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One-parameter orthogonality relations for basic hypergeometric series

Dedicated to Tom Koornwinder on the occasion of his 60th birthday

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ABSTRACT

The second order hypergeometric q-difference operator is studied for the value $c = -q$. For certain parameter regimes the corresponding recurrence relation can be related to a symmetric operator on the Hilbert space $\ell^2(\mathbb{Z})$. The operator has deficiency indices $(1,1)$ and we describe as explicitly as possible the spectral resolutions of the self-adjoint extensions. This gives rise to one-parameter orthogonality relations for sums of two ${}_2\varphi_1$-series. In particular, we find that the Ismail-Zhang q-analogue of the exponential function satisfies certain orthogonality relations.

1. INTRODUCTION

As is well known, special functions arise in several contexts in mathematics. One of the areas is the theory of self-adjoint operators on a Hilbert space, see e.g. Titchmarsh [18]. On the one hand, given an explicit self-adjoint operator, we can try to use special functions in order to obtain the spectral decomposition of the self-adjoint operator. On the other hand, given a family of interesting special functions, we can try to find a self-adjoint operator which has these interesting special functions as eigenfunctions. If we can give the spectral decomposition of the corresponding operator we can use this information to find e.g. orthogonality relations or a corresponding integral transform for the special function we have started with.

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In this paper we are in the second situation. The interesting special function is

$$(1.1)\qquad \mathcal{E}_q(z;t)=\frac{(-t;q^{\frac{1}{2}})_\infty}{(qt^2;q^2)_\infty}\,{}_2\varphi_1\left(\begin{matrix}q^{\frac{1}{4}}y,q^{\frac{1}{4}}/y\\ -q^{\frac{1}{2}}\end{matrix};q^{\frac{1}{2}},-t\right)$$

originally introduced by Ismail and Zhang [7, (1.22) with $a=i$, $b=-it$] up to a normalisation factor, and the expression used here can be found in Ismail and Stanton [5, Corr. 4.3], [6, Corr. 2.5], see also Suslov [17] for more information. Ismail and Zhang [7, (1.25)], see also [5], point out that formally

$$(1.2)\qquad \lim_{q\uparrow 1}\ \mathcal{E}_q(z;\frac{1}{2}t)\Big|_{t=(1-q)\lambda}=e^{\lambda z}.$$

This function has been studied intensively recently because it is the appropriate q-analogue of the exponential function well suited for the Askey-Wilson difference operator.

In the point of view of this paper, we study the function $\mathcal{E}_q(z,t)$ as a function of t. The parameter z occurs as the spectral parameter. Because of the expression (1.1) $\mathcal{E}_{q^2}$ is an eigenfunction of the second order hypergeometric q-difference operator. It is convenient to switch to q^2 in (1.1). This operator and its eigenfunctions have been studied in connection with representation theory of non-compact quantum groups, in particular the quantum analogue of $SU(1,1)$, see [8], [11], and [12] for a more general scheme. The parameter regimes for the basic hypergeometric function in these papers does not include the case corresponding to $\mathcal{E}_q$, so we have to perform the spectral analysis again. The crucial property is that the lower parameter $c=-q$.

It turns out that for specific values of the remaining parameters the second order hypergeometric q-difference operator can be realised as an unbounded symmetric operator on the Hilbert space $\ell^2(\mathbb{Z})$ of square integrable sequences. In particular, this occurs for $\mathcal{E}_{q^2}$. However, it turns out that the corresponding operator is not essentially self-adjoint, but it has deficiency indices $(1,1)$. We describe the self-adjoint extensions, which depend on one extra parameter, and we study the corresponding spectral decompositions. There is always continuous spectrum on $[-1,1]$, and the point spectrum is an infinite set tending to plus and/or minus infinity which is described as the zero set of some explicit function. For the case of the function $\mathcal{E}_{q^2}$ we establish that this set consists of two q^2-quadratic grids. The corresponding transforms do not give orthogonality relations for $\mathcal{E}_{q^2}$ but for a linear combination of two $\mathcal{E}_{q^2}$'s similar to the relation $2\cos\lambda x=e^{i\lambda x}+e^{-i\lambda x}$. So we can think of the result as a q-analogue of the Fourier-cosine transform instead of the Fourier transform. We perform the spectral analysis in somewhat greater generality, and the main result is Theorem 5.8 and its counterpart Theorem 6.3 for another parameter regime.

We also present the link with the recurrence relation for the big q-Jacobi functions [13] or the associated dual q-Hahn polynomials [4], and this leads to a quadratic transformation in which a ${}_2\varphi_1$-series in base q is given as a ${}_3\varphi_2$-series in base q^2. In particular, this gives a new expression for $\mathcal{E}_q$ as a sum of two ${}_3\varphi_2$-series in base q.

The plan of the paper is as follows. In §2 we recall the second order hypergeometric q-difference equation, its solutions and their interrelations. In §3 we discuss for which parameter regimes the recurrence relation can be interpretated as a symmetric operator on $\ell^2(\mathbb{Z})$. We recall the general theory of doubly infinite Jacobi operators on $\ell^2(\mathbb{Z})$ in §4. In §§5, 6 we work out the spectral decompositions of the self-adjoint extensions as explicitly as possible. In §5 we give detailed arguments, and we indicate the (similar) arguments in the easier case of §6. Finally, in §7 we indicate the link with the big q-Jacobi functions, and we derive the quadratic relation.

Notation. In this paper we follow the notation for basic hypergeometric series of Gasper and Rahman [3]. Our standing assumption on q is $0 < q < 1$. The series

$$ {}_{r+1}\varphi_r\left(\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix}; q, z\right) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_{r+1}; q)_k}{(b_1, \ldots, b_r, q; q)_k} z^k, \tag{1.3} $$

where $(a;q)_k = \prod_{i=0}^{k-1}(1 - aq^i)$, $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, and $(a_1, \ldots, a_r; q)_k = \prod_{i=1}^{k} (a_i; q)_k$. Generically the radius of convergence of the series in (1.3) is 1, but the series has a unique analytic continuation to $\mathbb{C}\backslash[1, \infty)$. We also use $\theta(z) = (z, q/z; q)_\infty$ for the (renormalised) Jacobi theta-function, and $\theta(a_1, \ldots, a_r) = \theta(a_1) \cdots \theta(a_r)$. The identity

$$ \theta(aq^k) = (-a)^{-k} q^{-\frac{1}{2}k(k-1)}\, \theta(a) \tag{1.4} $$

is useful.

Dedication and acknowledgement. The paper is dedicated to Tom Koornwinder from whom I have learnt very much about special functions and representation theory. In particular, his unpublished notes on spectral theory for ${}_2\varphi_1$-series have been very influential for this paper. I thank Mourad Ismail for discussions about the q-analogue of the exponential function defined in (1.1) that has triggered the research for this paper, for his kind hospitality during a visit when these discussions took place, and for his comments on a previous version of this paper.

2. The difference equation and its solutions

In this section we consider the second order hypergeometric q-difference operator for which the ${}_2\varphi_1$-series in the definition (1.1) is an eigenfunction. We discuss other solutions and their connection coefficients.

The hypergeometric difference equation is

$$ (c - abz)f(qz) + \big(-(c+q) + (a+b)z\big)f(z) + (q - z)f(z/q) = 0, \tag{2.1} $$

see [3, Exerc. 1.13], having

$$f(z) = {}_2\varphi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right)$$

as a solution. We are particularly interested in the case $c = -q$, cf. (1.1).

Lemma 2.1. *The difference equation*

$$2z f_k(z) = \frac{1 + a^2 t q^{k-1}}{a t q^{k-1}} f_{k+1}(z) - \frac{1 - q^{k-1} t}{a t q^{k-1}} f_{k-1}(z), \qquad k \in \mathbb{Z}, \tag{2.2}$$

is solved by, where $z = \frac{1}{2}(y + y^{-1})$,

$$\begin{aligned} u_k(z) &= {}_2\varphi_1\left(\begin{matrix} ay, a/y \\ -q \end{matrix}; q, tq^k\right), \\ v_k(z) &= (-1)^k \, {}_2\varphi_1\left(\begin{matrix} -ay, -a/y \\ -q \end{matrix}; q, tq^k\right), \\ F_k(y) &= (ay)^{-k} \, {}_2\varphi_1\left(\begin{matrix} ay, -ay \\ qy^2 \end{matrix}; q, -\frac{q^{2-k}}{a^2 t}\right), \qquad y^2 \notin q^{-\mathbb{N}}. \end{aligned}$$

Here, and in the sequel, we always assume that $a \neq 0$, $t \neq 0$.

Proof. This is a straightforward verification using (2.1). □

In §7 we also give expressions for the solutions of (2.2) in terms of ${}_3\varphi_2$-series using a quadratic transformation.

Remark 2.2. Note that the difference equation (2.2) has two obvious symmetries. The first is $a \leftrightarrow -a$, $z \leftrightarrow -z$, leaving all solutions unchanged. The second symmetry is $a \leftrightarrow -a$, $f_k \leftrightarrow (-1)^k f_k$, which interchanges the solutions $u_k \leftrightarrow v_k$ and leaves F_k unchanged.

Remark 2.3. We are mainly interested in the case that the coefficients in (2.2) do not vanish for $k \in \mathbb{Z}$, i.e. we assume $t \notin q^{\mathbb{Z}}$, $-ta^2 \notin q^{\mathbb{Z}}$. In case one of the coefficients does vanish, we can assume without loss of generality that $t = q$ or $a^2 t = -q$. In this case the recurrence can be split into two recurrence relations labeled by $\mathbb{N}$. So we have polynomial solutions, and the polynomials can be given explicitly in terms of symmetric Al-Salam–Chihara polynomials in base q for negative k and in base q^{-1} for positive k.

Since Lemma 2.1 describes four solutions (note that $F_k(y^{-1})$ is also a solution) to (2.1), whose solution space is two-dimensional, we find relations between the solutions.

Lemma 2.4. *The solutions of Lemma 2.1 are related by,* $z = \frac{1}{2}(y + y^{-1})$,

$$
\begin{aligned}
u_k(z) &= c(y;a,t)\,F_k(y) + c(y^{-1};a,t)\,F_k(y^{-1}),\\
v_k(z) &= c(y;-a,t)\,F_k(y) + c(y^{-1};-a,t)\,F_k(y^{-1}),\\
c(y;a,t) &= \frac{(a/y,-q/ay,ayt,q/ayt;q)_\infty}{(-q,y^{-2},t,q/t;q)_\infty},
\end{aligned}
\tag{2.3}
$$

and

$$
\begin{aligned}
F_k(y) &= d(y;a,t)\,u_k(z) + d(y;-a,t)\,v_k(z),\\
d(y;a,t) &= \frac{(-ay,qy/a,-at/qy,-q^2y/at;q)_\infty}{(-1,qy^2,-a^2t/q,-q^2/a^2t;q)_\infty},
\end{aligned}
\tag{2.4}
$$

Proof. The second equation of (2.3) follows from the first using the symmetries as in Remark 2.2. The first equation of (2.3) follows from [3, (4.3.2)]. The expansion (2.4) can be proved similarly, or by inverting (2.3). In the last case the addition formula for Jacobi theta functions has to be used, see e.g. [3, Exerc. 2.16], to find

$$
\det\begin{pmatrix} c(y;a,t) & c(y^{-1};a,t)\\ c(y;-a,t) & c(y^{-1};-a,t)\end{pmatrix} = \frac{-aty\,\theta(-a^2t,t^{-1},y^{-2},-1)}{(-q,-q,y^2,y^{-2},t,q/t,t,q/t;q)_\infty}
$$
$$
= \frac{2a}{y^{-1}-y}\frac{\theta(-a^2t)}{\theta(t)}. \qquad \square
$$

3. SYMMETRIC FORM OF THE DIFFERENCE EQUATION

Since we want to find a symmetric operator on the Hilbert space $\ell^2(\mathbb{Z})$ for which the $\mathcal{E}_{q^2}(z;tq^k)$ occur as eigenfunctions we need to find conditions on a and t such that we can rewrite (2.2) in a symmetric form. This is done in this section.

Let $f_k(z)$ satisfy (2.2), then $g_k(z) = \alpha_k f_k(z)$, for non-zero constants α_k, satisfies

$$
2z\,g_k(z) = \frac{\alpha_k}{\alpha_{k+1}}\frac{1+a^2tq^{k-1}}{atq^{k-1}}\,g_{k+1}(z) - \frac{\alpha_k}{\alpha_{k-1}}\frac{1-q^{k-1}t}{atq^{k-1}}\,g_{k-1}(z).
\tag{3.1}
$$

We need to determine if we can rewrite the recurrence in the symmetric form

$$
2z\,g_k(z) = a_k\,g_{k+1}(z) + a_{k-1}\,g_{k-1}(z), \qquad a_k > 0.
\tag{3.2}
$$

From the coefficient of $g_{k+1}(z)$ in equations (3.1) and (3.2) we find the first equality

$$
\bar{a}_{k-1} = \frac{\bar{\alpha}_{k-2}}{\bar{\alpha}_k}\frac{1+\bar{a}^2\bar{t}q^{k-2}}{\bar{a}\bar{t}q^{k-1}} = -\frac{\alpha_k}{\alpha_{k-1}}\frac{1-q^{k-1}t}{atq^{k-1}},
$$

where the second equality follows from the coefficient for $g_{k-1}(z)$ in (3.1) and (3.2). Hence,

$$
\left|\frac{\alpha_{k-1}}{\alpha_k}\right|^2 = -\frac{1-q^{k-1}t}{1+\bar{a}^2\bar{t}q^{k-2}}\frac{\bar{a}\bar{t}}{qat}
\tag{3.3}
$$

and we can make the appropriate choice for α_k precisely when the right hand side of (3.3) is strictly positive. Note that the choice for α_k is determined by (3.3) and one initial value, say for α_0, up to a phase factor. We can choose the phase factor such that the value a_k in (3.2) is indeed positive.

Lemma 3.1. *In the following cases (3.1) can be written in the symmetric form (3.2) with* $a_k > 0$, $\forall k \in \mathbb{Z}$*:*

(1) $a = \sqrt{q}\, e^{i\psi}$, $t = ire^{-i\psi}$ *with* $r \in \mathbb{R}\backslash\{0\}$,

(2) $t < 0$, $a = is$ *with* $s \in \mathbb{R}\backslash\{0\}$,

(3) $t \in \mathbb{R}_{>0}$, $a = is$, $s \in \mathbb{R}\backslash\{0\}$, *such that there exists* $k_0 \in \mathbb{Z}$ *with* $tq^{k_0+1} < 1 < tq^{k_0}$ *and* $s^2tq^{k_0} < 1 < s^2tq^{k_0-1}$.

Note that there is overlap between cases (1) and (2), and (1) and (3). For the remainder of the paper we stick to the cases (1) and (2), where in case (1) we moreover assume that $t \notin \mathbb{R}_{>0}$ in order to have the ${}_2\varphi_1$-series in Lemma 2.1 well-defined as analytic functions on $\mathbb{C}\backslash[1,\infty)$.

We fix the corresponding values of the coefficients a_k and α_k as follows. In case (1) we take

$$\begin{aligned} a_k &= |r|^{-1}q^{-k}\sqrt{1-2rq^k\sin\psi + r^2q^{2k}} = \left|\frac{1+ire^{i\psi}q^k}{irq^k}\right|, \\ \alpha_k &= e^{i\phi_k}q^{\frac{1}{2}k}, \qquad \phi_{k+1}-\phi_k \equiv \arg(1+ire^{i\psi}q^k) - \frac{1}{2}\pi\,\mathrm{sgn}(r) \bmod 2\pi, \end{aligned} \tag{3.4}$$

and in case (2) we take

$$\begin{aligned} a_k &= \frac{q^{\frac{1}{2}-k}}{|st|}\sqrt{(1-tq^k)(1-ts^2q^{k-1})} = \sqrt{(1-q^{-k}/t)(1-q^{1-k}/ts^2)}, \\ \alpha_k &= i^k s^k \sqrt{\frac{(q^{2-k}/s^2t;q)_\infty}{(q^{1-k}/t;q)_\infty}} = (i\,\mathrm{sgn}(s))^k q^{\frac{1}{2}k}\sqrt{\frac{(tq^k;q)_\infty\,\theta(s^2t/q)}{(ts^2q^{k-1};q)_\infty\,\theta(t)}}, \end{aligned} \tag{3.5}$$

using the θ-product identity (1.4).

It follows from Lemma 2.4 that in all cases of Lemma 3.1 we have

$$\overline{c(\bar{y};a,t)} = \frac{\theta(t)}{\theta(\bar{t})}\, c(y;-a,t). \tag{3.6}$$

4. GENERALITIES ON DOUBLY INFINITE JACOBI OPERATORS

In this section we recall some of the general theory for the spectral analysis of doubly infinite Jacobi operators on the Hilbert space $\ell^2(\mathbb{Z})$ given by (4.1). In the cases considered in this paper we have to deal with one-dimensional deficiency spaces, and the self-adjoint extensions are described. The results of this section can be found in [14], [9], [1, Ch. 7], [15] and for more generalities Dunford and Schwartz [2] can be consulted.

We consider next the corresponding operator L on the Hilbert space $\ell^2(\mathbb{Z})$ equipped with an orthonormal basis $\{e_k\}_{k\in\mathbb{Z}}$ defined by

(4.1) $\qquad 2L\,e_k = a_k\,e_{k+1} + a_{k-1}\,e_{k-1}$

with $a_k > 0$ as in (3.4) and (3.5).

The operator is initially defined on the dense domain $\mathcal{D}$ of finite linear combinations of the basis vectors e_k. The operator $(L, \mathcal{D})$ is a symmetric operator, and, since $a_k \in \mathbb{R}$, L commutes with conjugation. So the deficiency indices are equal, and since the solution space of $L\xi = z\xi$ is two-dimensional the deficiency indices are $(0,0)$, $(1,1)$, or $(2,2)$. In the cases (3.4) and (3.5) it follows that a_k is bounded for $k \to -\infty$. By Theorem 2.1 of Masson and Repka [14], see also [9, (4.2.2)], we find that deficiency indices are $(0,0)$ or $(1,1)$. The adjoint operator is $(L^*, \mathcal{D}^*)$ given by

$$L^*\Big(\sum_{k=-\infty}^{\infty} \xi_k\,e_k\Big) = \sum_{k=-\infty}^{\infty} (a_k\xi_{k+1} + a_{k-1}\xi_{k-1})\,e_k,$$
$$\mathcal{D}^* = \big\{\xi \in \ell^2(\mathbb{Z}) \mid L^*\xi \in \ell^2(\mathbb{Z})\big\}.$$

In the cases considered in this paper the deficiency indices are $(1,1)$.

Note that $g(z) = \sum_{k=-\infty}^{\infty} g_k(z)\,e_k$ is a solution to the eigenvalue equation $L^*\xi = z\,\xi$ precisely when $g_k(z)$ satisfies the recurrence relation (3.2). We denote by $\alpha f(z)$ the solutions to the eigenvalue equation of the form $\alpha f(z) = \sum_{k=-\infty}^{\infty} \alpha_k f_k(z)\,e_k$ with $f_k(z)$ a solution to the recurrence of Lemma 2.1 and α_k as in (3.4) or (3.5).

Recall the Wronskian (or Casorati determinant),

(4.2) $$[u, v]_k = \frac{1}{2}a_k\,(u_{k+1}\,v_k - u_k\,v_{k+1}), \qquad u = \sum_{k=-\infty}^{\infty} u_k\,e_k, \quad v = \sum_{k=-\infty}^{\infty} v_k\,e_k.$$

If moreover u and v satisfy the eigenvalue equation $L^*\,\xi = z\,\xi$, then $[u, v]_k$ is independent of $k \in \mathbb{Z}$. And u and v are linearly independent solutions of the eigenvalue equation if and only if the Wronskian $[u, v] \neq 0$. Note that we do not impose $u, v \in \ell^2(\mathbb{Z})$.

Since a_k is bounded as $k \to -\infty$ the space

$$S^-(z) = \Big\{\xi = \sum_{n=-\infty}^{\infty} \xi_n\,e_n \mid L^*\,\xi = z\,\xi,\ \sum_{n=-\infty}^{0} |\xi_n|^2 < \infty\Big\}$$

is one-dimensional for $z \in \mathbb{C}\backslash\mathbb{R}$. We assume it is spanned by $\Psi(z) = \sum_{k=-\infty}^{\infty} \Psi_k(z)\,e_k$ satisfying $\Psi_k(z) = \overline{\Psi_k(\bar{z})}$. Note that this condition can be imposed since L commutes with complex conjugation. The similarly defined space

$$S^+(z) = \Big\{\xi = \sum_{n=-\infty}^{\infty} \xi_n\,e_n \mid L^*\,\xi = z\,\xi,\ \sum_{n=0}^{\infty} |\xi_n|^2 < \infty\Big\}$$

is at most two-dimensional and at least one-dimensional for $z \in \mathbb{C}\backslash\mathbb{R}$. We show later that in cases (1) and (2) of Lemma 3.1 the space $S^+(z)$ is two-dimensional, so that the deficiency indices of $(L, \mathcal{D})$ are $(1,1)$. Indeed, $\dim\ker(L^* \pm i) = \dim S^+(\mp i) \cap S^-(\mp i) = 1$. The fact $\dim S^+(z) = 2$, $z \in \mathbb{C}\backslash\mathbb{R}$, follows from the fact that the asymptotic behaviour of $u_k(z)$ is the same as that of $v_k(z)$ (up to a sign $(-1)^k$) as $k \to \infty$, assuming we know that the solutions $u_k(z)$ and $v_k(z)$ are linearly independent, see §§5, 6. So we have $\dim\ker(L^* \pm i) = 1$, and $\Psi(\pm i) \in \ker(L^* \mp i)$. Then the self-adjoint extensions of $(L, \mathcal{D})$ are given by $(L^*, \mathcal{D}_\theta)$ with

$$\mathcal{D}_\theta = \Big\{\xi \in \mathcal{D}^* \mid \lim_{N\to\infty} [\xi, e^{i\theta}\Psi(i) + e^{-i\theta}\Psi(-i)]_N = 0\Big\}, \quad \theta \in [0, 2\pi). \tag{4.3}$$

Pick $\overline{\psi(\bar{z})} \in S^+(z) \cap \mathcal{D}_\theta$, then we can describe the resolvent for the self-adjoint operator $(L, \mathcal{D}_\theta)$ in terms of the Green function

$$G_{k,l}(z) = \frac{1}{[\Psi(z), \overline{\psi(\bar{z})}]} \begin{cases} \Psi_k(z)\overline{\psi_l(\bar{z})}, & k \le l, \\ \Psi_l(z)\overline{\psi_k(\bar{z})}, & l \le k, \end{cases} \tag{4.4}$$

and the resolvent $R(z) = (z - L)^{-1}$ is given by

$$R(z)\xi = \sum_{k=-\infty}^{\infty} \big(R(z)\xi\big)_k e_k, \qquad \big(R(z)\xi\big)_k = \sum_{l=-\infty}^{\infty} \xi_l G_{k,l}(z) = \langle \xi, \overline{G_{k,\cdot}(z)}\rangle.$$

Note that for $\xi, \eta \in \ell^2(\mathbb{Z})$

$$[\Psi(z), \overline{\psi(\bar{z})}]\, \langle R(z)\xi, \eta\rangle = \sum_{k\le l} \Psi_k(z)\overline{\psi_l(\bar{z})}\big(\xi_l\bar{\eta}_k + \xi_k\bar{\eta}_l\big)\big(1 - \frac{1}{2}\delta_{k,l}\big). \tag{4.5}$$

The corresponding spectral measure E of the self-adjoint operator $(L, \mathcal{D}_\theta)$ can be obtained from the resolvent by

$$E_{\xi,\eta}\big((x_1, x_2)\big) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} \frac{1}{2\pi i}\int_{x_1+\delta}^{x_2-\delta} \langle R(x - i\varepsilon)\xi, \eta\rangle - \langle R(x + i\varepsilon)\xi, \eta\rangle\, dx \tag{4.6}$$

for $\xi, \eta \in \ell^2(\mathbb{Z})$.

5. SPECTRAL DECOMPOSITION OF L IN THE FIRST CASE

In this section we calculate the spectral measure as explicitly as possible of the self-adjoint extensions of $(L, \mathcal{D})$ with L as in (4.1) with a_k given by (3.4). This depends on the parameter θ of the self-adjoint extension $(L^*, \mathcal{D}_\theta)$ of $(L, \mathcal{D})$. There is always continuous spectrum on the interval $[-1, 1]$, and an infinite series of discrete mass points tending to plus or minus ∞. The location of the discrete mass points depends on the choice of the self-adjoint extension. In this section we always have $a = q^{\frac{1}{2}}e^{i\psi}$ and $t = ire^{-i\psi}$ as in case (1) of Lemma 3.1, but we keep the notation a and t in order to keep the analogy with §5 in §6.

Using (3.4) we see that

$$a_k e^{i(\phi_{k+1}-\phi_k)} = \frac{1 + ire^{i\psi}q^k}{irq^k}. \tag{5.1}$$

Lemma 5.1. *There is a $\gamma \in \mathbb{R}$ such that $\overline{e^{i\gamma}\alpha_k F_k(\bar{y})} = e^{i\gamma}\alpha_k F_k(y)$.*

Proof Now

$$\overline{\alpha_k F_k(\bar y)} = \overline{\alpha_k}(q^{\frac12}e^{-i\psi}y)^{-k}\,{}_2\varphi_1\left(\begin{matrix} q^{\frac12}e^{-i\psi}y,\ -q^{\frac12}e^{-i\psi}y \\ qy^2 \end{matrix};q,-ie^{-i\psi}\frac{q^{1-k}}{r}\right)$$

$$(5.2)\qquad = e^{2ik\psi}\frac{(ie^{-i\psi}q^{1-k}/r;q)_\infty}{(-ie^{i\psi}q^{1-k}/r;q)_\infty}\overline{\alpha_k}F_k(y)$$

$$= e^{2ik\psi-2i\phi_k}\frac{(ie^{-i\psi}q^{1-k}/r;q)_\infty}{(-ie^{i\psi}q^{1-k}/r;q)_\infty}\alpha_k F_k(y)$$

where we use [3, (1.4.6)] in the second equation. From this calculation we only find $\overline{\alpha_k F_k(\bar y)} = C_k\alpha_k F_k(y)$ with $|C_k| = 1$. It remains to show that C_k is independent of k, and this follows from

$$\frac{C_{k+1}}{C_k} = e^{2i(\phi_k-\phi_{k+1})}e^{2i\psi}\frac{1-ie^{-i\psi}q^{-k}/r}{1+ie^{i\psi}q^{-k}/r}$$

$$= \left(e^{i(\phi_k-\phi_{k+1})}\frac{1+irq^k e^{i\psi}}{irq^k}\right)\left(e^{i(\phi_k-\phi_{k+1})}\frac{-irq^k}{1-ire^{-i\psi}q^k}\right) = \frac{a_k}{\overline{a_k}} = 1$$

by (5.1) and $a_k \in \mathbb{R}$. □

Remark 5.2. Note that the one-dimensional space $S^-(z)$, $z \in \mathbb{C}\backslash\mathbb{R}$, is spanned by $\alpha F(y)$, $|y| < 1$, with $z = \frac12(y+y^{-1})$, since $|\alpha_k F_k(y)| = \mathcal{O}(|y|^{-k})$ as $k \to -\infty$. Since the coefficients a_k are positive $S^-(z)$ is also spanned $\overline{\alpha F(\bar y)}$, so we see that $\overline{\alpha F(\bar y)} = C\alpha F(y)$ for some constant C. In Lemma 5.1 we have shown moreover that C is independent of z.

A straightforward corollary to Lemma 5.1 is

$$(5.3)\qquad \overline{e^{i\gamma}\theta(t)\alpha_k u_k(\bar z)} = \theta(t)e^{i\gamma}\alpha_k v_k(z)$$

using Lemma 2.4 and (3.6).

It follows from Lemma 5.1 that, in the notation of §4, we have $\Psi(z) = e^{i\gamma}\alpha F(y) = \overline{e^{i\gamma}\alpha F(\bar y)}$ with $z = \frac12(y+y^{-1})$ and $|y| < 1$.

As is clear from §4 we need to calculate various Wronskians in order to determine the domain of the self-adjoint extensions and the corresponding spectral measures. We state the results in the following lemma.

Lemma 5.3. *We have the following Wronskians;*

$$[\alpha F(y), \overline{\alpha F(\bar y^{-1})}] = \frac12(y^{-1}-y),$$

$$\lim_{N\to\infty}[\overline{\alpha u(\bar w)}, \alpha F(y)]_N = -\frac{q^{\frac12}}{ir}d(y;a,t),$$

$$\lim_{N\to\infty}[\overline{\alpha v(\bar w)}, \alpha F(y)]_N = \frac{q^{\frac12}}{ir}d(y;-a,t)$$

Proof. We first calculate the Wronskian

$$
\begin{aligned}
[\alpha F(y), \overline{\alpha F(\bar{y}^{-1})}]_k &= \frac{a_k}{2}\left(\alpha_{k+1}F_{k+1}(y)\,\overline{\alpha_k F_k(\bar{y}^{-1})} - \alpha_k F_k(y)\,\overline{\alpha_{k+1}F_{k+1}(\bar{y}^{-1})}\right) \\
&= \frac{1}{2}q^{k+\frac{1}{2}}\left(a_k e^{i(\phi_{k+1}-\phi_k)}\,F_{k+1}(y)\overline{F_k(\bar{y}^{-1})} - a_k e^{i(\phi_k-\phi_{k+1})}\,F_k(y)\overline{F_{k+1}(\bar{y}^{-1})}\right) \\
&= \frac{q^{k+\frac{1}{2}}}{2}\frac{(1+ire^{i\psi}q^k)}{irq^k}\left(q^{\frac{1}{2}}e^{i\psi}y\right)^{-k-1}\overline{\left(q^{\frac{1}{2}}e^{i\psi}\bar{y}^{-1}\right)^{-k}}(1+\mathcal{O}(q^{-k})) \\
&\quad - \frac{q^{k+\frac{1}{2}}}{2}\frac{(1-ire^{-i\psi}q^k)}{-irq^k}\left(q^{\frac{1}{2}}e^{i\psi}y\right)^{-k}\overline{\left(q^{\frac{1}{2}}e^{i\psi}\bar{y}^{-1}\right)^{-k-1}}(1+\mathcal{O}(q^{-k})) \\
&= \frac{1}{2}(y^{-1}-y)(1+\mathcal{O}(q^{-k}))
\end{aligned}
$$

using (5.1) twice and the expression for $F_k(y)$ in Lemma 2.1. Since the Wronskian is independent of k, we let $k \to -\infty$ to find the first statement of the lemma.

The next statement of the lemma follows from

$$
\begin{aligned}
&\lim_{N\to\infty}[\overline{\alpha u(\bar{w})}, \alpha F(y)]_N \\
&= \lim_{N\to\infty}\frac{1}{2}q^{N+\frac{1}{2}}\Big(a_N e^{i\phi_N - i\phi_{N+1}}\overline{u_{N+1}(\bar{w})}\big\{d(y;a,t)u_N(z) + d(y;-a,t)v_N(z)\big\} \\
&\qquad - a_N e^{i\phi_{N+1}-i\phi_N}\overline{u_N(\bar{w})}\big\{d(y;a,t)u_{N+1}(z) \\
&\qquad + d(y;-a,t)v_{N+1}(z)\big\}\Big) \\
&= \lim_{N\to\infty}\frac{1}{2}q^{N+\frac{1}{2}}\Big(\frac{1-ire^{-i\psi}q^N}{-irq^N}(1+\mathcal{O}(q^N))\big\{d(y;a,t)(1+\mathcal{O}(q^N)) \\
&\qquad + d(y;-a,t)(-1)^N(1+\mathcal{O}(q^N))\big\} \\
&\qquad - \frac{1+ire^{i\psi}q^N}{irq^N}(1+\mathcal{O}(q^N))\big\{d(y;a,t)(1+\mathcal{O}(q^N)) \\
&\qquad + d(y;-a,t)(-1)^{N+1}(1+\mathcal{O}(q^N))\big\}\Big) \\
&= -\frac{q^{\frac{1}{2}}}{ir}d(y;a,t)
\end{aligned}
$$

using (5.1) and Lemma 2.1. The last statement follows similarly. □

It follows from Lemma 5.3 that $\alpha F(y)$ and $\overline{\alpha F(\bar{y}^{-1})}$, and hence $\alpha F(y)$ and $\alpha F(y^{-1})$, are linearly independent solutions to the eigenvalue equation $L\xi = z\xi$ for $y^2 \neq 1$. Now Lemma 2.4 implies that $\alpha u(z)$ and $\alpha v(z)$ are linearly independent solutions to $L\xi = z\xi$. Since $\alpha u(z), \alpha v(z) \in S^+(z)$ we see that the deficiency indices of L are $(1,1)$ in case (1) of Lemma 3.1. For $\psi(z)$ in (4.4) we have a choice $\psi(z) = A\alpha u(z) + B\alpha v(z)$, where we have to choose $A, B \in \mathbb{C}$ such that $\overline{A\alpha u(\bar{z}) + B\alpha v(\bar{z})} \in \mathcal{D}_\theta$. In order to determine the possible choices for $A, B \in \mathbb{C}$ we use Lemma 5.3.

Lemma 5.4. *Let* $\lambda_0 = 1 - \sqrt{2}$, *then* $\psi(z) = A\alpha u(z) + B\alpha v(z) \in \mathcal{D}_\theta$ *for*

$$\begin{pmatrix} \bar{A} \\ \bar{B} \end{pmatrix} = \begin{pmatrix} E & F \\ F & E \end{pmatrix} \begin{pmatrix} e^{i\theta} \\ e^{-i\theta} \end{pmatrix}$$

$$\begin{cases} E = (i\lambda_0 q^{\frac{1}{2}} e^{i\psi}, -i\lambda_0 q^{\frac{1}{2}} e^{-i\psi}, rq^{-\frac{1}{2}}/\lambda_0, q^{\frac{3}{2}}\lambda_0/r; q)_\infty, \\ F = (-i\lambda_0 q^{\frac{1}{2}} e^{i\psi}, i\lambda_0 q^{\frac{1}{2}} e^{-i\psi}, -rq^{-\frac{1}{2}}/\lambda_0, -q^{\frac{3}{2}}\lambda_0/r; q)_\infty. \end{cases}$$

Of course, A and B are determined only up to a common scalar. Note that $E, F \in \mathbb{R}$ in Lemma 5.4, and hence $\bar{A} = B$. In this case we have, using (5.3),

$$\begin{aligned} \psi_k(z) &= A\alpha_k u_k(z) + \bar{A}\alpha_k v_k(z) = A\alpha_k u_k(z) + \frac{\overline{\theta(t)}}{\theta(t)} e^{-2i\gamma} \overline{A\alpha_k u_k(\bar{z})} \\ &= \frac{e^{-i\gamma}|A|}{\theta(t)} \left(e^{i(\gamma + \arg A)}\theta(t)\alpha_k u_k(z) + \overline{e^{i(\gamma+\arg A)}\theta(t)\alpha_k u_k(\bar{z})} \right). \end{aligned} \tag{5.4}$$

Proof. Note that $\frac{1}{2}(i\lambda_0 + (i\lambda_0)^{-1}) = i$ and $|i\lambda_0| < 1$, so $\Psi(i) = e^{i\gamma}\alpha F(i\lambda_0) = \overline{e^{i\gamma}\alpha F(-i\lambda_0)}$ and $\Psi(-i) = e^{i\gamma}\alpha F(-i\lambda_0) = \overline{e^{i\gamma}\alpha F(i\lambda_0)}$, so we can now relate A and B to the self-adjoint extension $(L^*, \mathcal{D}_\theta)$, see (4.3), by

$$\begin{aligned} 0 &= \lim_{N\to\infty} [\overline{\psi(\bar{z})}, e^{i\theta}\Psi(i) + e^{-i\theta}\Psi(-i)]_N \\ &= \lim_{N\to\infty} [\overline{A\,\alpha u(\bar{z}) + B\,\alpha v(\bar{z})}, e^{i\theta}\Psi(i) + e^{-i\theta}\Psi(-i)]_N \\ &= \frac{q^{\frac{1}{2}} e^{i\gamma}}{ir} \Big(\bar{B}\{e^{i\theta} d(i\lambda_0; -a, t) + e^{-i\theta} d(-i\lambda_0; -a, t)\} \\ &\quad - \bar{A}\{e^{i\theta} d(i\lambda_0; a, t) + e^{-i\theta} d(-i\lambda_0; a, t)\} \Big) \end{aligned} \tag{5.5}$$

using Lemma 5.3. The condition of (5.5) determines A and B uniquely in terms of $e^{i\theta}$ up to a common scalar constant. Observe that all functions d in (5.5) have a common denominator, so that we can take A and B as in the lemma. □

With A and $B = \bar{A}$ determined by Lemma 5.4 in terms of $e^{i\theta}$ we can determine the resolvent operator $R(z)$ for the corresponding self-adjoint extension $(L^*, \mathcal{D}_\theta)$. For the Green kernel, see (4.4), we need the Wronskian, with $z = \frac{1}{2}(y + y^{-1})$, $|y| < 1$,

$$[\Psi(z), \overline{\psi(\bar{z})}] = e^{i\gamma}\{\overline{A\,c(\bar{y}^{-1}; a, t)} + A\,\overline{c(\bar{y}^{-1}; -a, t)}\}\frac{1}{2}(y^{-1} - y) \tag{5.6}$$

by Lemmas 2.4 and 5.3.

We use the parametrisation $z = \frac{1}{2}(y + y^{-1})$, $|y| < 1$, for $\mathbb{C}\backslash\mathbb{R}$, and we want to take $z \to x \in \mathbb{R}$ in order to use (4.6) to determine the spectral measure of the self-adjoint extension $(L^*, \mathcal{D}_\theta)$. Note that $z \in [-1, 1]$ corresponds to y on the unit circle and $z \in (-\infty, -1]$, respectively $[1, \infty)$ corresponds to $y \in [-1, 0)$, respectively in $(0, 1]$. Letting z tend to $x = \frac{1}{2}(y_0 + y_0^{-1}) \in \mathbb{R}\backslash[-1, 1]$, $y_0 \in (-1, 1)$ from the upper or lower half plane both correspond to $y \to y_0$. However, for $x \in (-1, 1)$, put $x = \cos\chi$ with $0 < \chi < \pi$, for $\varepsilon \downarrow 0$, $z = x - i\varepsilon \to x$ corresponds to $y \to e^{i\chi}$ and $z = x + i\varepsilon \to x$ corresponds to $y \to e^{-i\chi}$. So we consider these cases separately. For the moment we assume $\xi, \eta \in \mathcal{D}$ so that all summations are actually finite, the general case $\xi, \eta \in \ell^2(\mathbb{Z})$ follows by continuity of the spectral projections $E(\mathcal{B})$, $\mathcal{B} \subset \mathbb{R}$ a Borel set.

Proposition 5.5. $[-1,1]$ *is contained in the continuous spectrum of* $(L, \mathcal{D}_\theta)$ *and for* $0 \le \chi_1 < \chi_2 \le \pi$ *the spectral measure is determined by*

$$\langle E([\cos\chi_2, \cos\chi_1])\xi, \eta\rangle = \frac{1}{2\pi}\int_{\chi_1}^{\chi_2} \frac{\langle \xi, \psi(\cos\chi)\rangle \overline{\langle \eta, \psi(\cos\chi)\rangle}}{\left|A\, c(e^{i\chi}; a, t) + \bar{A}\, c(e^{i\chi}; -a, t)\right|^2}\, d\chi$$

where A, $B = \bar{A}$ is determined by Lemma 5.4.

Proof. We first assume $x \in (-1, 1)$, $x = \cos\chi$, $0 < \chi < \pi$. Observe that for $\varepsilon \downarrow 0$, $\Psi_k(x - i\varepsilon) \to e^{i\gamma}\alpha_k F_k(e^{i\chi})$, $\Psi_k(x + i\varepsilon) \to e^{i\gamma}\alpha_k F_k(e^{-i\chi})$, and $\psi_k(x \pm i\varepsilon) \to \psi_k(\cos\chi)$, so by (4.5) and Lemma 5.3

$$\begin{aligned} &\lim_{\varepsilon\downarrow 0}\langle R(x - i\varepsilon)\xi, \eta\rangle - \langle R(x + i\varepsilon)\xi, \eta\rangle \\ &= 2\sum_{k\le l} A_k \frac{\overline{\psi_l(\cos\chi)}}{(e^{-i\chi} - e^{i\chi})}\left(\xi_l\bar{\eta}_k + \xi_k\bar{\eta}_l\right)\left(1 - \frac{1}{2}\delta_{k,l}\right) \end{aligned} \tag{5.7}$$

with

$$\begin{aligned} A_k &= \frac{\alpha_k F_k(e^{i\chi})}{\left(A\, c(e^{i\chi}; a, t) + A\, \overline{c(e^{i\chi}; -a, t)}\right)} + \frac{\alpha_k F_k(e^{-i\chi})}{\left(A\, c(e^{-i\chi}; a, t) + A\, \overline{c(e^{-i\chi}; -a, t)}\right)} \\ &= \frac{\left(A\, \overline{c(e^{-i\chi}; a, t)} + A\, \overline{c(e^{-i\chi}; -a, t)}\right)\alpha_k F_k(e^{i\chi}) + \left(A\, \overline{c(e^{i\chi}; a, t)} + A\, \overline{c(e^{i\chi}; -a, t)}\right)\alpha_k F_k(e^{-i\chi})}{\left(A\, c(e^{i\chi}; a, t) + A\, \overline{c(e^{i\chi}; -a, t)}\right)\left(A\, c(e^{-i\chi}; a, t) + A\, \overline{c(e^{-i\chi}; -a, t)}\right)} \\ &= \frac{\left(\bar{A}\,\alpha_k v_k(\cos\chi) + A\,\alpha_k u_k(\cos\chi)\right)}{\left(A\, c(e^{i\chi}; a, t) + \bar{A}\, c(e^{i\chi}; -a, t)\right)\left(\bar{A}\, c(e^{i\chi}; -a, t) + A\, c(e^{i\chi}; a, t)\right)} \\ &= \frac{\psi_k(\cos\chi)}{\left|A\, c(e^{i\chi}; a, t) + \bar{A}\, c(e^{i\chi}; -a, t)\right|^2} \end{aligned}$$

using (3.6) and Lemma 2.4 for the third equality. The above gives an explicit expression for A_k. If the expression $A_k\overline{\psi_l(\cos\chi)}$ is symmetric in k and l we can rewrite the sum over $k \le l$ in (5.7) as the product of a sum over k and a sum over l. From (5.4) we see that $\psi_k(x)\overline{\psi_l(x)}$ is symmetric in k and l for $x \in \mathbb{R}$, and hence $A_k\overline{\psi_l(\cos\chi)}$ is symmetric in k and l.

So we can antisymmetrise the sum in (5.7) and (5.7) equals

$$\frac{\left(\sum_{l=-\infty}^{\infty} \xi_l \overline{\psi_l(\cos\chi)}\right)\left(\sum_{k=-\infty}^{\infty} \psi_k(\cos\chi)\bar{\eta}_k\right)}{\left|A\, c(e^{i\chi}; a, t) + \bar{A}\, c(e^{i\chi}; -a, t)\right|^2}.$$

Using $dx = \frac{1}{2i}(e^{i\chi} - e^{-i\chi})d\chi$, dominated convergence and (4.6) we find for $0 < \chi_1 < \chi_2 < \pi$

$$\langle E\big((\cos\chi_2),\cos(\chi_1)\big)\xi,\eta\rangle=\frac{1}{2\pi}\int_{\chi_1}^{\chi_2}\frac{\langle\xi,\psi(\cos\chi)\rangle\overline{\langle\eta,\psi(\cos\chi)\rangle}}{\big|A\,c(e^{i\chi};a,t)+\bar{A}\,c(e^{i\chi};-a,t)\big|^2}\,d\chi.$$

By the previous calculation the proposition follows for the open interval $(-1,1)$. Since the spectrum is closed we see that ±1 are contained in the spectrum of $(L^*,\mathcal{D}_\theta)$. Since $(L^*,\mathcal{D}_\theta)$ is a self-adjoint operator ±1 can be in the continuous spectrum or in the point spectrum, see [2, Thm. 13.27]. The proposition follows by showing that the endpoints ±1 are not contained in the discrete spectrum. Note that for $y=\pm1$ the first Wronskian in Lemma 5.3 vanishes, so we need to construct a second independent solution to $\alpha F(\pm1)$ first. We consider the case $y=1$, the case $y=-1$ is being dealt with similarly. Put

$$H_k(y)=\frac{F_k(y)-F_k(1)}{y-1},\tag{5.8}$$

then it satisfies

$$\frac{1+a^2tq^{k-1}}{atq^{k-1}}\,H_{k+1}(y)-\frac{1-q^{k-1}t}{atq^{k-1}}\,H_{k-1}(y)=2\,H_k(y)+(1-y^{-1})\,F_k(y).$$

Taking $y\to1$ gives the solution $H_k(1)=\frac{\partial F_k(y)}{\partial y}\big|_{y=1}$ to (2.2) for the eigenvalue $z=1$. Now (5.8) gives the asymptotic behaviour

$$H_k(1)=\frac{\partial F_k(y)}{\partial y}\Big|_{y=1}=(-k)a^{-k}\big(1+\mathcal{O}(q^{-k})\big),\qquad k\to-\infty.\tag{5.9}$$

Using the asymptotic behaviour (5.9) we can calculate the Wronskian

$$[\alpha H(1),\overline{\alpha F(1)}]=-\frac{1}{2}$$

similar to the calculation of the first Wronskian of Lemma 5.3. So we have two linearly independent solutions of the eigenvalue equation $L^*\xi=\xi$. From the asymptotic behaviour (5.9) of αH_k and of $\overline{\alpha F(1)}$ as $k\to-\infty$, it follows that no linear combination of αH and $\overline{\alpha F(1)}$ can be an element of $\ell^2(\mathbb{Z})$. Hence, 1 is not in the point spectrum, and hence 1 is contained in the continuous spectrum. □

Note that with the choices for $\Psi(z)$ and $\psi(z)$ the function $[\Psi(z),\overline{\psi(\bar z)}]\,G_{k,l}(z)$ is analytic for $z\in\mathbb{C}\backslash[-1,1]$, and the same holds for $[\Psi(z),\overline{\psi(\bar z)}]\,\langle R(z)\xi,\eta\rangle$, where we still assume $\xi,\eta\in\mathcal{D}$. We next turn to the spectrum of $(L,\mathcal{D}_\theta)$ contained in $(-\infty,-1)\cup(1,\infty)$. Because of these remarks and the remarks in the paragraph preceding Proposition 5.5 and (4.5) we see that $E_{\xi,\eta}\big((x_1,x_2)\big)=0$ as long as (x_1,x_2) contains no zero of the Wronskian (5.6) using dominated convergence in (4.6). Note that the zeroes of the Wronskian are isolated, since the Wronskian (5.6) is meromorphic in y. So the only discrete mass points can occur at a zero of the Wronskian (5.6).

Proposition 5.6. *There is no continuous spectrum of* $(L^*, \mathcal{D}_\theta)$ *in* $(-\infty,-1)\cup(1,\infty)$*. The point spectrum of* $(L^*, \mathcal{D}_\theta)$ *occurs at the set*

$$S = \left\{ x_0 = \frac{1}{2}(y_0 + y_0^{-1}) \,\Big|\, |y_0| > 1,\ \bar{A}c(y_0; -a, t) + Ac(y_0; a, t) = 0 \right\}$$

and the spectral projection is determined by

$$\begin{aligned}&\langle E(\{x_0\})\xi, \eta\rangle \\ &= \operatorname{Res} \frac{\langle \xi, \psi(x_0)\rangle\langle \psi(x_0), \eta\rangle}{y\big(Ac(y^{-1}; a, t) + \bar{A}c(y^{-1}; -a, t)\big)\big(\bar{A}c(y; -a, t) + Ac(y; a, t)\big)}\Big|_{y=y_0}.\end{aligned}$$

Remark 5.7. From Proposition 5.6 and the discussion preceding it we see that the discrete set S is contained in $\mathbb{R}$. Moreover, the unboudedness of $(L^*, \mathcal{D}_\theta)$ and the boundedness of the continuous spectrum $[-1, 1]$, see Proposition 5.5, shows that the set S is unbounded.

Proof. From the remarks preceding Proposition 5.6 we see that we can only have discrete spectrum in $(-\infty,-1)\cup(1,\infty)$. Next assume that $x_0 \in (-\infty,-1)\cup(1,\infty)$ is a zero of the Wronskian (5.6). (Note that we have already dealt with the case $x_0 = \pm 1$ in Proposition 5.5.) Let $x_0 = \frac{1}{2}(y_0 + y_0^{-1})$ with $|y_0| > 1$. (Note that this is against the convention, but it makes formulas better looking.) Moreover, since $\langle R(z)\xi, \eta\rangle$ is meromorphic in a neighbourhood of x_0 we find that x_0 is an element of the point spectrum of $(L, \mathcal{D}_\theta)$ and

$$\text{(5.10)} \qquad \langle E(\{x_0\})\xi, \eta\rangle = \frac{1}{2\pi i}\oint_C \langle R(z)\xi, \eta\rangle\, dz = \operatorname{Res} \langle R(z)\xi, \eta\rangle\big|_{z=x_0}$$

where C is a small positively oriented contour enclosing x_0 once and no other singularities of the resolvent. From Lemma 2.4 and the fact that y_0^{-1} is a zero of the Wronskian (5.6) using (3.6) we find

$$\begin{aligned}\psi_k(x_0) &= A\alpha_k u_k(x_0) + \bar{A}\alpha_k v_k(x_0) = \big(Ac(y_0^{-1}; a, t) + \bar{A}c(y_0^{-1}; -a, t)\big)\,\alpha_k F_k(y_0^{-1}) \\ &= \big(Ac(y_0^{-1}; a, t) + \bar{A}c(y_0^{-1}; -a, t)\big)\, e^{-i\gamma}\Psi_k(x_0).\end{aligned}$$

In particular, this implies $\psi(x_0) \in \ell^2(\mathbb{Z})$. Using this in (5.10) and switching from z to y gives the desired expression for $\langle E(\{x_0\})\xi, \eta\rangle$. $\square$

Combining Propositions 5.5 and 5.6 proves the following theorem, which summarises the results of this section.

Theorem 5.8. *The spectral decomposition of the self-adjoint extension* $(L^*, \mathcal{D}_\theta)$ *defined by (4.3) of* $(L, \mathcal{D})$ *as defined in (4.1) with* a_k *as in (3.4) and* $a = q^{\frac{1}{2}} e^{i\psi}$*,* $t = ire^{-i\psi} \notin \mathbb{R}_{>0}$*,* $r \in \mathbb{R}\backslash\{0\}$*, is given by*

$$\langle L^*\xi,\eta\rangle = \frac{1}{2\pi}\int_0^{\pi} \frac{\cos\chi\,(\mathcal{F}_\theta\xi)(\cos\chi)\overline{(\mathcal{F}_\theta\eta)(\cos\chi)}}{\left|A\,c(e^{i\chi};a,t)+\bar{A}\,c(e^{i\chi};-a,t)\right|^2}\,d\chi$$
$$+\sum_{x_0\in S}\operatorname{Res}\frac{x_0\,(\mathcal{F}_\theta\xi)(x_0)\overline{(\mathcal{F}_\theta\eta)(x_0)}}{y\big(Ac(y^{-1};a,t)+\bar{A}c(y^{-1};-a,t)\big)\big(\bar{A}c(y;-a,t)+Ac(y;a,t)\big)}\Big|_{y=y_0}$$

where $\xi\in\mathcal{D}_\theta$, $\eta\in\ell^2(\mathbb{Z})$, A, $B=\bar{A}$ is determined in Lemma 5.4 by $e^{i\theta}$, $c(\cdot;a,t)$ is defined in Lemma 2.4, the set of discrete mass points is given by

$$S=\Big\{x_0=\frac{1}{2}(y_0+y_0^{-1})\;\Big|\;|y_0|>1,\ \bar{A}c(y_0;-a,t)+Ac(y_0;a,t)=0\Big\},$$

the corresponding Fourier transform is

$$(\mathcal{F}_\theta\xi)(x)=\langle\xi,\psi(x)\rangle=\langle\xi,A\alpha u(x)+\bar{A}\alpha v(x)\rangle$$

with α_k defined by (3.4), and $u_k(x)$, $v_k(x)$ as in Lemma 2.4.

We recast Theorem 5.8 into two immediate corollaries.

Corollary 5.9. *With the notation of Theorem 5.8 the orthogonality relations*

$$\delta_{k,l}=\frac{1}{2\pi}\int_0^{\pi}\frac{\psi_k(\cos\chi)\overline{\psi_l(\cos\chi)}}{\left|A\,c(e^{i\chi};a,t)+\bar{A}\,c(e^{i\chi};-a,t)\right|^2}\,d\chi$$
$$+\sum_{x_0\in S}\operatorname{Res}\frac{\psi_k(x_0)\overline{\psi_l(x_0)}}{y\big(Ac(y^{-1};a,t)+\bar{A}c(y^{-1};-a,t)\big)\big(\bar{A}c(y;-a,t)+Ac(y;a,t)\big)}\Big|_{y=y_0}$$

hold, and the functions $\{\psi_k\}_{k\in\mathbb{Z}}$ form an orthonormal basis of the corresponding weighted L^2-space, and $\mathcal{F}_\theta$ is a unitary isomorphism from $\ell^2(\mathbb{Z})$ to the corresponding weighted L^2-space.

Corollary 5.10. *With the notation of Theorem 5.8 we have the following transform pair; for $\xi=\sum_{k=-\infty}^{\infty}\xi_k e_k\in\ell^2(\mathbb{Z})$*

$$\xi_l=\frac{1}{2\pi}\int_0^{\pi}\frac{(\mathcal{F}_\theta\xi)(\cos\chi)\overline{\psi_l(\cos\chi)}}{\left|A\,c(e^{i\chi};a,t)+\bar{A}\,c(e^{i\chi};-a,t)\right|^2}\,d\chi$$
$$+\sum_{x_0\in S}\operatorname{Res}\frac{(\mathcal{F}_\theta\xi)(x_0)\overline{\psi_l(x_0)}}{y\big(Ac(y^{-1};a,t)+\bar{A}c(y^{-1};-a,t)\big)\big(\bar{A}c(y;-a,t)+Ac(y;a,t)\big)}\Big|_{y=y_0}.$$

Remark 5.11. Note that the spectral measure for the continuous spectrum in Theorem 5.8 is rather explicit, and that the description of the discrete mass points in Theorem 5.8 is indirect. For the special case of the Ismail-Zhang q-analogue of the exponential function defined in (1.1) we can describe the discrete mass points a bit more explicitly. This special case corresponds to $\psi=0 \bmod \pi$. Without loss of generality we can assume $\psi=0$ by [3, (1.4.6)], and take $a=q^{\frac{1}{2}}$ and $t=ir$, $r\in\mathbb{R}\backslash\{0\}$. First observe that in the definition of $A=e^{i\theta}E+e^{-i\theta}F$ in Lemma 5.4 we can replace E and F by, recall $\lambda_0=1-\sqrt{2}$, $E=$

$\theta(rq^{-\frac{1}{2}}/\lambda_0)$ and $F = \theta(-rq^{-\frac{1}{2}}/\lambda_0)$ by cancelling a common factor. In this case the c-functions have a common factor and we have

$$\begin{aligned} &A\, c(y; q^{\frac{1}{2}}, ir) + \bar{A}\, c(y; -q^{\frac{1}{2}}, ir) \\ &= \frac{(q^{\frac{1}{2}}/y, -q^{\frac{1}{2}}/y; q)_\infty}{(-q;q)_\infty \theta(ir)(y^{-2};q)_\infty} \Big(\theta(yq^{\frac{1}{2}}ir)A + \theta(-yq^{\frac{1}{2}}ir)\bar{A}\Big). \end{aligned} \tag{5.11}$$

So the spectral measure for the continuous part can be read off from (5.11). For the discrete spectrum we have to find the zeroes of (5.11) as function of y for $|y| > 1$, so we have to solve $\theta(yq^{\frac{1}{2}}ir)A = -\theta(-yq^{\frac{1}{2}}ir)\bar{A}$. Put $y = e^{2\pi i w}$ and consider

$$g(w,\tau) = \frac{\theta(e^{2\pi i w}q^{\frac{1}{2}}ir)}{\theta(-e^{2\pi i w}q^{\frac{1}{2}}ir)}, \qquad q = e^{\pi i \tau}, \quad \tau \in i\mathbb{R}_{>0}, \tag{5.12}$$

so that the equation is rewritten as $g(w,\tau) = -\bar{A}/A$. It follows from (1.4) that $g(w,\tau)$ is an elliptic function with periods 1 and τ. From [19, Ch. XX, XXI] we see that the order of the elliptic function g is 2, so that the equation $g(w,\tau) = -\bar{A}/A$ has 2 solutions in each fundamental parallellogram. Since the solutions in the y-coordinate are real we find $w \in i\mathbb{R} \cup \frac{1}{2} + i\mathbb{R}$ (modulo 1). By period τ it follows that the discrete mass points are of the form $x_n^{(i)} = \frac{1}{2}(y_i q^{-2n} + y_i^{-1}q^{2n})$, $n \in \mathbb{Z}$, with $|x_n^{(i)}| > 1$, where $i = 1,2$. So in particular, in this case the discrete mass points are located on two q^2-quadratic grids.

In this case we have from (5.4)

$$\begin{aligned} \psi_k(x) &= \frac{2e^{-i\gamma}|A|}{\theta(ir)} \Re\Big[q^{\frac{1}{2}k} e^{i(\phi_k+\gamma+\arg A)} \frac{\theta(ir)(-r^2q^{2+4k};q^4)_\infty}{(irq^k;q)_\infty} \mathcal{E}_{q^2}(x; -irq^k)\Big], \\ & x \in \mathbb{R}, \end{aligned} \tag{5.13}$$

so that we can look upon the orthogonality relations of Corollary 5.9 or the integral transform of Corollary 5.10 as a q-analogue of the Fourier cosine transform for the q-exponential $\mathcal{E}_q$ as defined in (1.1) using (1.2).

Remark 5.12. It is of interest to be able to calculate the $\mathcal{F}_\theta$ transforms of specific vectors and next use Corollary 5.10 to get explicit transforms, even though the precise location of the discrete mass points is not known. Results already present in the literature can be used for this. Since the corresponding formulas are well known we leave it to the reader to fill in the details. As a first example, the $\mathcal{F}_\theta$-transform of the vector $\xi = \sum_{k=-\infty}^{\infty} z^k \alpha_k\, e_k$ can be expressed in terms of infinite q-shifted factorials using the generating function [10, Lemma 3.3 with $k = 1$], [16, Lemma 2.2 with $k = 1$]. Since this is not an $\ell^2(\mathbb{Z})$-vector some care has to be taken, but using an approximation argument plus the absolute convergence of the sum defining $\mathcal{F}_\theta\xi$ for z in a certain annulus, we can find the result.

Using a generalisation of Rahman's summation formulas, see [10, Prop. 3.1], [16, Thm. 2.1 with $k = l = 1$] it is possible to calculate the Poisson kernel, i.e. the

$\mathcal{F}_\theta$-transform of the vector $\sum_{k=-\infty}^{\infty} z^k \psi_k(x')\, e_k$ for a different value for the argument. For the arguments in the interval $[-1,1]$ the Poisson kernel can be expressed in term of eight very-well-poised ${}_8W_7$-series. For the case $a = q^{\frac{1}{2}}$, $t = ir$, i.e. for the situation corresponding to $\mathcal{E}_{q^2}$, the situation simplifies greatly, and the eight ${}_8W_7$-series can be combined to only two ${}_8W_7$-series by [3, (2.10.1)]. To evaluate the Poisson kernels in the discrete mass points we express $\psi_k(x_0)$ in terms of $\alpha F(y_0^{-1})$ as a single ${}_2\varphi_1$, and use the connection coefficients of Lemma 2.4 before applying the same summation formulas again. The procedure sketched above can be generalised to the $\mathcal{F}_\theta$-transform of the vector

$$\sum_{k=-\infty}^{\infty} z^k\, {}_{r+1}\varphi_r\left(\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, tq^k\right) e_k$$

using [16, Thm. 2.1, with $k = r$, $l = 1$].

6. SPECTRAL DECOMPOSITION OF L IN THE SECOND CASE

In this section we calculate the spectral measure as explicitly as possible of the self-adjoint extensions of $(L, \mathcal{D})$ with L as in (4.1) with a_k given by (3.5). As in §6, this depends on the parameter θ of the self-adjoint extension $(L^*, \mathcal{D}_\theta)$ of $(L, \mathcal{D})$. There is always continuous spectrum on the interval $[-1,1]$, and an infinite series of discrete mass points tending to plus or minus ∞. The location of the discrete mass points depends on the choice of the self-adjoint extension. In this section we always have $a = is$, $s \in \mathbb{R}\backslash\{0\}$ and $t < 0$ as in case (2) of Lemma 3.1, but we keep the notation a and t in order to keep the analogy with §5. The case considered in this section is slightly easier than the case considered in §5, so we only state the results and indicate the proofs by analogy to §5.

So, in this section $t < 0$, $a = is$, $s \in \mathbb{R}\backslash\{0\}$, see case (2) of Lemma 3.1 and a_k and α_k are given in (3.5). In this case it is straightforward to see that $\overline{\alpha_k F_k(\bar{y})} = \alpha_k F_k(y)$. Using Lemma 2.4 and (3.6) this implies

$$\overline{\alpha_k\, u_k(\bar{z})} = \alpha_k\, v_k(z). \tag{6.1}$$

We also have that $\alpha_k F_k(y) = y^{-k}(1 + \mathcal{O}(q^{-k}))$, so that $S^-(z)$ is spanned by $\Psi(z) = \alpha F(y)$ with $z = \frac{1}{2}(y + y^{-1})$ with $|y| < 1$. The statement analogous to Lemma 5.3 is the following lemma, whose proof is similar to the proof of Lemma 5.3.

Lemma 6.1. *We have the following Wronskians;*

$$[\alpha F(y), \alpha F(y^{-1})] = \frac{1}{2}(y^{-1} - y),$$

$$\lim_{N\to\infty} [\overline{\alpha u(\bar{w})}, \alpha F(y)]_N = \frac{iq}{st}\frac{\theta(s^2t/q)}{\theta(t)}\, d(y; a, t),$$

$$\lim_{N\to\infty} [\overline{\alpha v(\bar{w})}, \alpha F(y)]_N = -\frac{iq}{st}\frac{\theta(s^2t/q)}{\theta(t)}\, d(y; -a, t).$$

It follows from Lemma 6.1 that $\alpha F(y)$ and $\overline{\alpha F(\bar{y}^{-1})}$, and hence $\alpha F(y)$ and

$\alpha F(y^{-1})$, are linearly independent solutions to the eigenvalue equation $L\xi = z\xi$ for $y^2 \neq 1$. Now Lemma 2.4 implies that $\alpha u(z)$ and $\alpha v(z)$ are linearly independent solutions to $L\xi = z\xi$. Since $\alpha u(z), \alpha v(z) \in S^+(z)$ we see that the deficiency indices of L are $(1,1)$ in case (2) of Lemma 3.1. Again, for $\psi(z)$ in (4.4) we have a choice $\psi(z) = A\alpha u(z) + B\alpha v(z)$, where we have to choose $A, B \in \mathbb{C}$ such that $\overline{A\alpha u(\bar{z}) + B\alpha v(\bar{z})} \in \mathcal{D}_\theta$. In order to determine the possible choices for $A, B \in \mathbb{C}$ we use Lemma 6.1.

Lemma 6.2. *Let* $\lambda_0 = 1 - \sqrt{2}$, *then* $\psi(z) = A\alpha u(z) + B\alpha v(z) \in \mathcal{D}_\theta$ *for*

$$\begin{pmatrix} \bar{A} \\ \bar{B} \end{pmatrix} = \begin{pmatrix} E & F \\ F & E \end{pmatrix} \begin{pmatrix} e^{i\theta} \\ e^{-i\theta} \end{pmatrix} \qquad \begin{cases} E = (s\lambda_0, \lambda_0 q/s, -st/q\lambda_0, -q^2\lambda_0/st; q)_\infty, \\ F = (-s\lambda_0, -\lambda_0 q/s, st/q\lambda_0, q^2\lambda_0/st; q)_\infty. \end{cases}$$

Of course, A and B are determined only up to a common scalar. Note that $E, F \in \mathbb{R}$ in Lemma 6.2, and hence $\bar{A} = B$. In this case we have, using (6.1),

$$\psi_k(z) = A\alpha_k u_k(z) + \bar{A}\alpha_k v_k(z) = A\alpha_k u_k(z) + \overline{A\alpha_k u_k(\bar{z})}, \tag{6.2}$$

so that for $z = x \in \mathbb{R}$ we have $\psi_k(x) = 2\Re[A\alpha_k u_k(x)]$ is real-valued.

Completely analagous to Proposition 5.5 we obtain that $[-1,1]$ is contained in the continuous spectrum of $(L, \mathcal{D}_\theta)$ and for $0 \le \chi_1 < \chi_2 \le \pi$ the spectral measure is determined by the same formula as in Proposition 5.5, but with $a = is$, $t < 0$ and where A, $B = \bar{A}$ are determined by Lemma 6.2.

The expression (5.6) for the Wronskian has to be replaced by, again $z = \frac{1}{2}(y + y^{-1})$, $|y| < 1$,

$$[\Psi(z), \overline{\psi(\bar{z})}] = \{\overline{A\, c(\bar{y}^{-1}; a, t)} + A\, \overline{c(\bar{y}^{-1}; -a, t)}\}\frac{1}{2}(y^{-1} - y), \tag{6.3}$$

and since for the discrete spectrum only the zeroes of the Wronskian play a role, we see that Proposition 5.6 goes through in this case with A and $\bar{A} = B$ defined by Lemma 6.2 in this case.

Combining these results then gives the spectral decomposition of the self-adjoint extension $(L^*, \mathcal{D}_\theta)$ of $(L, \mathcal{D})$ as in (4.1) with a_k defined by (3.5).

Theorem 6.3. *The spectral decomposition of the self-adjoint extension* $(L^*, \mathcal{D}_\theta)$ *defined by (4.3) of* $(L, \mathcal{D})$ *as defined in (4.1) with* a_k *as in (3.4) and* $a = is$, $s \in \mathbb{R}\backslash\{0\}$, $t \in \mathbb{R}_{<0}$, *is given by the same formula as in Theorem 5.8 except that* A *and* $\bar{A} = B$ *are defined by Lemma 6.2. Moreover, Corollary 5.9 and Corollary 5.10 remain valid in this case.*

7. QUADRATIC TRANSFORMATION

In this section we relate some of the solutions discussed in Lemma 62 to ${}_3\varphi_2$-series of base q^2. The resulting transformation of Proposition 7.1 can be considered as a non-terminating analogue of Singh's quadratic transformation [3, (3.10.13)]. The reason for this is that the symmetric Al-Salam–Chihara polynomials of even degree can be expressed in terms of continuous dual q^2-Hahn polynomials, and the resulting transform is a special case of Singh's transformation. It does not seem possible to obtain the result of Proposition 7.1 as a special or limiting case of Singh's transformation.

The recurrence relation (2.2) has no term involving $f_k(z)$ in the right hand side, so we can iterate the recurrence to obtain a three term recurrence for the even and odd degree $f_k(z)$'s. For convenience put

$$c_k = \frac{1+a^2tq^{k-1}}{atq^{k-1}} = a(1+q^{1-k}/at), \qquad d_k = -\frac{1-q^{k-1}t}{atq^{k-1}} = \frac{1}{a}(1-q^{1-k}/t),$$

and iterate (2.2) to find

$$(2z)^2 f_k(z) = c_k c_{k+1} f_{k+2}(z) + \big(c_k d_{k+1} + d_k c_{k-1}\big) f_k(z) + d_k d_{k-1} f_{k-2}(z). \tag{7.1}$$

So from (7.1) we find a three-term recurrence relation for the even and odd degree f_k's. The recurrence (7.1) can be matched to the one studied by Gupta, Ismail and Masson [4], where a lot of solutions are discussed. The recurrence relation (7.1) is also studied in detail in [13] as a linear operator on a suitable Hilbert space.

We recall from [13, §2-4] that

$$\begin{aligned} &\Phi_\gamma(xq^k; a, b, c; q) \\ &= \frac{(-q^{1-k}/bcx, -q^{1-k}\gamma/ax; q)_\infty}{(-q^{1-k}/abx, -q^{1-k}/acx; q)_\infty} (a\gamma)^{-k} {}_3\varphi_2\left(\begin{matrix} q\gamma/a, b\gamma, c\gamma \\ -q^{1-k}\gamma/ax, q\gamma^2 \end{matrix} ; q, \frac{-q^{1-k}}{bcx} \right) \end{aligned} \tag{7.2}$$

is a solution of

$$\begin{aligned} (\gamma+\gamma^{-1}) F(xq^k) &= a\left(1+\frac{q^{-k}}{abx}\right)\left(1+\frac{q^{-k}}{acx}\right) F(xq^{k+1}) \\ &- \left(q^{-k}\left(\frac{1}{bx}+\frac{1}{cx}+\frac{q}{abcx}+\frac{1}{ax}\right) + \frac{q^{-2k}}{x^2abc}(1+q)\right) F(xq^k) \\ &+ \frac{1}{a}\left(1+\frac{q^{1-k}}{bcx}\right)\left(1+\frac{q^{-k}}{x}\right) F(q^{k-1}x). \end{aligned} \tag{7.3}$$

Many other solutions and their connections are known, see [4], [13], but we only need this solution. The main result of this section is the following proposition.

Proposition 7.1. *For* $|z| < \min(1, |a|^2)$

$${}_2\varphi_1\left(\begin{matrix} ay, -ay \\ qy^2 \end{matrix} ; q, -\frac{z}{a^2} \right) = \frac{(z, qzy^2/a; q^2)_\infty}{(-z/a; q)_\infty} {}_3\varphi_2\left(\begin{matrix} q^2y^2/a, -y^2, -qy^2 \\ qzy^2/a, q^2y^4 \end{matrix} ; q^2, z \right).$$

Proof. A straightforward calculation starting from (7.1) shows that a^kR_k with $R_k = f_{2k}(z)$, $z = \frac{1}{2}(y+y^{-1})$, satisfies (7.3) in base q^2 with (a,b,c,x) of (7.3) specialised to $(a, -1, -q, -t/q)$ and $\gamma = y^2$. So the solutions of (7.3) are related to the ones in Lemma 2.1 for k replaced by $2k$. Since the solution space is two-dimensional, we cannot immediately give direct relations. However, the space of subdominant or minimal solutions is one-dimensional, see [4, Thm. 1], and spanned by $F_{2k}(y)$, $|y|<1$, and $a^{-k}\Phi_{y^2}(-tq^{2k-1};a,-1,-q;q^2)$. For the renormalised recurrence in case of §§5, 6, this is just the statement that $S^-(z)$ is one-dimensional.

So these two solutions only differ by a constant C which can be determined by considering the limit behaviour for $k\to-\infty$. The limit behaviour follows from the explicit expression in (7.2) and Lemma 2.1. This gives $C=1$, and canceling common factors gives, for $k\in\mathbb{Z}$ and $|y|<1$, the relation

$$\begin{aligned}
&{}_2\varphi_1\left(\begin{matrix} ay,-ay \\ qy^2\end{matrix};q,-\frac{q^{2-2k}}{a^2t}\right)\\
&=\frac{(q^{2-2k}/t,q^{3-2k}y^2/at;q^2)_\infty}{(-q^{3-2k}/at,-q^{2-2k}/at;q^2)_\infty}\,{}_3\varphi_2\left(\begin{matrix} q^2y^2/a,-y^2,-qy^2 \\ q^{3-2k}y^2/at,q^2y^4\end{matrix};q^2,\frac{q^{2-2k}}{t}\right).
\end{aligned}\tag{7.4}$$

Multiplying (7.4) by $(q^2y^4;q^2)_\infty=(qy^2,-qy^2;q)_\infty$ we see that both sides become analytic in y. By analytic continuation (7.4) remains valid for all $y^2\notin q^{-\mathbb{N}}$.

This proves the proposition for $z=\frac{q^{2-2k}}{t}$, and by analytic continuation in z the result follows. □

Remark 7.2. (i) There are more choices possible for the parameters in (7.3) to match the recurrence (7.3) to the recurrence (7.1) for the even and odd degree f_k's. All other possible choices lead to the same result, In particular, going over the proof for the odd degree f_k's leads to (7.4) with t replaced by qt.

(ii) In [13] the spectral analysis of the operator arising from the recurrence relation (7.3) has been studied on a suitable Hilbert space larger than $\ell^2(\mathbb{Z})$. For the values of a and t as considered in §§5, 6 this operator, say S, is, up to a shift by a constant, the square of $(L,\mathcal{D})$, so that Theorems 5.8, 6.3 also gives the spectral decomposition of S. This shows that for the choices of the parameters in (7.3) as in the proof of Proposition 7.1 the spectral decomposition is explicit, cf. the remarks on p. 193 and p. 200 of [13]. Note that the parameters of (7.3) used here are not contained in the parameter set considered in [13].

(iii) Using Proposition 7.1 and the connection coefficients of Lemma 2.4 and [13, Prop. 4.4, Prop. 5.5] we can rewrite any solution of (2.2) in terms of solutions of (7.3) in base q^2 with (a,b,c,x) replaced by $(a,-1,-q,-t/q)$ or $(a,-1,-q,-t)$. Apart from the case discussed in Proposition 7.1 this only gives more-term transformations. As an example we give the expression for the Ismail-Zhang q-analogue of the exponential $\mathcal{E}_q$ defined in (1.1). Using Lemma 2.4 and (1.4), or [3, (4.3.2)], and Proposition 7.1 in base $q^{\frac12}$ gives, $z=\frac12(y+y^{-1})$,

$$\begin{aligned}
&\frac{(q^{\frac14}/y,-q^{\frac14}/y,-q^{\frac14}ty,-q^{\frac14}/ty;q^{\frac12})_\infty(-q/t,-q^{\frac54}y^2/t;q)_\infty}{(-q^{\frac12},y^{-2},-q^{\frac12}/t;q^{\frac12})_\infty(q^{\frac34}/t;q^{\frac12})_\infty(qt^2;q^2)_\infty}\,{}_3\varphi_2\left(\begin{matrix} q^{\frac34}y^2,-y^2,-q^{\frac12}y^2 \\ -q^{\frac54}y^2/t,qy^4\end{matrix};q,-\frac{q}{t}\right)\\
&\qquad+\mathrm{Idem}(y\leftrightarrow y^{-1})=\mathcal{E}_q(z;t)
\end{aligned}$$

REFERENCES

[1] Berezanskiĭ, J.M. – Expansions in Eigenfunctions of Selfadjoint Operators, Transl. Math. Monographs 17, Amer. Math. Soc., 1968.

[2] Dunford, N. and J.T. Schwartz – Linear Operators II: Spectral Theory, Interscience, 1963.

[3] Gasper, G. and M. Rahman – Basic Hypergeometric Series, Encycl. Math. Appl. 35, Cambridge Univ. Press, 1990.

[4] Gupta, D.P., M.E.H. Ismail and D.R. Masson – Contiguous relations, basic hypergeometric functions, and orthogonal polynomials. III. Associated continuous dual q-Hahn polynomials, J. Comp. Appl. Math. **68**, 115–149 (1996).

[5] Ismail, M.E.H. and D. Stanton – q-integral and moment representations for q-orthogonal polynomials, Canad. J. Math. **54**, 709–735 (2002).

[6] Ismail, M.E.H. and D. Stanton – Applications of q-Taylor theorems, preprint (2002).

[7] Ismail, M.E.H. and R. Zhang – Diagonalization of certain integral operators, Adv. Math. **109**, 1–33 (1994).

[8] Kakehi, T. – Eigenfunction expansion associated with the Casimir operator on the quantum group $SU_q(1,1)$, Duke Math. J. **80**, 535–573 (1995).

[9] Koelink, E. – Spectral theory and special functions, in "Proceedings of the 2000 SIAG OP-SF summer school on Orthogonal Polynomials and Special Functions (Laredo, Spain)", (eds. F. Marcellán, W. Van Assche and R. Álvarez-Nodarse), Nova Science, to appear. `math.CA/0107036` (2001).

[10] Koelink, E. and H. Rosengren, – Transmutation kernels for the little q-Jacobi function transform, Rocky Mountain J. Math. **32**, 703–738 (2002).

[11] Koelink, E. and J.V. Stokman with an appendix by M. Rahman – Fourier transforms on the quantum SU(1,1) group, Publ. Res. Inst. Math. Sci., Kyoto Univ. **37**, 621–715 (2001).

[12] Koelink, E. and J.V. Stokman – The Askey-Wilson function transform scheme, 221–241 in "Special Functions 2000" (eds. J. Bustoz, M.E.H. Ismail, S.K. Suslov), NATO Science Series II, Vol. 30, Kluwer, 2001.

[13] Koelink, E. and J.V. Stokman – The big q-Jacobi function transform, Constr. Approx. **19**, 191–235 (2003).

[14] Masson, D.R. and J. Repka – Spectral theory of Jacobi matrices in $\ell^2(\mathbb{Z})$ and the $su(1,1)$ Lie algebra, SIAM J. Math. Anal. **22**, 1131–1146 (1991).

[15] Pruitt, W.E. – Bilateral birth and death processes, Trans. Amer. Math. Soc. **107**, 508–525 (1963).

[16] Rosengren, H. – A bilateral series involving basic hypergeometric functions, in "Theory and Applications of Special Functions. A Volume Dedicated to Mizan Rahman", (eds. M.E.H. Ismail and E. Koelink), Kluwer, to appear. `math.CA/0302114` (2003)

[17] Suslov, S.K – An Introduction to Basic Fourier Series, Developments in Math. **9**, Kluwer Acad. Publ., 2003.

[18] Titchmarsh, E.C – Eigenfunction Expansions Associated with Second-order Differential Equations, Part I, 2nd ed, Oxford Univ. Press, 1962.

[19] Whittaker, E.T. and G.N. Watson, A Course of Modern Analysis, 4th ed., Cambridge Univ. Press, 1927.

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