Note

On the fixed edge of planar graphs with minimum degree five

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Abstract

An edge $e$ of a finite and simple graph $G$ is called a fixed edge of $G$ if $G - e + e' \cong G$ implies $e' = e$. In this paper, we show that planar graphs with minimum degree 5 contain fixed edges, from which we prove that a class of planar graphs with minimum degree one is edge reconstructible.

We consider simple and finite graphs $G = (V(G), E(G))$. Undefined concepts and notations are all from [2]. An edge $e \in E(G)$ is called a fixed edge of $G$ if $G - e + e' \cong G$ implies $e' = e$. A graph $H$ is an edge-reconstruction of graph $G$ if there exists a bijection: $\phi : E(G) \rightarrow E(H)$ such that $G - e \cong H - \phi(e)$ for all $e \in E(G)$; $G$ is edge reconstructible if every edge-reconstruction of $G$ is isomorphic to $G$. The edge form of the reconstruction conjecture [3] claims that every graph with at least four edges is edge-reconstructible. The edge $e$ is a forced edge of $G$ if $G - e + e'$ is an edge-reconstruction of $G$ implies $e' = e$. Obviously, $G$ is edge-reconstructible if it has a forced edge, and a forced edge is also a fixed edge. Thus we may study the edge-reconstruction of some graphs by showing that a fixed edge is a forced edge. In fact, if a subgraph $B$ of $G$ has following properties: for each edge $e \in E(B)$, $G - e + e'$ being an edge-reconstruction of $G$ implies that the ends of $e'$ belong to $V(B)$ and $B - e + e' \cong B$, then a fixed edge of $B$ is a forced edge of $G$. A similar problem was first investigated by Sheehan [5, 6], in which, the graphs with no fixed edge was said to be 1-free. In [7], we conjectured that almost all graphs contain a fixed edge. In this paper, we show that each planar graph of minimum degree 5 contains a fixed edge and, by using this result, we prove an edge-reconstruction theorem.

Let $H$ be a graph. Specifying a vertex, say $v$, of $H$ as root, we obtain a rooted graph, denoted by $H \{v\}$. Two rooted graphs $H \{v\}, H' \{v'\}$ are isomorphic if there exists an

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isomorphism \( \psi \) from \( H \) to \( H' \) such that \( \psi(v) = v' \). An edge \( e \in E(H) \) is a fixed edge of \( H \setminus v \) if \( H - e + e' \{v\} \cong H \setminus v \) implies \( e' = e \). It is easy to verify that each rooted tree contains a fixed edge.

Let \( G \) be a 2-connected planar graph, and let \( G_1 \) be a plane representation of \( G \). Let \( f \) be a face of \( G_1 \); then the boundary \( b(f) \) of \( f \) is a cycle of \( G \). If \( b(f) \) has \( k \) edges, then \( f \) is called a \( k \)-face of \( G_1 \). A vertex of degree \( k \) is called a \( k \)-vertex.

**Theorem A** (Lauri [4]). Let \( G \) be a planar graph with connectivity \( \kappa(G) \geq 2 \) and minimum degree \( \delta(G) = 5 \). Then either \( G \) has two adjacent 5-vertices, or there exists a 5-vertex \( v \in V(G) \) such that \( v \) is only incident to 3-faces in any plane representation of \( G \).

**Theorem 1.** Every planar graph of minimum degree 5 contains a fixed edge.

**Proof.** Let \( G \) be a planar graph, \( \delta(G) = 5 \). If \( G \) contains two adjacent 5-vertices then the edge joining them is a fixed edge. We suppose that no two 5-vertices of \( G \) are adjacent and that \( G \) is connected. We consider two cases:

*Case 1: \( \kappa(G) \geq 2 \).

By Theorem A, there exists a 5-vertex \( v \in V(G) \) such that \( v \) is only incident to 3-faces in any plane representation of \( G \). Let \( N(v) = \{v_0, v_1, v_2, v_3, v_4\} \), and let the face boundaries incident to \( v \) in a plane representation be \( vv_i v_{i+1} \), \( i = 0, 1, 2, 3, 4 \) (mod 4). Then \( vv_i v_{i+1} \), \( i = 0, 1, 2, 3, 4 \), are the face boundaries in any planar representation of \( G \), therefore \( vv_i v_{i+1} \), \( i = 1, 2, 3 \), are the face boundaries in any plane representation of \( G - v_0 \).

Let \( e' \) satisfy that \( G - v_0 + e' \cong G \). Then \( e' \) must be incident to \( v \). If the cycle \( C = vv_4 v_0 v_1 \) is the boundary of a face in every representation of \( G - v_0 \), then \( e' = v_0 v_1 \), and \( v_0 v_1 \) is a fixed edge of \( G \). Otherwise, there exists a representation \( G' \) of \( G - v_0 \), in which \( C \) is not a boundary of any face. Then there exist at least two non-skew bridges, say \( B_1, B_2 \) on \( C \) in \( G' \). Clearly, \( A(C, B_j) = V(C) \cap V(B_j) \neq \{v_4, v_0\} \) or \( \{v_0, v_1\} \), and \( A(C, B_j) \neq \{v, v_4, v_0, v_1\}, j = 1, 2 \). Without loss of generality, assume \( \{v, v_1, v_4\} \subset A(C, B_1) \). Then \( \{v_2, v_3\} \subset V(B_1) \) and \( A(C, B_2) \subset \{v_4, v_0, v_1\} \) and \( \{v_4, v_1\} \subset A(C, B_2) \). Therefore there exists a path \( P_{G - v_0}(v_1, v_4) \) joining \( v_1, v_4 \) in \( B_2 \) which does not pass through \( v_0 \) and \( v_1 \). Using the same method, we can show that, if \( v_0 \) is not a fixed edge, then there must be a path \( P_{G - v_1}(v_0, v_2) \) which does not pass through \( v \) and \( v_1 \). Since \( G \) is a planar graph, \( P_{G - v_0}(v_1, v_4) \) and \( P_{G - v_1}(v_0, v_2) \) must have a common vertex. This implies that \( v_0 \in V(B_1) \). Hence \( A(C, B_1) = \{v, v_4, v_0, v_1\} \), a contradiction. Therefore one of \( v_0 \) and \( v_1 \) must be a fixed edge.

*Case 2: \( \kappa(G) = 1 \).

Let \( B \) be an end-block of \( G \) with a minimum number of edges, where the minimality is taken over all end-blocks of \( G \), and let \( v \) be the cut vertex contained in \( B \). (An end-block of \( G \) is a block containing only one cut vertex.) Then for any \( u \in V(B) \setminus \{v\} \),
Let $G$ be a connected graph with $\delta(G) = 1$ and containing a cycle. Then $G$ has an edge-disjoint decomposition (called its tree decomposition) \cite{1}:

$$G = G^* \cup T_1 \cup T_2 \cup \cdots \cup T_k,$$

where $G^*$ is a maximal subgraph of $G$ with minimum degree \( \geq 2 \), and $T_i$ are disjoint rooted trees each rooted in distinct vertices $x_i \in V(G^*)$ and having no other vertices in $V(G^*)$; $G^*$ is termed the trunk of $G$ and $T_1 \cup T_2 \cup \cdots \cup T_k$ its tree-growth. We use $m(G, T_i \{x_i\})$ to denote the number of rooted trees which are isomorphic to $T_i \{x_i\}$ in the tree-growth of $G$. The following lemmas are clearly true.

**Lemma 2.** Let $\delta(G) = 1$ and $H$ an edge reconstruction of $G$. Then the trunk of $H$ is isomorphic to the trunk of $G$.

**Lemma 3.** Let $\delta(G) = 1$, let $T_i \{x_i\}$ be a rooted tree in the tree-growth of $G$, and let $H$ be an edge-reconstruction of $G$. Then $m(G, T_i \{x_i\}) = m(H, T_i \{x_i\})$.

**Lemma 4.** If $\delta(G) = 1$, $\delta(G^*) \geq 3$ and $G^*$ has a fixed edge, then $G$ is edge reconstructible.

As a corollary of Theorem 1 and Lemmas 2 and 4, we have following edge reconstruction theorem for a class of planar graphs.
Theorem 5. Let $G$ be a planar graph with $\delta(G) = 1$ and $\delta(G^*) = 5$. Then $G$ is edge reconstructible.

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References