On exponential stability of nonlinear differential systems with
time-varying delay

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ABSTRACT

General nonlinear differential systems with time-varying delays are considered. Several explicit criteria for exponential stability are presented. An example is given to illustrate the obtained results. To the best of our knowledge, the results of this note are new.

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1. Introduction and preliminaries

Delay differential equations have numerous applications in science and engineering. They are used as models for a variety of phenomena in the life sciences, physics and technology, chemistry and economics, see e.g. [1,2].

Problems of stability of time-delay systems have been studied intensively during the past decades, see e.g. [3–10] and references therein. Recently, problems of exponential stability of differential systems with delays have attracted much attention from researchers, see e.g. [5,7–10]. In this note, we present some new explicit criteria for exponential stability of a class of general nonlinear time-varying differential systems with time-varying delays.

Let $N$ be the set of all natural numbers. For given $m \in N$, let us denote $\mathcal{M} := \{1, 2, \ldots, m\}$. For integers $l, q \geq 1$, $\mathbb{R}^l$ denotes the $l$-dimensional vector space over $\mathbb{R}$ and $\mathbb{R}^{l \times q}$ stands for the set of all $l \times q$-matrices with entries in $\mathbb{R}$. Inequalities between real matrices or vectors will be understood componentwise, i.e. for two real matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{R}^{l \times q}$, we write $A \geq B$ iff $a_{ij} \geq b_{ij}$ for $i = 1, \ldots, l$, $j = 1, \ldots, q$. In particular, if $a_{ij} > b_{ij}$ for $i = 1, \ldots, l$, $j = 1, \ldots, q$, then we write $A \gg B$ instead of $A \geq B$. We denote by $\mathbb{R}^{l \times q}_{\geq 0}$ the set of all nonnegative matrices $A \geq 0$. Similar notations are adopted for vectors. For $x \in \mathbb{R}^n$ and $p \in \mathbb{R}^{l \times q}$ we define $|x| = (|x_i|)$ and $|p| = (|p_{ij}|)$. A norm $\| \cdot \|$ on $\mathbb{R}^n$ is said to be monotonic if $\|x\| \leq \|y\|$ whenever $x, y \in \mathbb{R}^n$, $|x| \leq |y|$. Every $p$-norm on $\mathbb{R}^n$ ($\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$, $1 \leq p < \infty$ and $\|x\|_\infty = \max_{i=1,2,\ldots,n} |x_i|$), is monotonic. In what follows, a norm of vectors on $\mathbb{R}^n$ is monotonic.

For any matrix $M \in \mathbb{R}^{n \times n}$ the spectral abscissa of $M$ is denoted by $\mu(M) = \max\{s : \lambda \in \sigma(M)\}$, where $\sigma(M) := \{z \in \mathbb{C} : \det(zI_n - M) = 0\}$ is the spectrum of $M$.

A matrix $M \in \mathbb{R}^{n \times n}$ is called a Metzler matrix if all off-diagonal elements of $M$ are nonnegative. We now summarize in the following theorem some properties of Metzler matrices.

**Theorem 1.1** ([11]). Suppose that $M \in \mathbb{R}^{n \times n}$ is a Metzler matrix. Then

(i) (Perron–Frobenius) $\mu(M)$ is an eigenvalue of $M$ and there exists a nonnegative eigenvector $x \neq 0$ such that $Mx = \mu(M)x$.

(ii) Given $\alpha \in \mathbb{R}$, there exists a nonzero vector $x \geq 0$ such that $Mx \geq \alpha x$ if and only if $\mu(M) \geq \alpha$.
(iii) \((\ell_n - M)^{-1}\) exists and is nonnegative if and only if \(t > \mu(M)\).
(iv) Given \(B \in \mathbb{R}^{n \times n}_+, C \in \mathbb{C}^{n \times n}\). Then
\[
|C| \leq B \implies \mu(M + C) \leq \mu(M + B).
\]

The following is immediate from Theorem 1.1 and is used in what follows.

**Theorem 1.2.** Let \(M \in \mathbb{R}^{n \times n}\) be a Metzler matrix. Then the following statements are equivalent

(i) \(\mu(M) < 0\);
(ii) \(Mp \ll 0\) for some \(p \in \mathbb{R}^n_+\);
(iii) \(M\) is invertible and \(M^{-1} \leq 0\);
(iv) For given \(b \in \mathbb{R}^n, b \gg 0\), there exists \(x \in \mathbb{R}^n_+\) such that \(Mx + b = 0\);
(v) For any \(x \in \mathbb{R}^n_+ \setminus \{0\}\), the row vector \(x^tM\) has at least one negative entry.

Let \(\mathbb{R}^n\) be endowed with the norm \(\|\cdot\|\) and let \(C([\alpha, \beta], \mathbb{R}^n)\) be the Banach space of all continuous functions on \([\alpha, \beta]\) with values in \(\mathbb{R}^n\) normed by the maximum norm \(\|\phi\| = \max_{\theta \in [\alpha, \beta]} \|\phi(\theta)\|\). In particular, we write \(\mathcal{C}\) instead of \(C([\alpha, \beta], \mathbb{R}^n)\) and denote \(\mathcal{C}_r \coloneqq \{\phi \in \mathcal{C} : \|\phi\| \leq r\}\), for given \(r > 0\).

### 2. Exponential stability of nonlinear differential systems with time-varying delay

Consider a nonlinear differential system with time-varying delays of the form
\[
\dot{x}(t) = A(t)x(t) + F\left(t; x(t), x(t - h_1(t)), \ldots, x(t - h_m(t)), \int_{-h(t)}^0 B(s)x(t + s)ds\right),
\]
where \(t \geq \sigma \geq 0\) and

(i) \(h_k(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+, k \in m\) are given continuous functions such that \(0 < h_k(t) \leq h_k, \ 0 < h(t) \leq h, \ h \geq h_k \ \forall k \in m\), for some positive numbers \(h, h_k, k \in m\);
(ii) \(A(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n}\) and \(B(\cdot) : [-h, 0] \rightarrow \mathbb{R}^{n \times n}\) are given continuous functions.
(iii) \(F(\cdot; \cdot, \ldots, \cdot) : \mathbb{R}_+ \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \rightarrow \mathbb{R}^n\), is a given continuous function such that \(F(t; 0, \ldots, 0) = 0, \ \forall t \geq 0\) and \(F(t; u_1, u_2, \ldots, u_{m+2})\) is (locally) Lipschitz continuous with respect to \(u_1, u_2, \ldots, u_{m+2}\) on each compact subset of \(\mathbb{R}_+ \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n\).

Then (i)–(iii) imply that for a fixed \(\sigma \geq 0\) and a given \(\phi \in \mathcal{C}\), there exists a unique local solution of (1) satisfying the initial condition
\[
x(\sigma + s) = \phi(s), \quad s \in [-h, 0].
\]

This solution is defined and continuous on \([\sigma - h, \gamma]\) for some \(\gamma > \sigma\) and satisfies (1) for every \(t \in [\sigma, \gamma]\) see e.g. [3, 12]. It is denoted by \(x(\cdot; \sigma, \phi)\). Furthermore, if the interval \([\sigma - h, \gamma]\) is the maximum interval of existence of the solution \(x(\cdot; \sigma, \phi)\) then \(x(\cdot; \sigma, \phi)\) is said to be noncontinuable. The existence of a noncontinuable solution follows from Zorn’s lemma and the maximum interval of existence must be open.

**Definition 2.1.** (i) The zero solution of (1) is said to be **locally exponentially stable** if there exist positive numbers \(r, K, \beta\) such that for each \(\sigma \in \mathbb{R}_+\) and each \(\phi \in \mathcal{C}_r\), the solution \(x(\cdot; \sigma, \phi)\) of (1)–(2) exists on \([\sigma - h, \infty)\) and furthermore satisfies
\[
\|x(t; \sigma, \phi)\| \leq Ke^{-\beta(t - \sigma)}, \quad \forall t \geq \sigma.
\]
(ii) The zero solution of (1) is said to be **globally exponentially stable** if there exist positive numbers \(K, \beta\) such that for each \(\sigma \in \mathbb{R}_+\) and each \(\phi \in \mathcal{C}\), the solution \(x(\cdot; \sigma, \phi)\) of (1)–(2) exists on \([\sigma - h, \infty)\) and furthermore satisfies
\[
\|x(t; \sigma, \phi)\| \leq Ke^{-\beta(t - \sigma)}\|\phi\|, \quad \forall t \geq \sigma.
\]

When the zero solution of (1) is locally exponentially stable, globally exponentially stable then we also say that (1) is locally exponentially stable, globally exponentially stable, respectively.

We are now in the position to state the main result of this note whose proof is given in the **Appendix**.
Theorem 2.2. Let \( A(t) := (a_{ij}(t)), \ t \geq 0 \). Suppose there exist \( A_0 := (a_{ij}^{(0)}) \in \mathbb{R}^{n \times n} \) and \( A_k := (a_{ij}^{(k)}) \in \mathbb{R}_+^{n \times n}, \ k \in m+2 \) so that
\[
a_{ij}(t) \leq a_{ij}^{(0)}, \quad \forall t \geq 0, \ i \in \mathbb{n}; \quad |a_{ij}(t)| \leq a_{ij}^{(0)}, \quad \forall t \geq 0, \ i \neq j, \ i, j \in \mathbb{n}
\] (3)

and
\[
|F(t; u_1, \ldots, u_{m+2})| \leq \sum_{k=1}^{m+2} A_k[u_k], \quad \forall t \geq 0; \ \forall u_1, \ldots, u_{m+2} \in \mathbb{R}^n.
\] (4)

If
\[
M := A_0 + \sum_{k=1}^{m+1} A_k + \int_0^0 A_{m+2} |B(s)| \, ds
\]
satisfies one of the equivalent conditions (i)–(v) of Theorem 1.2 then (1) is locally exponentially stable.

In addition, if the function \( F \) is positive homogeneous of degree one with respect to \( u_1, u_2, \ldots, u_{m+2} \), that is, \( F(t; \alpha u_1, \ldots, \alpha u_{m+2}) = \alpha F(t; u_1, \ldots, u_{m+2}) \), for any \( \alpha \geq 0, t \geq 0, u_1, u_2, \ldots, u_{m+2} \in \mathbb{R}^n \), then (1) is globally exponentially stable.

Remark 2.3. It is important to note that if \( F(t; u_1, u_2, \ldots, u_{m+2}) \) is (globally) Lipschitz continuous with respect to \( m+2 \) times \( u_1, u_2, \ldots, u_{m+2} \) on \( \mathbb{R}^m \times \cdots \times \mathbb{R}^n \) and \( F(t; 0, 0, \ldots, 0) = 0, \forall t \geq 0 \), then (4) holds automatically for some \( A_k \in \mathbb{R}^{n \times n}, k \in m+2 \).

In particular, when \( A(\cdot) \) is a constant matrix-valued function, we get the following.

Theorem 2.4. Let \( A(t) \equiv A := (a_{ij}), \forall t \geq 0 \). Suppose there exist \( A_k := (a_{ij}^{(k)}) \in \mathbb{R}_+^{n \times n}, k \in m+2 \) so that (4) holds.

If
\[
M := \text{diag}(a_{11}, a_{22}, \ldots, a_{mm}) + |A - \text{diag}(a_{11}, a_{22}, \ldots, a_{mm})| + \sum_{k=1}^{m+1} A_k + \int_{-h}^0 A_{m+2} |B(s)| \, ds
\]
satisfies one of the equivalent conditions (i)–(v) of Theorem 1.2 then (1) is locally exponentially stable. In addition, if the function \( F \) is positive homogeneous of degree one with respect to \( u_1, u_2, \ldots, u_{m+2} \), then (1) is globally exponentially stable.

To state the next result, we now consider a differential system with delays of the form
\[
\dot{x}(t) = A(t)x(t) + \sum_{k=1}^{m} A_k(t)x(t - h_k(t)) + G \left( t; \int_{-h(t)}^0 B(s)x(t + s) \, ds \right),
\] (5)

where

(i) \( h_k(\cdot), h(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+, k \in m \) are given continuous functions such that \( 0 < h_k(t) \leq h, 0 < h(t) \leq h, h \geq h_k \forall k \in m \), for some positive numbers \( h, h_k, k \in m \);

(ii) \( A(\cdot), A_k(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+^{n \times n}, k \in m \) and \( B(\cdot) : [-h, 0] \rightarrow \mathbb{R}_+^{n \times n} \) are given continuous functions.

(iii) \( G(\cdot; \cdot) : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n \), is a given continuous function satisfying \( G(t; 0) = 0, \forall t \geq 0 \) and is Lipschitz continuous with respect to the second argument on each compact subset of \( \mathbb{R}_+ \times \mathbb{R}^n \).

Then the following is immediate from Theorem 2.2.

Theorem 2.5. Let \( A(t) := (a_{ij}(t)), \ t \geq 0 \). Suppose there exist \( A_0 := (a_{ij}^{(0)}) \in \mathbb{R}^{n \times n} \) and \( A_k \in \mathbb{R}_+^{n \times n}, k \in m+1 \) so that
\[
a_{ij}(t) \leq a_{ij}^{(0)}, \quad \forall t \geq 0; \quad |a_{ij}(t)| \leq a_{ij}^{(0)}, \quad \forall t \geq 0, \ i \neq j, \ i, j \in \mathbb{n}
\]

and
\[
|A_k(t)| \leq A_k, \quad \forall t \geq 0, k \in m; \quad |G(t; u)| \leq A_{m+1} |u|, \quad \forall t \geq 0, \ u \in \mathbb{R}^n.
\]

If \( M := A_0 + \sum_{k=1}^{m} A_k + \int_{-h}^0 A_{m+1} |B(s)| \, ds \) satisfies one of the equivalent conditions (i)–(v) of Theorem 1.2 then (5) is locally exponentially stable. In addition, if the function \( G \) is positive homogeneous of degree one with respect to the second argument then (5) is globally exponentially stable.

In particular, when \( A(\cdot) \equiv A_0 \in \mathbb{R}_+^{n \times n} \) is a Metzler matrix, \( A_k(\cdot) \equiv A_k \in \mathbb{R}_+^{n \times n}, k \in m \) are nonnegative matrices and \( G(\cdot; \cdot) \equiv 0, \) Theorem 2.5 reduces to a very recent result in [7, Theorem 1]. The proof of Theorem 1 given in [7] is complicated. We give below a simple proof which also works for much more general classes of differential systems with delays, see the Appendix.
Remark 2.6. (i) In Theorem 2.5, the assumption $|A_k(t)| \leq A_k, \forall t \geq 0, k \in m$ can be relaxed by $\sum_{k=1}^{m} |A_k(t)| \leq M_0, \forall t \geq 0$, for some $M_0 \in \mathbb{R}_+^n$. In this case, the matrix $M$ is defined by $M := A_0 + M_0 + \int_{-h}^{0} A_{m+1} |B(s)| ds$. For details, see the proof given in the Appendix.

(ii) In the standard book [12, page 154], it has been shown that the equation solution of (1)–(2). We divide the proof into two steps.

Example 2.7. Consider the differential equation with time-varying delays

$$\dot{x}(t) = -a(t)x(t) - \sum_{k=1}^{m} b_k(t)x(t - h_k(t)),$$

is globally exponentially stable for all bounded continuous functions $a, b_k, h_k, k \in m$, provided $a(t) \geq \delta > 0, \sum_{k=1}^{m} |b_k(t)| \leq \gamma \delta, 0 < \gamma < 1, 0 \leq h_k(t) \leq h$, for all $t \in \mathbb{R}$. The proof given in [12] relies completely upon a Razumikhin-type theorem. However, this is immediate from Theorem 2.5 and (i).

A similar example has been found in [13, Example 5.1, page 74]. By applying a Razumikhin-type theorem to the equation $\dot{x}(t) = -ax(t) + b(t)x(t - h), t \geq t_0$ where $a > 0, h > 0$ and $b(t)$ is a continuous function, the authors showed that the equation under consideration is globally exponentially stable if $\sup_{t \geq 0} |b(t)| < a$. One again, this assertion is clear by Theorem 2.5.

We illustrate Theorem 2.2 by a simple example.

Example 2.7. Consider the differential equation with time-varying delays

$$\dot{x}(t) = (-2 + \sin t)x(t) + \sqrt{ae^{-t}x(t)^2 + b(cos \sqrt{t})^2x(t - h(t))^2 + c} \left( \int_{-h(t)}^{0} e^x(t + s) ds \right)^2$$

where $a, b, c \geq 0$ are parameters and $h(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a given continuous function satisfying $0 < h(t) \leq h, \forall t \geq 0$ for some $h > 0$.

Clearly, (6) is of the form (1) with $a(t) := -2 + \sin t, t \geq 0$; and $F(t; u_1, u_2, u_3) := \sqrt{ae^{-t}u_1^2 + b(cos \sqrt{t})^2u_2^2 + cu_3^2}, t \geq 0, u_1, u_2, u_3 \in \mathbb{R}$. Furthermore, it is easy to see that $a(t) := -2 + \sin t \leq -1, \forall t \geq 0$ and $F$ is positive homogeneous of degree one with respect to $u_1, u_2, u_3$ and satisfies

$$F(t; u_1, u_2, u_3) = \sqrt{ae^{-t}u_1^2 + b(cos \sqrt{t})^2u_2^2 + cu_3^2} \leq \sqrt{a}|u_1| + \sqrt{b}|u_2| + \sqrt{c}|u_3|$$

and

$$|F(t; u_1, u_2, u_3) - F(t; v_1, v_2, v_3)| \leq \sqrt{a}|u_1 - v_1| + \sqrt{b}|u_2 - v_2| + \sqrt{c}|u_3 - v_3|,$$

for any $t \geq 0$, and any $u_i, v_i \in \mathbb{R}, i \in \{1, 2, 3\}$. Therefore all hypotheses of Theorem 2.2 hold. Thus, (6) is globally exponentially stable if

$$-1 + \sqrt{a} + \sqrt{b} + \sqrt{c} \int_{-h}^{0} e^s ds < 0,$$

or equivalently

$$-1 + \sqrt{a} + \sqrt{b} + \sqrt{c}(1 - e^{-h}) < 0.$$

Appendix. Proof of Theorem 2.2

Since $M$ is a Metzler matrix, any two of (i)–(v) of Theorem 1.2 are equivalent. We first show that (1) is locally exponentially stable provided (iv) of Theorem 1.2 holds. Let $\phi \in \mathcal{C}$ be given and let $x(t) := x(t; \sigma, \phi), t \in [\sigma - h, \gamma)$ be a noncontinuable solution of (1)–(2). We divide the proof into two steps.

Step I. There exists $\beta > 0$ such that for any $\sigma \geq 0$ and any $r > 0$ and any $\phi \in \mathcal{C}$, we have

$$\|x(t; \sigma, \phi)\| \leq Ke^{-\beta(t-\sigma)}, \forall t \in [\sigma, \gamma),$$

where $K$ depends on $\beta, r$.

By (iv) of Theorem 1.2, there exists $p \in \mathbb{R}_+^m$ such that

$$\left( A_0 + \sum_{k=1}^{m+1} A_k + \int_{-h}^{0} A_{m+2} |B(s)| ds \right) p \ll 0.$$
By continuity, (8) also holds for some $p := (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$, $\alpha_i > 0$, $\forall i \in \mathbb{N}$. Furthermore, (8) implies that

$$
\left( A_0 + A_1 + \sum_{k=2}^{m+1} e^{\beta k} A_k + \int_{-\infty}^0 e^{-\beta t} A_{m+2} |B(s)| ds \right) p \leq -\beta (\alpha_1, \ldots, \alpha_n)^T,
$$

(9)

for some sufficiently small $\beta > 0$. Fix $r > 0$ and choose $K > 0$ such that $|\phi(t)| \leq Ke^{-\beta t} p$ for any $t \in [-h, 0]$ and for any $\phi \in C_r$. Define $u(t) := Ke^{-\beta(t-\sigma)} p$, $t \in [\sigma - h, \infty)$. Set $x(t) := x(t; \sigma, \phi)$, $t \in [\sigma - h, \gamma)$. Then, we have $|x(t)| \leq u(t)$, $\forall t \in [\sigma - h, \sigma]$. We claim that $|x(t)| \leq u(t)$ for any $t \in [\sigma, \gamma)$. Assume on the contrary that there exists $t_0 > \sigma$ such that $|x(t_0)| \leq u(t_0)$. Set $t_1 := \inf\{t \in [\sigma, \gamma) : |x(t)| \leq u(t)\}$. By continuity, $t_1 > \sigma$ and there is $i_0 \in \mathbb{N}$ such that

$$
|x(t)| \leq u(t), \quad \forall t \in [\sigma, t_1), \quad |x_{i_0}(t_1)| > u_{i_0}(t_1), \quad |x_{i_0}(t)| > u_{i_0}(t), \quad \forall t \in (t_1, t_1 + \epsilon),
$$

(10)

for some $\epsilon > 0$. For every $i \in \mathbb{N}$, we have

$$
\frac{d}{dt} |x_i(t)| = \text{sgn}(x_i(t)) \dot{x}_i(t) \leq a_{ii}(t) |x_i(t)| + \sum_{j=1, j \neq i}^{n} a_{ij}(t) |x_j(t)| + \int_{-h(t)}^{0} B(s)x(t + s)ds.
$$

for almost any $t \in [\sigma, \gamma)$. Then (3) implies

$$
\frac{d}{dt} |x_i(t)| \leq \sum_{j=1, j \neq i}^{n} a_{ij}(0) |x_j(t)| + \int_{-h(t)}^{0} B(s)x(t + s)ds,
$$

for almost any $t \in [\sigma, \gamma)$. It follows that for any $t \in [\sigma, \gamma)$

$$
D^+ |x_i(t)| := \limsup_{h \to 0^+} \frac{|x_i(t + h)| - |x_i(t)|}{h} = \limsup_{h \to 0^+} \frac{1}{h} \int_{t}^{t+h} \frac{d}{ds} |x_i(s)| ds \leq \sum_{j=1, j \neq i}^{n} a_{ij}(0) |x_j(t)| + \int_{-h(t)}^{0} B(s)x(t + s)ds,
$$

where $D^+$ denotes the Dini upper-right derivative. Let $A_{m+2} |B(s)| := (c_{ij(s)})$, $s \in [-h, 0]$. Taking (4) into account, we have for any $t \in [\sigma, \gamma)$

$$
D^+ |x_i(t)| \leq \sum_{j=1, j \neq i}^{n} a_{ij}(0) |x_j(t)| + \sum_{j=1}^{n} a_{ij}(1) |x_j(t)| + \sum_{k=2}^{m+1} \sum_{j=1}^{n} a_{ij}(k) |x_j(t - h_k(t))| + \int_{-h(t)}^{0} c_{ij}(s) |x_j(t + s)| ds.
$$

In particular, it follows from (9) and (10) that

$$
D^+ |x_{i_0}(t_1)| \leq a_{i_0i_0}(0) Ke^{-\beta(t_1-\sigma)} \alpha_{i_0} + \sum_{j=1, j \neq i_0}^{n} a_{i_0j}(0) Ke^{-\beta(t_1-\sigma)} \alpha_j + \sum_{j=1}^{n} a_{i_0j}(1) Ke^{-\beta(t_1-\sigma)} \alpha_j + \sum_{k=2}^{m+1} \sum_{j=1}^{n} a_{i_0j}(k) Ke^{-\beta(t_1-\sigma)} e^{\beta h} \alpha_j + \sum_{j=1}^{n} \int_{-h}^{0} c_{i_0j} Ke^{-\beta(t_1-\sigma)} e^{-\beta t} \alpha_j ds
$$

$$
\leq Ke^{-\beta(t_1-\sigma)} \left( \sum_{j=1}^{n} a_{i_0j}(0) \alpha_j + \sum_{j=1}^{n} a_{i_0j}(1) \alpha_j + \sum_{k=2}^{m+1} \sum_{j=1}^{n} a_{i_0j}(k) e^{\beta h} \alpha_j + \sum_{j=1}^{n} \int_{-h}^{0} c_{i_0j} e^{-\beta t} \alpha_j ds \right)
$$

(10)

$$
\leq -\beta Ke^{-\beta(t_1-\sigma)} \alpha_{i_0} = D^+ u_{i_0}(t_1).
$$

However, this conflicts with (10). Therefore

$$
|x(t; \sigma, \phi)| \leq u(t) = Ke^{-\beta(t-\sigma)} p, \quad \forall \sigma \geq 0; \quad \forall \phi \in C_r; \quad \forall t \in [\sigma, \gamma).
$$

By the monotonicity of vector norms, this yields

$$
\|x(t; \sigma, \phi)\| \leq Ke^{-\beta(t-\sigma)}, \quad \forall \sigma \geq 0; \quad \forall \phi \in C_r; \quad \forall t \in [\sigma, \gamma),
$$

for some $K_1 > 0$.  


Step II. We claim that \( \gamma = \infty \) and so (1) is locally exponentially stable.

Seeking a contradiction, we assume that \( \gamma < \infty \). Then it follows from (7) that \( x(\cdot; \sigma, \phi) \) is bounded on \([\sigma, \gamma]\). Furthermore, this together with (1) and (4) imply that \( \dot{x}(\cdot) \) is bounded on \([\sigma, \gamma]\). Thus \( x(\cdot) \) is uniformly continuous on \([\sigma, \gamma]\). Therefore, \( \lim_{t \to \gamma^-} x(t) \) exists and \( x(\cdot) \) can be extended to a continuous function on \([\sigma, \gamma]\). Moreover, the closure of \( \{x_t : t \in [\sigma, \gamma]\} \) is a compact set in \( C \), by Arzéla–Ascoli theorem [14]. Note that

\[
\{(t, x_t) : t \in [\sigma, \gamma]\} \subset [\sigma, \gamma] \times \text{the closure of } \{x_t : t \in [\sigma, \gamma]\}.
\]

Thus, the closure of \( \{(t, x_t) : t \in [\sigma, \gamma]\} \) is a compact set in \( \mathbb{R}_+ \times C \). Since \((\gamma, x_\gamma) \) belongs to this compact set, one can find a solution of (1) through this point to the right of \( \gamma \). This contradicts the noncontinuability hypothesis on \( x(\cdot) \). Thus \( \gamma \) must be equal to \( \infty \).

Finally, we show that (1) is globally exponentially stable provided \( F \) is positive homogeneous of degree one with respect to \( u_1, u_2, \ldots, u_{m+2} \). Let \( \phi \in C \) be given. Since \( F \) is positive homogeneous of degree one with respect to \( u_1, u_2, \ldots, u_{m+2} \), it follows that \( \frac{1}{\|\phi\|} \leq K e^{-\beta(t-\sigma)} \), \( \forall t \geq \sigma \), or equivalently, \( \|x(t; \sigma, \phi)\| \leq K e^{-\beta(t-\sigma)} \|\phi\| \), \( \forall t \geq \sigma \). Here \( K, \beta \) are independent with \( \phi \) and thus (1) is globally exponentially stable. This completes the proof.

References