions of any adjacent prime ideals of $K \otimes_k A$ are adjacent in $\mathrm{Spec}(A)$. Then, since $K \otimes_k A$ is an integral extension of A that satisfies GD, the result follows, since the contraction map from $\mathrm{Spec}(K \otimes_k A)$ to $\mathrm{Spec}(A)$ preserves height. □

Theorem 4.9. *Let A be a Noetherian domain that is a k-algebra and K a field extension of k such that* $\mathrm{t.d.}(K:k) < \infty$. *Then $K \otimes_k A$ is a stably strong* S-*ring. If, in addition, $K \otimes_k A$ satisfies MPC and $A[X]$ is catenarian, then $K \otimes_k A$ is universally catenarian.*

Proof. Recall first that a Noetherian ring A is universally catenarian if and only if $A[X]$ is catenarian [22]. We have $K \otimes_k A \cong K \otimes_{k(X_1,\dots,X_t)} S^{-1}A[X_1,\dots,X_t]$, where $t := \mathrm{t.d.}(K:k)$ and $S := k[X_1,\dots,X_t] \setminus \{0\}$. Since $S^{-1}A[X_1,\dots,X_t]$ is Noetherian, it suffices to handle the case where K is algebraic over k. Thus, in that case, $K \otimes_k A$ is an integral extension of a Noetherian domain A. Let P_0 be a minimal prime ideal of $K \otimes_k A$. By GD, $P_0 \cap A = (0)$. It follows that $(K \otimes_k A)/P_0$ is an integral extension of A. Hence, by [19, Proposition 4.20], $(K \otimes_k A)/P_0$ is a stably strong S-domain, whence $K \otimes_k A$ is a stably strong S-ring. Now, assume that $K \otimes_k A$ satisfies MPC and $A[X]$ is catenarian. Let P_0 be a minimal prime ideal of $K \otimes_k A$. As above $(K \otimes_k A)/P_0$ is an integral extension of A. By [22, Theorem 3.8], $(K \otimes_k A)/P_0$ is a universally catenarian domain. It follows that $K \otimes_k A$ is a universally catenarian ring. The proof is complete. □

Corollary 4.10. *Let K and L be field extensions of k such that* $\mathrm{t.d.}(K:k) < \infty$. *Then $K \otimes_k L$ is universally catenarian.*

Proof. $K \otimes_k L$ is LFD by Lemma 4.2, and satisfies MPC by [25, Corollary 4]. Theorem 4.9 completes the proof. □

Corollary 4.11. *Let A be a one-dimensional k-algebra and K an algebraic field extension of k. Then the following conditions are equivalent*:

(i) *$K \otimes_k A$ is a stably strong* S-*ring*;
(ii) *$K \otimes_k A$ is a strong* S-*ring*;
(iii) *A is a strong* S-*ring*;
(iv) *A satisfies* (P_2).

If, in addition, $K \otimes_k A$ satisfies MPC, then the following conditions are equivalent and the assertions (i)–(iv) *are equivalent to each of* (v)–(viii):

(v) *$K \otimes_k A$ is universally catenarian*;
(vi) *$(K \otimes_k A)[X]$ is catenarian*;
(vii) *$K \otimes_k A$ is an* S-*ring*;
(viii) *A is an* S-*ring*.

Proof. By Proposition 2.3 and Proposition 2.5, we have (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv).

(iv) $\Rightarrow$ (ii) Assume that (iv) holds. Let P be a minimal prime ideal of $K \otimes_k A$ and $p := P \cap A$. If $K \otimes_k A/P$ is a field, then it is an S-domain. If $\dim(K \otimes_k A/P) = 1$, then $\dim(A/p) = 1$ (since $K \otimes_k A/P$ is an integral extension of A/p); therefore A/p is an S-domain by Proposition 2.5, whence $K \otimes_k A/P$ is an S-domain by [19, Theorem 4.2]. We conclude that $K \otimes_k A$ is a strong S-ring. Thus, the statements (i)–(iv) are equivalent. On the other hand, the assertions (v)–(vii) are equivalent, by Proposition 2.5. Also, (vii) $\Leftrightarrow$ (viii), by Lemma 3.8. Apply Proposition 2.5 to complete the proof. □

Proposition 4.12. *Let A be a two-dimensional k-algebra and K an algebraic field extension of k such that $K \otimes_k A$ satisfies MPC. Then $K \otimes_k A$ is a strong* S-*ring* (*resp.*, *catenarian*) *if and only if so is A.*

Proof. The "only if" assertion follows from Proposition 2.3. Conversely, we first show that the contractions of any pair of adjacent prime ideals of $K \otimes_k A$ are adjacent in $\mathrm{Spec}(A)$. In fact, let $P \subset Q$ be a pair of adjacent prime ideals in $K \otimes_k A$, $p := P \cap A$ and $q := Q \cap A$. If $\mathrm{ht}(P) = 1$, then $\mathrm{ht}(p) = 1$ and hence $\mathrm{ht}(q/p) = 1$, since $\dim(A) = 2$. In the remaining case, P is a minimal prime ideal of $K \otimes_k A$. Since $K \otimes_k A$ satisfies MPC, P is the unique minimal prime ideal contained in Q. Then $\mathrm{ht}(Q) = \mathrm{ht}(Q/P) = 1$. It follows that $\mathrm{ht}(q/p) \leqslant \mathrm{ht}(q) = \mathrm{ht}(Q) = 1$, since $K \otimes_k A$ is an integral extension of A that satisfies GD. Then $\mathrm{ht}(q/p) = 1$. Hence, the "strong S-ring" assertion follows immediately. As the contraction map from $\mathrm{Spec}(K \otimes_k A)$ to $\mathrm{Spec}(A)$ preserves height, the "catenarian" assertion also holds. □

Next, we state the main theorem of this section. It generates new families of stably strong S-rings and universally catenarian rings.

Theorem 4.13. *Let A be an LFD k-algebra and K a field extension of k such that either* t.d.$(A : k) < \infty$ *or* t.d.$(K : k) < \infty$. *Let B be a transcendence basis of K over k, and let L be the separable algebraic closure of $k(B)$ in K. Assume that $[L : k(B)] < \infty$. If A is a stably strong* S-*ring* (*resp.*, *universally catenarian and $K \otimes_k A$ satisfies MPC*), *then $K \otimes_k A$ is a stably strong* S-*ring* (*resp.*, *universally catenarian*).

The proof of this theorem requires the following preparatory result.

Proposition 4.14. *Let A be an LFD k-algebra and K a purely transcendental field extension of k such that either* t.d.$(A : k) < \infty$ *or* t.d.$(K : k) < \infty$. *If A is a stably strong* S-*ring* (*resp.*, *universally catenarian*), *then $K \otimes_k A$ is a stably strong* S-*ring*, (*resp.*, *universally catenarian*).

Proof. First note that the stably strong S-property and universal catenarity are stable under formation of rings of fractions. Let $K = k(B)$, where B is a transcendence basis of K over k. If B is a finite set $\{X_1, \ldots, X_n\}$, then $K \otimes_k A \cong S^{-1}A[X_1, \ldots, X_n]$, where $S := k[X_1, \ldots, X_n] \setminus \{0\}$. Clearly, $K \otimes_k A$ is a stably strong S-ring (resp., universally catenarian), if A is. Hence, without loss of generality, B is an infinite set and t.d.$(A) < \infty$. Let

$$T = K \otimes_k A = \varinjlim_{E \text{finite},\ E \subseteq B} T_E,$$

where $T_E := k_E \otimes_k A \subseteq T$ and $k_E := k(E)$. Let us point out that, for any finite subset E of B and any prime ideal P_E of T_E, $P_E T$ is a prime ideal of T. Indeed, let E be a finite subset of B and P_E a prime ideal of T_E. Then $T \cong K \otimes_{k_E} T_E$ and $P_E T = P_E(K \otimes_{k_E} T_E) = K \otimes_{k_E} P_E$. Thus $T/P_E T \cong (K \otimes_{k_E} T_E)/(K \otimes_{k_E} P_E) \cong K \otimes_{k_E} (T_E/P_E)$. Note that if F is a field, $L = F(X_1, \ldots, X_n)$ and D is a domain containing F, then $L \otimes_F D (\cong D[X_1, \ldots, X_n]_{F[X_1,\ldots,X_n] \setminus \{0\}})$ is a domain. It follows that $T/P_E T$ is an integral domain, as desired, since it is a directed union of the domains $k_F \otimes_{k_E} (T_E/P_E)$, where F is a finite subset of B containing E.

Let $P \in \mathrm{Spec}(T)$ and $P_E := P \cap T_E$, for each finite subset E of B. We claim that there exists a finite subset E of B such that $P = P_E T$. Suppose by way of contradiction that for each finite subset E of B we have $P_E T \subset P$. Let F be a finite subset of B. Assume that $P_E T = P_F T$ for each finite subset E of B that contains F. Let $x \in P$. Since $x \in T = \varinjlim T_E$, there exists a finite subset E_1 of B such that $x \in T_{E_1}$. Then $x \in P_{E_1} T$. Thus $x \in P_{E_1 \cup F} T = P_F T$. It follows that $P = P_F T$, a contradiction. Consequently, there exists a finite subset E of B such that $F \subset E$ and $P_F T \subset P_E T$. Hence, by iterating the above argument, we can construct an infinite chain of prime ideals $P_{E_1} T \subset P_{E_2} T \subset \cdots \subset P_{E_n} T \subset \cdots \subset P$, where the E_j are finite subsets of B. This is a contradiction, since, by Proposition 4.1, T is LFD. Therefore there exists a finite subset E of B such that $P = P_E T$, proving the claim.

Let $P \subset Q$ be a chain of prime ideals of T. Then there exists a common finite subset E of B such that $P = P_E T$ and $Q = Q_E T$. We claim (*): $P \subset Q$ is saturated in $\mathrm{Spec}(T)$ if and only if $P_E \subset Q_E$ is saturated for each finite subset E of B such that

$P = P_ET$ and $Q = Q_ET$. Indeed, assume that $P \subset Q$ is saturated and consider a finite subset E of B such that $P = P_ET$ and $Q = Q_ET$. Let J be a prime ideal of T_E such that $P_E \subseteq J \subseteq Q_E$. Then $P_ET = P \subseteq JT \subseteq Q_ET = Q$. Since $\mathrm{ht}(Q/P) = 1$ and JT is a prime ideal of T, we obtain that either $JT = P = P_ET$ or $JT = Q = Q_ET$. Since $T_E \hookrightarrow T$ is a faithfully flat homomorphism, we conclude that either $J = P_E$ or $J = Q_E$ (see condition (i) in [2, Exercise 16, p. 45]). Then $P_E \subset Q_E$ is saturated. Conversely, suppose that $P_E \subset Q_E$ is saturated for each finite subset E of B such that $P = P_ET$ and $Q = Q_ET$. Let P' be a prime ideal of T such that $P \subseteq P' \subseteq Q$. There exists a finite subset F of B satisfying $P = P_FT$, $P' = P'_FT$ and $Q = Q_FT$. Then $P_F \subseteq P'_F \subseteq Q_F$. By hypothesis, $P_F \subset Q_F$ is saturated, so either $P'_F = P_F$ or $P'_F = Q_F$. Hence, either $P' = P$ or $P' = Q$. Then $P \subset Q$ is saturated. This establishes the claim.

Now assume that A is a stably strong S-ring and let $P \subset Q$ be a saturated chain in $\mathrm{Spec}(T)$. Then $P_E \subset Q_E$ is saturated for each finite subset E of B such that $P = P_ET$ and $Q = Q_ET$. Hence $P_E[X] \subset Q_E[X]$ is saturated, for each finite subset E of B such that $P = P_ET$ and $Q = Q_ET$. We have

$$T[X] = (K \otimes_k A)[X] = K \otimes_k (A[X]) \cong \lim_{\substack{\rightarrow \\ E\text{finite},\ E\subseteq B}} (T_E[X]).$$

In view of the equivalence (*), replacing T by $T[X]$, P by $P[X]$ and Q by $Q[X]$, we conclude that $P[X] \subset Q[X]$ is saturated. Therefore, T is a strong S-ring. Let $n \geqslant 1$ be an integer. Since $T[X_1,\dots,X_n] \cong K \otimes_k (A[X_1,\dots,X_n])$ and $A[X_1,\dots,X_n]$ is a stably strong S-ring, by repeating the earlier argument with A replaced by $A[X_1,\dots,X_n]$, we can show that $K \otimes_k (A[X_1,\dots,X_n]) \cong T[X_1,\dots,X_n]$ is a strong S-ring. Hence T is a stably strong S-ring.

Now, suppose that A is universally catenarian. We first recall (use $E := \emptyset$ in an earlier part of the proof) that $(K \otimes_k A)/(K \otimes_k p) \cong K \otimes_k (A/p)$ is a domain, for any prime ideal p of A. Furthermore, as $A \subset T$ satisfies GD, one can easily check that $\mathrm{Min}(T) = \{K \otimes_k p\colon\ p \in \mathrm{Min}(A)\}$. It follows that $K \otimes_k A$ satisfies MPC, since A satisfies MPC by hypothesis. Moreover, T is LFD by Proposition 4.1. Let $P \subset Q$ be a saturated chain of prime ideals of T. Then $P_E \subset Q_E$ is saturated for each finite subset E of B such that $P = P_ET$ and $Q = Q_ET$. Take a finite subset $E = \{X_1,\dots,X_n\}$ of B and set $S_E = k[X_1,\dots,X_n] \setminus \{0\}$. Then $T_E \cong S_E^{-1}A[X_1,\dots,X_n]$ is (universally) catenarian, by the hypothesis on A. Hence, $\mathrm{ht}(Q_E) = 1 + \mathrm{ht}(P_E)$ for each finite subset E of B such that $P = P_ET$ and $Q = Q_ET$. On the other hand, we claim that $\mathrm{ht}(P) = \sup\{\mathrm{ht}(P_E)$: E is a finite subset of B such that $P = P_ET\}$ and $\mathrm{ht}(Q) = \sup\{\mathrm{ht}(Q_E)$: E is a finite subset of B such that $Q = Q_ET\}$.

Indeed, let E be a finite subset of B such that $P = P_ET$. Since the homomorphism $T_E \hookrightarrow T$ satisfies GD, we have $\mathrm{ht}(P_E) \leqslant \mathrm{ht}(P)$. Hence $\sup\{\mathrm{ht}(P_E)$: E is a finite subset of B such that $P = P_ET\} \leqslant \mathrm{ht}(P)$. Since T is LFD, $\mathrm{ht}(P)$ is finite. Let $P_0 \subset P_1 \subset \cdots \subset P_h = P$ be a chain of prime ideals of T such that $h = \mathrm{ht}(P)$. There exists a common finite subset E of B such that $P_i = P_{iE}T$, for $i = 0,\dots,h$. Then $P_{0E} \subset P_{1E} \subset \cdots \subset P_{hE}$ is a chain of distinct prime ideals in T_E, since the homomorphism $T_E \to T$ is faithfully flat. Hence $h = \mathrm{ht}(P) \leqslant \mathrm{ht}(P_{hE}) = \mathrm{ht}(P_E)$. It follows that $\mathrm{ht}(P) \leqslant \sup\{\mathrm{ht}(P_E)$: E is a finite subset of B such that $P = P_ET\}$. This establishes the above claim. We conclude that $\mathrm{ht}(Q) = 1 + \mathrm{ht}(P)$. Hence T is catenarian. Since $T[X_1,\dots,X_n] \cong K \otimes_k (A[X_1,\dots,X_n])$, an argument similar to the above, with A replaced by $A[X_1,\dots,X_n]$, shows that T is universally catenarian and the proof is complete. $\square$

Proof of Theorem 4.13. We have $K\otimes_k A \cong K\otimes_{k(B)}(k(B)\otimes_k A) \cong K\otimes_L(L\otimes_{k(B)}(k(B)\otimes_k A))$. Since $[L : k(B)] < \infty$, we have $K = k(B)(x_1,\dots,x_n)$ for some $x_1,\dots,x_n \in L$. So $L \cong k(B)[X_1,\dots,X_n]/I$, for some prime ideal I of $k(B)[X_1,\dots,X_n]$. It follows that $L\otimes_{k(B)}(k(B)\otimes_k A) \cong (k(B)\otimes_k A)[X_1,\dots,X_n]/J$, where $J = I\otimes_{k(B)}(k(B)\otimes_k A)$. By Proposition 4.14, $k(B)\otimes_k A$ is a stably strong S-ring (resp., universally catenarian) if A is. Thus, if A is a stably strong S-ring (resp., universally catenarian), $L\otimes_{k(B)}(k(B)\otimes_k A)$ is so (we have just used the easy fact that the class of stably strong S-rings is closed under formation of factor rings). Then, by Proposition 4.5, the result follows, since K is a purely inseparable extension of L. $\square$

5. Examples

This section displays some examples showing that several results of Section 4 concerning the strong S-property and catenarity of $K\otimes_k A$ fail, in general, when the field extension K is no longer algebraic over k. Our last example, Example 5.5, shows clearly that the study of the spectrum of $A\otimes_k B$ becomes more intricate if one moves beyond the context where at least one of A,B is a field extension of k.

In order to provide some background for the present section, we recall the following definitions and results from [26]. A domain A is called an AF-domain if A is a k-algebra of finite transcendence degree over k such that $\mathrm{ht}(p) + \mathrm{t.d.}(A/p : k) = \mathrm{t.d.}(A : k)$ for each $p \in \mathrm{Spec}(A)$. Finitely generated k-algebras (that are domains) and field extensions of finite transcendence degree over k are AF-domains. Let A be a k-algebra, p a prime ideal of A and $0 \leqslant d \leqslant s$ be integers. Set

$$\triangle(s,d,p) := \mathrm{ht}(p[X_1,\dots,X_s]) + \min\left(s, d + \mathrm{t.d.}\left(\frac{A}{p} : k\right)\right),$$

$$D(s,d,A) := \max\{\triangle(s,d,p):\ p \in \mathrm{Spec}(A)\}.$$

Wadsworth's main two results relative to the Krull dimension of tensor products of AF-domains read as follows. If A is an AF-domain and R is any k-algebra, then $\dim(A\otimes_k R) = D(\mathrm{t.d.}(A : k), \dim(A), R)$ [26, Theorem 3.7]. If, in addition, R is an AF-domain, then $\dim(A\otimes_k R) = \min(\dim(A) + \mathrm{t.d.}(R : k), \mathrm{t.d.}(A : k) + \dim(R))$ [26, Theorem 3.8].

We turn now to our examples. It is still an open problem to know whether $K\otimes_k A$ is a strong S-ring (resp., catenarian) when K is an algebraic field extension of k and A is a strong S-ring (resp., catenarian such that $K\otimes_k A$ satisfies MPC). However, for the case where K is a transcendental field extension of k, the answer is negative, as illustrated by the following two examples.

Example 5.1. Let k be a field. There exists a strong S-domain A that is a k-algebra such that $L\otimes_k A$ is a strong S-ring for any algebraic field extension L of k, while $K\otimes_k A$ is not a strong S-ring for some transcendental field extension K of k.

Our example draws on [8, Example 3], which we assume that the reader has at hand. Let k be a field and k' an algebraic closure of k. Let (V_1, M_1') be the valuation domain of the Y_3-adic valuation on $k'(Y_1, Y_2)[Y_3]$. Let V^* be a discrete rank-one valuation domain of $k'(Y_1, Y_2)$ of the form $k' + N$ and let V be the pullback $\varphi^{-1}(V^*)$, where φ: $V_1 \to k'(Y_1, Y_2)$ is the canonical homomorphism. It is easily seen that V is a rank-two valuation domain of the form $k' + M_1$. Moreover, if p_1 is the height 1 prime ideal of V, $V_{p_1} = V_1$. Finally, let W be the valuation domain of the $(Y_3 + 1)$-adic valuation on $k'(Y_1, Y_2)[Y_3]$. Then W is a DVR of the form $k'(Y_1, Y_2) + M_2$. Set $A = k' + M$, where $M = M_1 \cap M_2$. It is shown in [8, Example 3] that A is a two-dimensional local strong S-domain with the following features: $\dim A[X, Y] = 5$ (hence $A[X]$ is not a strong S-domain), the quotient field of A is $k'(Y_1, Y_2, Y_3)$, and the prime ideals of A are $(0) \subset p \subset M$ with $A_p = V_1$. By Proposition 4.6, $L \otimes_k A$ is a strong S-ring, for any algebraic field extension L of k. On the other hand, by [26, Theorem 3.7], $\dim((k(X) \otimes_k A)[Y]) = \dim(k(X)[Y] \otimes_k A) = D(2, 1, A)$, since $k(X)[Y]$ is an AF-domain. We have

$$\begin{aligned}
\triangle(2,1,(0)) &= \min(2, 1 + \text{t.d.}(A : k)) \\
&= \min(2,4) = 2. \\
\triangle(2,1,p) &= \text{ht}(p[X,Y]) + \min\left(2, 1 + \text{t.d.}\left(\frac{A}{p} : k\right)\right) \\
&= \text{ht}(pA_p[X,Y]) + \min\left(2, 1 + \text{t.d.}\left(\frac{A_p}{pA_p} : k\right)\right) \\
&= \text{ht}(pA_p) + \min\left(2, 1 + \text{t.d.}\left(\frac{V_1}{M_1'} : k\right)\right) \quad (\text{since } A_p \text{ is a DVR}) \\
&= 1 + \min(2,3) = 3. \\
\triangle(2,1,M) &= \text{ht}(M[X,Y]) + \min\left(2, 1 + \text{t.d.}\left(\frac{A}{M} : k\right)\right) \\
&= \dim A[X,Y] - 2 + \min(2,1) = 4.
\end{aligned}$$

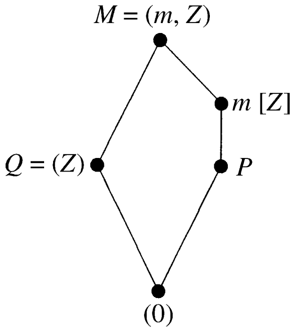
Hence $\dim(k(X) \otimes_k A)[Y]) = 4$. Furthermore, $\dim(k(X) \otimes_k A) = D(1, 0, A)$. We have

$$\begin{aligned}
\triangle(1,0,(0)) &= \min(1, \text{t.d.}(A : k)) \\
&= \min(1,3) = 1. \\
\triangle(1,0,p) &= \text{ht}(p[X]) + \min\left(1, \text{t.d.}\left(\frac{A}{p} : k\right)\right) \\
&= \text{ht}(pA_p[X]) + \min(1,2) \\
&= \text{ht}(pA_p) + 1 = 2 \quad (\text{since } A_p \text{ is a DVR}). \\
\triangle(1,0,M) &= \text{ht}(M[X]) + \min\left(1, \text{t.d.}\left(\frac{A}{M} : k\right)\right) \\
&= \text{ht}(M) + \min(1,0) = 2 \quad (\text{since } A \text{ is a strong S-domain}).
\end{aligned}$$

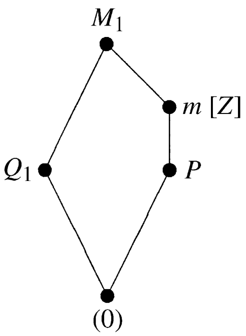
Hence, $\dim(k(X)) \otimes_k A) = 2$. Consequently, $\dim((k(X) \otimes_k A)[Y]) = 4 \neq 1 + 2 = 1 + \dim(k(X) \otimes_k A)$. Let $K = k(X)$. Therefore, by [18, Theorem 39], $K \otimes_k A$ is not a strong S-ring. □

Example 5.2. Let k be a field. There exists a catenarian domain A that is a k-algebra such that $L \otimes_k A$ is a catenarian for any algebraic field extension L of k, while $K \otimes_k A$ is not catenarian for some transcendental field extension K of k.

Let k be a field and k' an algebraic closure of k. Let $V := k'(X_1, X_2)[Y]_{(Y)} = k'(X_1, X_2) + m$, where $m := YV$. Let $A := k'(X_1) + m$. Clearly, A is catenarian while $A[Z]$ is not catenarian, as the following chains of prime ideals of $A[Z]$ are saturated:



where P is an upper to (0) (cf. [8, Example 5]). By Proposition 4.6, $L \otimes_k A$ is catenarian for any algebraic field extension L of k. On the other hand, $S^{-1}A[Z]/S^{-1}m[Z] \cong (k(Z) \otimes_k A)/(k(Z) \otimes_k m) \cong k(Z) \otimes_k \frac{A}{m} \cong k(Z) \otimes_k k'(X_1)$; let $S = k[Z] \setminus \{0\}$. Therefore $\dim(S^{-1}A[Z]/S^{-1}m[Z]) = 1$ by [23, Theorem 3.1]. Hence $S^{-1}m[Z]$ is not a maximal ideal of $S^{-1}A[Z]$, whence there exists an upper M_1 to m such that $M_1 \cap S = \emptyset$. By [9, Theorem B, p. 167], $l(M) = l(M_1)$, where $l(M)$ (resp., $l(M_1)$) denotes the set of lengths of saturated chains of prime ideals between (0) and M (resp., M_1). Then there exist two saturated chains of prime ideals in $A[Z]$ of the form:



where Q_1 in an upper to (0). Consequently, $K \otimes_k A \cong S^{-1}A[Z]$ is not catenarian, where $K := k(Z)$.

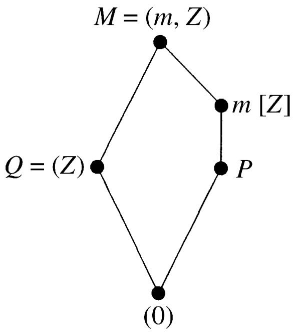
The next two examples show that Proposition 4.4 fails in general when K is no longer algebraic over k.

Example 5.3. There exists a k-algebra A which is not an S-domain and a field extension K of k such that $1 \leqslant \text{t.d.}(K : k) < \infty$ and $K \otimes_k A$ is a strong S-ring.

Let $V := k(X)[Y]_{(Y)} = k(X) + m$, where $m := YV$, and let $A := k + m$. We have $\text{ht}(m) = 1$ and $\text{ht}(m[Z]) = \text{ht}(m[Z,T]) = 2$ [8, Example 5]. Thus, A is not an S-domain. Let $K := k(Z)$. We claim that $K \otimes_k A \cong S^{-1}A[Z]$ is a strong S-domain, where $S := k[Z] \setminus \{0\}$. Notice first that $S^{-1}m[Z]$ is a maximal ideal of $S^{-1}A[Z]$, as $S^{-1}A[Z]/S^{-1}m[Z] \cong (k(Z) \otimes_k A)/(k(Z) \otimes_k m) \cong k(Z) \otimes_k (A/m) \cong k(Z)$. Now, let $P \subset Q$ be a pair of adjacent prime ideals of $A[Z]$ that are disjoint from S. Two cases are possible. If $P = (0)$, then $\text{ht}(Q) = 1$. Since $k(Z) \otimes_k A \cong S^{-1}A[Z]$ is an S-domain, $\text{ht}(Q[T]) = 1$. If P is an upper to (0), Q necessarily contracts to m in A and hence $Q = m[Z]$, since $Q \cap S = \emptyset$ and $S^{-1}m[Z] \in \text{Max}(S^{-1}A[Z])$. Therefore $(0) \subset P \subset m[Z] = Q$ is a saturated chain in $\text{Spec}(A[Z])$. Then $(0) \subset P[T] \subset m[Z,T] = Q[T]$ is a saturated chain in $\text{Spec}(A[Z,T])$. Consequently, in both cases, $P[T] \subset Q[T]$ is saturated. It follows that $K \otimes_k A \cong S^{-1}A[Z]$ is a strong S-domain, as desired. $\square$

Example 5.4. There exists a k-algebra A which is not a catenarian domain and a field extension K of k such that $1 \leqslant \text{t.d.}(K : k) < \infty$ and $K \otimes_k A$ is catenarian.

Let $V := k(X)[Y]_{(Y)} = k(X) + m$, where $m := YV$. Let $R := k + m$. Clearly, R is a one-dimensional integrally closed domain. There exist two saturated chains of prime ideals of $R[Z]$, as in Example 5.2, of the form:



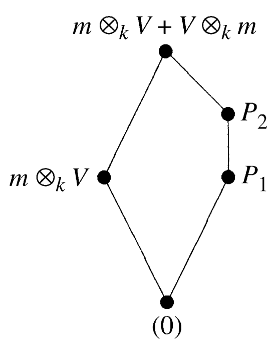
Let $A := R[Z]$. Then A is not catenarian. We next prove that $K \otimes_k A \cong S^{-1}R[Z,T]$ is catenarian, where $K := k(T)$ and $S := k[T] \setminus \{0\}$.

Notice first that $\text{ht}(m[Z,T]) = 2$ [8, Example 5]. Further, one may easily check, via [26, Theorem 3.7], that $\dim(K \otimes_k A) = \dim(k(T)[Z] \otimes_k R) = D(2,1,R) = 3$, since $k(T)[Z]$ is an AF-domain. Now, let $P \subseteq Q$ be a pair of prime ideals of $R[Z,T]$ such that $Q \cap S = \emptyset$. We claim that $\text{ht}(Q) = \text{ht}(P) + \text{ht}(Q/P)$. Without loss of generality, we may assume that $\text{ht}(Q) = 3$. Necessarily, Q contracts to m in $R = k + m$. Moreover, Q cannot be an upper to an upper to m in $R[Z,T]$; otherwise $\text{ht}(Q) = 4$. Hence, either $Q = M_1[T]$ or $Q = M_2[Z]$, where M_1 is an upper to m in $R[Z]$ and M_2 is an upper to m in $R[T]$. Assume that $Q = M[T]$, where M is an upper to m in $R[Z]$. In case $P \cap R = m$, we are done, since here $P = m[Z, T]$. We may then assume that $P \cap R = (0)$. Three cases are possible. If P is an upper to an upper to (0) in $R[Z, T]$, then $\mathrm{ht}(P) = 2$, and we are done. If $P = P_1[Z]$, where P_1 is an upper to (0) in $R[T]$, then $P \cap R[T] = P_1 \subset Q \cap R[T] = (M \cap R)[T] = m[T]$. Hence $P = P_1[Z] \subset m[Z, T]$. Thus $\mathrm{ht}(Q/P) = 2$ and $\mathrm{ht}(P) = 1$, as desired. Assume now that $P = P_2[T]$, where P_2 is an upper to (0) in $R[Z]$. We have $Q = M[T]$ is an upper to $m[T]$ in $(R[T])[Z]$ and P is an upper to (0) in $(R[T])[Z]$. If $\mathrm{ht}(Q/P) = 1 < \mathrm{ht}((m[T])[Z]) = 2$, then by [9, Proposition 2.2] and [14, Proposition 1.1, p. 742], $P \subset m[Z, T]$ (since $R[T]$ is integrally closed), a contradiction. Thus, $\mathrm{ht}(Q/P) = 2$ and $\mathrm{ht}(P) = 1$, as desired. A similar argument applies to the case where $Q = M[Z]$, where M is an upper to m in $R[T]$. Consequently, $K \otimes_k A$ is catenarian. $\square$

To emphasize the importance of K being a field in Theorem 4.13, we close this section with an example of two discrete rank-one valuation domains, hence universally catenarian, the tensor product of which is not catenarian.

Example 5.5. There exists a discrete rank-one valuation domain V such that $\mathrm{t.d.}(V : k) < \infty$ and $V \otimes_k V$ is not catenarian.

Consider the k-algebra homomorphism φ: $k[X, Y] \to k[[t]]$ such that $\varphi(X) = t$ and $\varphi(Y) = s := \sum_{n \geqslant 1} t^{n!}$. Since s is known to be transcendental over $k(t)$, φ is injective. This induces an embedding $\bar{\varphi} : k(X, Y) \to k((t))$ of fields. Put $V = \bar{\varphi}^{-1}(k[[t]])$. It is easy to check that V is a discrete rank-one valuation overring of $k[X, Y]$ of the form $k + m$, where $m := XV$. For convenience, put $A = B := V$. We have $\dim(A \otimes_k B) = \dim(V \otimes_k V) = \dim(V) + \mathrm{t.d.}(V : k) = 1 + 2 = 3$ [26, Corollary 4.2] and $\mathrm{ht}(m \otimes_k V) = \mathrm{ht}(m[X, Y]) = \mathrm{ht}(m) = 1$ [4, Lemma 1.4]. Since $\mathrm{ht}((m \otimes_k V + V \otimes_k m)/(m \otimes_k V)) \leqslant \dim((V \otimes_k V)/(m \otimes_k V)) = \dim(V) = 1$, we obtain $\mathrm{ht}((m \otimes_k V + V \otimes_k m)/(m \otimes_k V)) = 1$. On the other hand, in view of [26, Proposition 2.3], the height of no prime ideal of $A \otimes_k B$ contracting to (0) in A and to (0) in B can reach $\dim(A \otimes_k B) = 3$, since $\dim(k(X, Y) \otimes_k k(X, Y)) = 2$. Therefore, $\mathrm{ht}(m \otimes_k V + V \otimes_k m) = 3$. Hence $\mathrm{Spec}(V \otimes_k V)$ contains the following two saturated chains:



where $P_i \cap A = P_i \cap B = (0)$, for $i = 1, 2$. Consequently, $V \otimes_k V$ is not catenarian.

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