Applied Mathematics Letters 23 (2010) 479–483

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Left-continuity of t-norms on the n-dimensional Euclidean cube ☆

Guojun Wang [a,b,*], Wei Wang [c]

[a] Institute of Mathematics, Shaanxi Normal University, Xi'an 710062, China
[b] Shanghai Key Laboratory of Trustworthy Computing, Shanghai 200062, China
[c] Information and Education Technology Center, Xi'an University of Finance and Economics, Xi'an 710061, China

ARTICLE INFO

Article history:
Received 10 April 2009
Received in revised form 1 December 2009
Accepted 4 December 2009

Keywords:
t-norm
Euclidean cube
Left-continuity
Direct-sup-preserving

ABSTRACT

Left-continuity of t-norms on the unit interval [0, 1] is equivalent to the property of sup-preserving, but this equivalence does not hold for t-norms on the n-dimensional Euclidean cube $[0, 1]^n$ for $n \geq 2$. Based on the concept of direct poset we prove that a t-norm on $[0, 1]^n$ is left-continuous if and only if it preserves direct sups.

1. Introduction

An increasing mapping $f : [0, 1] \longrightarrow [0, 1]$ is left-continuous if and only if f is sup-preserving, i.e., $f(\sup Z) = \sup f(Z) (Z \subseteq [0, 1])$ [1], especially, a t-norm T [2] on [0, 1] is left-continuous if and only if T preserves sups (or, T is infinitely distributive, or, equivalently, T satisfies the residuation principle [1]). But this is no longer true for t-norms on the n-dimensional Euclidean cube $[0, 1]^n$ as shown by the Counterexample 4.1 of the present paper. Then a natural question arises: does there exist any interrelations between left-continuity and certain kind of sup-preserving property for t-norms on $[0, 1]^n (n \geq 2)$? The aim of the present paper is to give a positive answer to this question for t-norms on $[0, 1]^n$. We prove that a t-norm T on $[0, 1]^n$ is left-continuous if and only if T preserves direct sups.

2. Preliminaries

Throughout this paper we assume that L is a complete lattice and 1, 0, are the largest element and the least element of L, respectively.

Definition 2.1 (*[3,2]*). A triangular norm T (briefly t-norm) on L is a binary operator which is commutative, associative, monotone and has the neutral element 1.

For the sake of convenience, we use $a \otimes b$ instead of $T(a, b)$, then $\otimes$ is a t-norm on L if the following conditions are satisfied:

(i) $a \otimes b = b \otimes a$;
(ii) $(a \otimes b) \otimes c = a \otimes (b \otimes c)$;

☆ Project supported by the NSF of China under the Grant 10771129 and the 211 Construction Found of Shaanxi Normal University.
* Corresponding author at: Institute of Mathematics, Shaanxi Normal University, Xi'an 710062, China. Tel.: +86 29 85308626.
E-mail address: gjwang@snnu.edu.cn (G. Wang).

0893-9659/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2009.12.007

(iii) if $b \le c$, then $a \otimes b \le a \otimes c$;
(iv) $a \otimes 1 = a$,

where $a, b, c \in L$.

Definition 2.2. Let $\otimes$ be a t-norm on L.
(i) If for every non-empty subset Z of L

$$a \otimes \sup\{z \mid z \in Z\} = \sup\{a \otimes z \mid z \in Z\}, \quad a \in L, \tag{1}$$

then we say $\otimes$ is sup-preserving, or $\otimes$ preserves sups.
(ii) Let D be a non-empty subset of L, if $\forall a, b \in L$ there exists $c \in L$ such that $a \le c$ and $b \le c$, then D is said to be a directed set. If (1) holds for directed set Z, then we say that $\otimes$ is direct-sup-preserving (briefly, dsup-preserving), or $\otimes$ preserves direct sups (briefly, $\otimes$ preserves dsups).

Definition 2.3. Let $\otimes$ be a t-norm on L and d be a metric on L. If $\forall \beta \in L$

$$\lim_{x \to \beta, x \le \beta} (a \otimes x) = a \otimes \beta, \quad a \in L. \tag{2}$$

Then $\otimes$ is called left-continuous.

3. Triangular norms on the n-dimensional Euclidean cube $[0, 1]^n$

Definition 3.1. Let $[0, 1]^n$ be the n-dimensional Euclidean cube, define partial order $\le$ on $[0, 1]^n$ pointwisely, i.e., $\forall x, y \in [0, 1]^n$,

$$x \le y \quad \text{iff } x_i \le y_i \ (i = 1, \ldots, n), \tag{3}$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$. Then $([0, 1]^n, \le)$ is a complete lattice, and $1 = (1, \ldots, 1)$, $0 = (0, \ldots, 0)$ are the greatest element and least element of $[0, 1]^n$, respectively. Moreover, the metric d on $[0, 1]^n$ is defined as follows:

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}, \tag{4}$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$.

In the rest of the present paper, $\forall a \in [0, 1]^n$, a_i means the ith coordinate of a.

Note that suppose Z is a non-empty subset of the unit interval $[0, 1]$ and $\sup Z = \beta$, then $\forall \varepsilon > 0$ there exists $z_o \in Z$ such that $\mid \beta - z_o \mid < \varepsilon$. But this is not true for non-empty subset Z of $[0, 1]^n$ whenever $n \ge 2$. In fact, suppose that $n = 2$, $Z = \{a, b\}$, where $a = (0, 1)$, $b = (1, 0)$, then $\beta = \sup Z = (1, 1)$, but $d(\beta, a) = 1 = d(\beta, b)$, hence there is no element $z_o \in Z$ such that $d(\beta, z_o) < \varepsilon$ when $\varepsilon < 1$. However, the situation for directed subset is much better as shown by the following Lemma 3.2. To prove it, we need a simple Lemma 3.1.

Lemma 3.1. *Suppose that $x, y, z \in [0, 1]^n$, then*

$$x \le y \le z \Longrightarrow d(y, z) \le d(x, z). \tag{5}$$

Proof. It follows from (4) and $x \le y \le z$ that $d(y, z) = \sqrt{(z_1 - y_1)^2 + \cdots + (z_n - y_n)^2} \le \sqrt{(z_1 - x_1)^2 + \cdots + (z_n - x_n)^2} = d(x, z)$. □

Lemma 3.2. *Let D be a directed subset of $[0, 1]^n$ and $\beta = \sup D$, then $\forall \varepsilon > 0$ there exists $x^* \in D$ such that $d(\beta, x^*) < \varepsilon$. Moreover, if $y \in D$ and $x^* \le y$, then $d(\beta, y) < \varepsilon$.*

Proof. Suppose that D is a directed subset of $[0, 1]^n$, $\beta = \sup D$ and $\varepsilon > 0$. Since the order on $[0, 1]^n$ is pointwisely defined, it follows that

$$\sup\{x_i \mid x \in D\} = \beta_i, \quad i = 1, \ldots, n. \tag{6}$$

Hence there exists $x(i) \in D$ such that

$$0 \le \beta_i - (x(i))_i < \frac{\varepsilon}{\sqrt{n}}, \quad i = 1, \ldots, n. \tag{7}$$

Since D is a directed subset it follows that there exists $x^* \in D$ such that

$$x(i) \le x^* \le \beta, \quad i = 1, \ldots, n. \tag{8}$$

Therefore we have from (8) and (7) that

$$0 \le \beta_i - x_i^* < \frac{\varepsilon}{\sqrt{n}}, \quad i = 1, \ldots, n.$$

Hence

$$d(\beta, x^*) = \sqrt{(\beta_1 - x_1^*)^2 + \cdots + (\beta_n - x_n^*)^2} < \sqrt{n\left(\frac{\varepsilon}{\sqrt{n}}\right)^2} = \varepsilon.$$

Suppose $y \in D$ and $x^* \leq y$, then it follows from Lemma 3.1 that $d(\beta, y) \leq d(\beta, x^*) < \varepsilon$. □

4. A necessary and sufficient condition for t-norms on $[0, 1]^n$ being left-continuous

To prove the main result, we need one more lemma.

Lemma 4.1. *Suppose that*

$$(0, 1]^n = \{x \in [0, 1]^n \mid 0 \ll x\},$$

where $x \ll y$ is defined by $x_i < y_i (i = 1, \ldots, n)$. Let $\otimes$ be a dsup-preserving t-norm on $[0, 1]^n$ and be left-continuous on $(0, 1]^n$, then $\otimes$ is left-continuous on $[0, 1]^n$.

Proof. We prove Lemma 4.1 by induction. If $n = 1$, $\otimes$ is left-continuous on $(0, 1]$, then it is left-continuous on $[0, 1]$ because $x \leq 0$ if and only if $x = 0$ and $\otimes$ is certainly left-continuous at 0.

Now assume that Lemma 4.1 is valid for $n = k$ and $\otimes$ is dsup-preserving on $[0, 1]^{k+1}$ and is left-continuous on $(0, 1]^{k+1}$. For $i = 1, 2, \ldots, k + 1$, let

$$E_i = \{x \in [0, 1]^{k+1} | x_i = 0\},$$

and

$$E = \bigcup_{i=1}^{k+1} E_i,$$

then each E_i is a lower set in $I^{k+1}$and $E \cup (0, 1]^{k+1} = I^{k+1}$. It is clear that each E_i is a k-dimensional boundary side face of $[0, 1]^{k+1}$and thus E_i and $[0, 1]^k$ are isometric. Let $\beta \in E$, then there exists $i \leq k + 1$ such that $\beta \in E_i$. Without any loss of generality we can suppose that $i = k + 1$, i.e., $\beta \in E_{k+1}$. It is clear that for $x \in [0, 1]^{k+1}$, $x \leq \beta$ imply that $x \in E_{k+1}$. Since E_{k+1} and $[0, 1]^k$ are isometric, $\otimes$ is dsup-preserving and left-continuous on $(0, 1]^k \subset (0, 1]^{k+1}$, therefore it follows from the induction hypothesis that $\otimes$ is left-continuous on E_{k+1}. Since $E \cup (0, 1]^{k+1} = [0, 1]^{k+1}$, $\otimes$ is left-continuous on $[0, 1]^{k+1}$. This proves Lemma 4.1. □

Theorem 4.1. *Let $\otimes$ be a t-norm on $[0, 1]^n$, then $\otimes$ is left-continuous if and only if $\otimes$ preserves dsups.*

Proof. Suppose that $\otimes$ is left-continuous, D is a directed subset of $[0, 1]^n$, $\beta = \sup D$ and

$$\sup\{a \otimes x \mid x \in D\} \neq a \otimes \beta. \tag{9}$$

Let

$$\alpha = \sup\{a \otimes x \mid x \in D\}, \tag{10}$$

then it follows from (9) and the monotonicity of $\otimes$ and $\beta = \sup D$ that $\alpha < a \otimes \beta$. Hence

$$d(\alpha, a \otimes \beta) = \delta > 0. \tag{11}$$

Suppose that $x, y \in D$, then there exists $z \in D$ such that $x \leq z$ and $y \leq z$, hence $a \otimes x \leq a \otimes z$ and $a \otimes y \leq a \otimes z$, and $\{a \otimes z \mid x \in D\}$ is a directed subset of $[0, 1]^n$. Then it follows from (10) and Lemma 3.2 that there exists $x^* \in D$ such that $d(a \otimes x^*, \alpha) < \frac{\delta}{2}$. Since $\otimes$ is left-continuous it follows that there is $\varepsilon > 0$ such that

$$d(\beta, x) < \varepsilon, x \leq \beta \Longrightarrow d(a \otimes x, a \otimes \beta) < \frac{\delta}{2}. \tag{12}$$

Let x be any element of D satisfying the above condition. Since D is directed there exists $\bar{x} \in D$ such that $x \leq \bar{x}$, $x^* \leq \bar{x}$. Then it follows from (10) that $a \otimes x^* \leq a \otimes \bar{x} \leq \alpha$, hence it follows from Lemma 3.1 that

$$d(a \otimes \bar{x}, \alpha) \leq d(a \otimes x^*, \alpha) < \frac{\delta}{2}.$$

Moreover, since $\bar{x} \in D$, $\beta = \sup D$, we have $\bar{x} \leq \beta$, $a \otimes x \leq a \otimes \bar{x} \leq a \otimes \beta$ and it follows from Lemma 3.1 and (12) that

$$d(a \otimes \bar{x}, a \otimes \beta) \leq d(a \otimes x, a \otimes \beta) < \frac{\delta}{2}. \tag{13}$$

Then we have from (11) and (13) that

$$d(\alpha, a \otimes \beta) \leq d(\alpha, a \otimes \bar{x}) + d(a \otimes \bar{x}, a \otimes \beta) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

This contradicts (11). Hence (9) does not hold and $\otimes$ is dsup-preserving.

Conversely, suppose that $\otimes$ is dsup-preserving on $[0, 1]^n$, we are to prove that $\otimes$ is left-continuous on $[0, 1]^n$. By Lemma 4.1, we only need to prove that $\otimes$ is left-continuous on $(0, 1]^n$.

Suppose that $\beta \in (0, 1]^n$, let $D = \{x \in (0, 1]^n \mid x \ll \beta\}$, then it is easy to verify that D is a directed subset of $(0, 1]^n$ and $\beta = \sup D$. Since $\otimes$ preserves dsups we have

$$a \otimes \beta = a \otimes \sup D = \sup\{a \otimes x \mid x \in D\}. \tag{14}$$

Note that $\{a \otimes x \mid x \in D\}$ is a directed subset of $[0, 1]^n$, it follows from (14) and Lemma 3.2 that for any given positive number ε there exists an $x^* \in D$ such that

$$d(a \otimes x^*, a \otimes \beta) < \varepsilon. \tag{15}$$

Then $\forall y \in D$, $x^* \leq y$ implies that $a \otimes x^* \leq a \otimes y \leq a \otimes \beta$, hence by Lemma 3.1 and (15) we have

$$d(a \otimes y, a \otimes \beta) \leq d(a \otimes x^*, a \otimes \beta) < \varepsilon.$$

This proves that

$$x^* \leq y \ll \beta \Longrightarrow d(a \otimes y, a \otimes \beta) < \varepsilon. \tag{16}$$

Since $x^* \ll \beta$ we have

$$\min\{\beta_1 - x_1^*, \ldots, \beta_n - x_n^*\} = \delta > 0. \tag{17}$$

Suppose that $z \in (0, 1]^n$, define

$$\downarrow z = \{x \in [0, 1]^n \mid x \leq z\}, \Downarrow z = \{x \in [0, 1]^n \mid x \ll z\},$$

then it is clear that $\Downarrow z \neq \emptyset$, $\Downarrow z \subseteq \downarrow z$ and $\Downarrow z$ is a dense subset of $\downarrow z$ with respect to the topology generated by the Euclidean metric d on $(0, 1]^n$. Now suppose that $x \in (0, 1]^n$, $x \leq \beta$ and $d(\beta, x) < \delta$, then

$$x \in \mathbf{B}(\beta, \delta) = \{\gamma \in (0, 1]^n \mid d(\beta, \gamma) < \delta\}.$$

Since x is an interior point of $\mathbf{B}(\beta, \delta)$ there exists $\eta > 0$ such that $\mathbf{B}(x, \eta) \subseteq \mathbf{B}(\beta, \delta)$. Since $x \in (0, 1]^n$, $\Downarrow x \neq \emptyset$. Choose $y \in \Downarrow x \cap \mathbf{B}(x, \eta)$, then $y \ll x$ and it follows from $x \leq \beta$ that $y \ll \beta$. Moreover, it follows from $y \in \mathbf{B}(x, \eta) \subseteq \mathbf{B}(\beta, \delta)$ that $d(\beta, y) < \delta$, hence

$$\beta_i - y_i = \sqrt{(\beta_i - y_i)^2} \leq d(\beta, y) < \delta, \quad i = 1, \ldots, n. \tag{18}$$

From (17) and (18) we have $x^* < y$, and hence we have from (16) that $d(a \otimes y, a \otimes \beta) < \varepsilon$. Since $y \leq x \leq \beta$, it follows from Lemma 3.1 that $d(a \otimes x, a \otimes \beta) < \varepsilon$. Now we have proved that if $\beta \in (0, 1]^n$, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \in (0, 1]^n, x \leq \beta, d(\beta, x) \leq \delta \Longrightarrow d(a \otimes x, a \otimes \beta) < \varepsilon.$$

Therefore $\otimes$ is left-continuous on $(0, 1]^n$. This completes the proof. □

The following example shows that left-continuity of t-norms on $[0, 1]^n$ does not imply the property of sup-preserving.

Counterexample 4.1. Define a binary relation $\otimes$ on $[0, 1]^n$ as follows:

$$(x_1, x_2) \otimes (y_1, y_2) = ((x_1 + x_2 + y_1 + y_2 - x_2y_2 - 2) \vee 0, x_2y_2). \tag{19}$$

It is clear that $\otimes$ is commutative and increasing, and has $(1, 1)$ as its neutral element. Denote the right hand side of (19) by (x_1^*, x_2^*), then

$$((x_1, x_2) \otimes (y_1, y_2)) \otimes (z_1, z_2) = ((x_1^* + x_2^* + z_1 + z_2 - x_2^*z_2 - 2) \vee 0, x_2^*z_2), \tag{20}$$

where

$$x_1^* + x_2^* + z_1 + z_2 - x_2^*z_2 - 2 = (x_1 + x_2 + y_1 + y_2 - x_2y_2 - 2) \vee 0 + z_1 + z_2 - x_2y_2z_2 - 2, x_2^*z_2 = x_2y_2z_2. \tag{21}$$

We use (u_1, u_2) to denote the right hand side of (20), and let $a = x_1 + x_2 + y_1 + y_2 - x_2y_2 - 2$, then in case $a \geq 0$ we have from (21) that

$$(u_1, u_2) = ((x_1 + x_2 + y_1 + y_2 + z_1 + z_2 - x_2y_2z_2 - 4) \vee 0, x_2y_2z_2). \tag{22}$$

If $a < 0$, then

$$(x_1 + x_2 + y_1 + y_2 - x_2y_2 - 2) \vee 0 + z_1 + z_2 - x_2y_2z_2 - 2 = z_1 + z_2(1 - x_2y_2) - 2 \le 0,$$

and

$$\begin{aligned} x_1 + x_2 + y_1 + y_2 + z_1 + z_2 - x_2y_2z_2 - 4 &= (x_1 + x_2 + y_1 + y_2 - x_2y_2 - 2) + z_1 + z_2 + x_2y_2 + 2 - x_2y_2z_2 - 4 \\ &< z_1 + x_2y_2 + z_2(1 - x_2y_2) - 2 \le 0, \end{aligned}$$

hence (22) still holds. Similarly, we have

$$(x_1, x_2) \otimes ((y_1, y_2) \otimes (z_1, z_2)) = (u_1, u_2),$$

therefore $\otimes$ is associative and hence a t-norm on $[0, 1]^2$. It is clear that $\otimes$ is continuous. But $\otimes$ is not sup-preserving. The verification of this fact is simpler than that given in [1]. In fact, let $a = (0.5, 0.5)$, $Z = \{b, c\}$, where $b = (1.0)$, $c = (0, 1)$, then $\sup Z = (1, 1)$, and Z is not a directed subset of $[0, 1]^2$. Since

$$a \otimes \sup Z = a \otimes (1, 1) = a = (0.5, 0.5),$$

and by (19),

$$a \otimes (0, 1) = (0.5, 0.5) \otimes (0, 1) = (0, 0.5),$$
$$a \otimes (1, 0) = (0.5, 0.5) \otimes (1, 0) = (0, 0),$$

hence

$$\sup\{a \otimes z \mid z \in Z\} = (0, 0.5) \vee (0, 0) = (0, 0.5).$$

Therefore (1) does not hold, i.e., $\otimes$ is not sup-preserving.

5. Conclusion

In the present paper we have proved that a t-norm on the n-dimensional Euclidean cube $[0, 1]^n$ is left-continuous if and only if it is dsup-preserving. Moreover, we have constructed a simpler counterexample showing that continuity of t-norms on $[0, 1]^2$ does not guarantee the property of sup-preserving.

Acknowledgements

The authors wish to express their sincere thanks to the anonymous referees for their suggestions which improve the presentation of this paper.

References

[1] G.J. Wang, Introduction to Mathematical Logic and Resolution Principle, Science Press, Beijing, 2008.
[2] B. Schweizer, A. Sklar, Probability Metric Space, North-Holland, New york, 1983.
[3] E.P.K lement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Academic Publishers, Dordrent, 2000.