Entire functions that share one value with their linear differential polynomials

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Abstract
This paper studies the uniqueness problem on entire function that share a finite, nonzero value with their linear differential polynomials and proves some theorems which generalize some results given by Jank, Mues and Volkmann, P. Li, J.P. Wang and H.X. Yi.

Keywords: Linear differential polynomials; Meromorphic function; Uniqueness; Normal family

1. Introduction and main results

Let \( f(z) \) be a nonconstant meromorphic function in the complex plane \( \mathbb{C} \). We shall use the standard notations in Nevanlinna's value distribution theory of meromorphic functions such as \( T(r, f) \), \( N(r, f) \) and \( m(r, f) \) (see e.g. [14]). The notation \( S(r, f) \) is defined to be any quantity satisfying \( S(r, f) = o(T(r, f)) \) as \( r \to \infty \) possibly outside a set of \( E \) of finite linear measure.

Let \( \zeta \) be a family of holomorphic functions on a domain \( D \subset \mathbb{C} \). We say that \( \zeta \) is normal in \( D \) if every sequence of functions \( \{f_n\} \subset \zeta \) contains either a subsequence which converges to an analytic function \( f \) uniformly on each compact subset of \( D \) or a subsequence which converges to \( \infty \) uniformly on each compact subset of \( D \).

Let \( f(z) \) and \( g(z) \) denote some nonconstant meromorphic functions. We say \( f \) and \( g \) share a value \( b \) IM (CM) iff \( f - b = 0 \Leftrightarrow g - b = 0 \) \((f - b = 0 \Rightarrow g - b = 0)\), ignoring multiplicities (counting multiplicities) [14]. If \( f - a = 0 \) when \( g - a = 0 \) and the order of each zero \( z_0 \) of \( f - a \) is greater than or equal to the order of the zero \( z_0 \) of \( g - a \), we will denote this by \( g - a = 0 \rightarrow f - a = 0 \) (see [17]).

Schwick (see [11]) seems to have been the first to draw a connection between normality criteria and shared values. He proved

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Theorem A. Let \( \zeta \) be a family of meromorphic functions defined in \( D \), and let \( a, b, c \) be three distinct complex numbers. If \( f(z) \) and \( f'(z) \) share \( a, b, c \) in \( D \) for every \( f \in \zeta \), then \( \zeta \) is normal in \( D \).

To state the next theorem we require the following definitions.

Definition 1. Let \( k (\geq 2) \) be an integer; let
\[
L(f) = a_k f^{(k)} + \cdots + a_2 f'' + a_1 f',
\] (1.1)
where \( a_1, a_2, \ldots, a_k \neq 0 \) are constants.

Definition 2. Let \( a \in \mathbb{C} \cup \{\infty\} \), we denote by \( N_r(f - a) \) the counting function of those \( a \)-points of \( f \) whose multiplicities are not less than 2 where each \( a \)-point is counting according to its multiplicity.

Definition 3. Let \( a, b \in \mathbb{C} \cup \{\infty\} \), we denote by \( N(r, f = a | g \neq b) \) the counting function of those \( a \)-points of \( f \), counted according to multiplicity, which are not the \( b \)-points of \( g \).

Definition 4. Let \( a, b \in \mathbb{C} \cup \{\infty\} \), we denote by \( N(r, f = a | g = b) \) the counting function of those \( a \)-points of \( f \), counted according to multiplicity, which are the \( b \)-points of \( g \).

Definition 5. Let \( a \in \mathbb{C} \cup \{\infty\} \), we denote by \( N(r) \) the counting function of those multiple \( a \)-points of \( f' \), counted according to multiplicity, which are the \( a \)-points of \( f \).

Lin and Yang (see [7]) considered the problem on holomorphic function that share a finite nonzero value with their linear differential polynomials and derived the following result.

Theorem B. Let \( \zeta \) be a family of holomorphic functions defined in \( D \); let \( a \) be a complex number; and let \( L(f) \) be defined as (1.1). If for every \( f \in \zeta \), we have \( f = a \iff L = a \), and \( f' = L' = a \) whenever \( f = a \), then \( \zeta \) is normal in \( D \).

Remark 1. In Theorem B, \( a_k = 1 \). If the hypotheses “\( f = a \iff L = a \)” is replaced by \( f = a \Rightarrow L = a \), then the conclusion is not, in general, true. We have a counterexample.

Example 1. Let \( \zeta = \{f_n\} \) on the unit disc \( \Delta \), where
\[
f_n(z) = Ae^{az} + a.
\]
Clearly, \( f = a \Rightarrow f' = L = L' = a \), but
\[
f_n'(0) = \frac{An}{1 + |A + a|^2} \to \infty,
\]
thus we get \( \zeta \) is not normal on \( \Delta \).

In this paper, we use a lemma of J. Chang, M. Fang and L. Zalcman (see [3]) to prove

Theorem 1. Let \( \zeta \) be a family of holomorphic functions in a domain \( D \); let \( a \) be a nonzero finite value; let \( M \) be a positive number; and let \( L(f) \) be defined as (1.1). If for every \( f \in \zeta \), \( f(z) = a \Rightarrow f'(z) = a \), \( f'(z) = a \Rightarrow |L(f)| \leq M \), then \( \zeta \) is normal in \( D \).

In 1977, Rubel and Yang (see [9]) proved the following well-known theorem.

Theorem C. Let \( f \) be a nonconstant entire function. If \( f(z) \) and \( f'(z) \) share values \( a \) and \( b \) CM, then \( f = f' \).

In 1986, Junk, Mues and Volkmann [5] proved the following result.
Theorem D. Let $f$ be a nonconstant entire function, and let $a \neq 0$ be a finite constant. If $f = a \Leftrightarrow f' = a$, $f = a \rightarrow f'' = a$, then $f = f'$. 

Remark 2. From the hypothesis of Theorem D, it can be easily seen that the value $a$ is shared by $f$ and $f'$ CM.

The following counterexample (see [18]) shows that Theorem D is, in general, not true if the $f''$ of Theorem D is replaced by $f^{(k)} (k \geq 3)$.

Let $k (\geq 3)$ be a positive integer, and let $a$ be a $(k - 1)$th root of unity satisfying $a \neq 1$. Set $f(z) = e^{az} + a - 1$. It is easy to know that $f$, $f'$ and $f^{(k)}$ share the value $a$ CM, but $f \neq f'$ and $f \neq f^{(k)}$.

Theorems C and D suggest the following question of Yi and Yang.

Question 1. (See [4]..) Let $f$ be a nonconstant meromorphic function, let $a$ be a finite, nonzero constant, and let $n$ and $m$ ($n < m$) be positive integers. If $f$, $f^{(n)}$ and $f^{(m)}$ share a CM, where $n$ and $m$ are not both even or both odd, must $f = f^{(n)}$?

An example (see [15]) given by L. Yang shows that the answer to the above question is, in general, negative. Very recently, Wang and Yi [12] obtained the following theorem.

Theorem E. Let $f$ be a nonconstant entire function, let $a$ be a finite nonzero constant. If $f$ and $f'$ share the value $a$ CM, and if $f^{(k)} = a$ whenever $f = a$, the $f$ assumes the form

$$f(z) = Ae^{\lambda z} + a - \frac{a}{\lambda},$$

where $A$, $\lambda$ are nonzero constants and $\lambda^{k-1} = 1$.

From the hypothesis of Theorem E, we can easily get $f' = a \rightarrow f^{(k)} = a$, if the hypothesis “$f$ and $f'$ share a CM” is replaced by “$f = a \Rightarrow f' = a$,” what happens? We use the theory of normal families to prove the following theorem.

Theorem 2. Let $f$ be a nonconstant entire function; let $a (\neq 0)$ be a constant; and let $k (\geq 2)$ be an integer. If $f(z) = a \Rightarrow f'(z) = a$. $f'(z) = a \rightarrow f^{(k)}(z) = a$, then $f'(z) = f^{(k)}(z)$.

Clearly, Theorem 2 answers Question 1 and improves Theorems D and E.

In 1999, P. Li [6] obtained the next theorem.

Theorem F. Let $f$ be a nonconstant entire function; let $L(f)$ be defined as (1.1); and let $a (\neq 0)$ be a constant. If $f$ and $f'$ share the value $a$ IM, and $L(f) = L'(f) = a$ whenever $f = a$, then $f(z) = f'(z) = L(f)$.

Remark 3. In Theorem F, the hypothesis “$f$ and $f'$ share the value $a$ IM” suggests that $N(r, \frac{1}{f-a}) \neq S(r, f)$. If the hypothesis is replaced by $f = a \Rightarrow f' = a$, we cannot get $N(r, \frac{1}{f-a}) \neq S(r, f)$. Thus it does not seem that the new problem can be proved by using the methods in Theorem F.

Again, using the theory of normal families, we prove

Theorem 3. Let $f$ be a nonconstant entire function; let $a (\neq 0)$ be a constant. If $f = a \Rightarrow f' = a$, $f' = a \Rightarrow L(f) = L''(f) = a$, then one of the following cases holds:

(i) $f = Ae^{\lambda z} + a - \frac{a}{\lambda}$ and $f' = L(f) = L''(f)$;

(ii) $f = Ae^{\lambda z} + a$ and $f' = L(f) = L''(f)$,

where $\lambda^2 = 1$ and $A$ is a nonzero constant.

In the same way, we can get the conclusion easily.
Lemma 3. Let $f$ be a nonconstant entire function; let $a$ ($\neq 0$) be a constant. If $f = a \Rightarrow f' = a$, $f' = a \Rightarrow L(f) = L'(f) = a$, then one of the following cases holds:

(i) $f = Ae^z$ and $f = f' = L(f) = L'(f)$;
(ii) $f = Ae^z + a$ and $f' = L(f) = L'(f)$.

where $A$ is a nonzero constant.

Remark 4. If the hypothesis “$f'(z) = a \Rightarrow L(z) = L'(z) = a$” is replaced by “$f(z) = a \Rightarrow L(z) = L'(z) = a$,” the conclusion is not generally true, we give the following negative example.

Example 2. (See [13].) Let $f = 1 + 6e^{3z} + 2e^{3z/2}$, and let $L(z) = -9e^{3z} - 6e^{3z/2}$. It is obvious that $L(z)$ is a linear combination of $f'$ and $f''$. One can easily check that $f = 1 \Rightarrow f' = L(f) = L'(f) = 1$. But $f$ does not assume the conclusion of Theorem 3.

From Theorem 4, we can obtain the following corollary immediately.

Corollary 1. Let $f$ be a nonconstant entire function; let $a$ ($\neq 0$) be a constant; and let $k$ ($> 2$) be an integer. If $f(z) = a \Rightarrow f'(z) = a$ and $f'(z) = a \Rightarrow f^{(k)}(z) = f^{(k+1)}(z) = a$, then one of the following cases holds:

(i) $f = Ae^z$ and $f = f' = f^{(k)} = f^{(k+1)}$;
(ii) $f = Ae^z + a$ and $f' = f^{(k)} = f^{(k+1)}$.

where $A$ is a nonzero constant.

2. Some lemmas

Lemma 1. (See [8].) Let $\xi$ be a family of functions holomorphic on the unit disc, all of whose zeros have multiplicity at least $k$, and suppose that there exists $A \geq 1$ such that $|f^{(k)}| \leq A$ whenever $f = 0$, then if $\xi$ is not normal, there exist, for each $0 \leq \alpha \leq k$,

(a) a number $0 < r < 1$;
(b) points $z_n$, $z_n < 1$;
(c) functions $f_n \in \xi$; and
(d) positive number $\rho_n \to \infty$ such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi)$ locally uniformly, where $g$ is a nonconstant entire function on $C$, all of whose zeros have multiplicity at least $k$, such that $g^2(\xi) \leq g^2(0) = kA + 1$.

Here, as usual, $g^2(\xi) = \frac{|g'(\xi)|}{1 + |g(\xi)|^2}$ is the spherical derivative.

Lemma 2. (See [11].) Let $P$ be a nonzero polynomial; let $k$ be a positive integer; and let $g \neq 0$ be a solution of the equation $g^{(k)} = P g$. Then $\rho(g) = 1 + \deg(P)/k$.

Lemma 3. (See [2].) Let $f$ be an entire function, let $M$ be a positive number, if $f^2(z) \leq M$ for any $z \in \mathbb{C}$, then $f$ is of exponential type.

Lemma 4. (See [10].) Let $\xi$ be a family of meromorphic functions in a domain $D$, then $\xi$ is normal in $D$ if and only if the spherical derivatives of functions $f \in \xi$ are uniformly bounded on compact subsets of $D$.

Lemma 5. (See [16].) Let $Q(z)$ be a nonconstant polynomial. Then every solution $F$ of the differential equation $F^{(k)} - e^{Q(z)} F = 1$ is an entire function of infinite order.
Lemma 6. Let $P(z) \not\equiv 0$ be a polynomial and $Q(z)$ be a nonconstant polynomial. Then every solution $F$ of the differential equation $F^{(k)} - P(z)e^{Q(z)}F = 1$ is an entire function of infinite order.

Proof. The proof of this lemma is the same as that of Lemma 5 (see [16]).

Lemma 7. Let $f$ be a transcendental entire function with $\rho(f) \leq 1$; let $L(f)$ be defined as (1.1); let $h$ be a positive number and $a$ be a nonzero constant. If $f(z) = 0 \Rightarrow f'(z) = a$, $f'(z) = a \Rightarrow |L(f)| \leq h$ and $N(r, \frac{f-a}{f}) = O(\log r)$, then $\frac{f-a}{f} = c$, where $c$ is a nonzero constant.

Proof. Form $f(z) = 0 \Rightarrow f'(z) = a$, we get $f(z)$ only has simply zeros. Let $\mu = f' - a$, (2.1) then $\mu$ is an entire function. $f$ is a transcendental function, we get $\mu \not\equiv 0$, then $T(r, \mu) = m(r, \mu) \leq m(r, a) + S(r, f) \leq T(r, f) + S(r, f)$, from this we can get $\rho(\mu) \leq \rho(f) \leq 1$, where $\rho(f)$ denotes the order of $f$,

$N\left(r, \frac{1}{\mu}\right) = N\left(r, \frac{f}{f' - a}\right) = O(\log r) \quad (r \not\in E)$.

So $\mu$ has finite zeros. We set $\mu = P(z)e^{bz}$, where $P(z)$ is a polynomial and $b$ is a constant. From (2.1) we have

$f' - P(z)e^{bz}f = a$. (2.2)

Let $F = \frac{f}{a}$, then

$F' - P(z)e^{bz}F = 1$. (2.3)

If $b \neq 0$, by Lemma 6 we have the order of $f$ is infinite, which is a contradiction. Thus we get $b = 0$ and

$f' = P(z)f + a$. (2.4)

From (2.4) we obtain

$f^{(k)}(z) = P_1(z)f + P_2(z)$, (2.5)

where $P_1(z)$ and $P_2(z)$ are polynomials, $\deg(P_1) = k\deg(P)$, $\deg(P_2) = (k-1)\deg(P)$.

Thus, by (1.1), we have

$L(f) = Q_1(z)f + Q_2(z)$, (2.6)

where $Q_1(z)$ and $Q_2(z)$ are polynomials, $\deg(Q_1) = k\deg(P)$, $\deg(Q_2) = (k-1)\deg(P)$.

We consider two cases:

Case 1. $f$ has finite zeros, from (2.1) we can get $f' - a$ also has finite zeros, so $f$ is a polynomial, which is a contradiction.

Case 2. $f$ has infinite zeros $z_1, z_2, \ldots, z_n, \ldots$, and

$|z_1| \leq |z_2| \leq \cdots \leq |z_n| \leq \cdots, \quad |z_n| \to \infty \quad (n \to \infty)$.

From (2.5), we have $L(z_n) = Q_2(z_n)$. By $|L(z_n)| \leq h$, we see that $Q_2(z)$ is a constant. Thus $P(z)$ is a constant, let $P(z) = c$, $c$ is a nonzero constant. From (2.4) we obtain

$f' - a = c$. (2.7)

This completes the proof of Lemma 7. □
Lemma 8. (See [3].) Let $g$ be a nonconstant entire function with $\rho(g) \leq 1$; and let $k \geq 2$ be an integer; let $a$ be a nonzero finite value. If $g = 0 \Rightarrow g' = a$ and $g' = a \Rightarrow g^{(k)} = 0$, then

$$g(z) = a(z - z_0),$$

where $z_0$ is a constant.

Lemma 9. (See [19].) Let $f_1$ and $f_2$ be nonconstant meromorphic functions satisfying

$$\mathcal{N}(r, f) + \mathcal{N}(r, \frac{1}{f_i}) = S(r), \quad i = 1, 2.$$  \hspace{1cm} (3.1)

Then either

$$N_0(r, 1; f_1, f_2) = S(r)$$

or there exist two integers $s, t$ ($|s| + |t| > 0$) such that

$$f_1^s f_2^t = 1$$

where $N_0(r, 1; f_1, f_2)$ denotes the reduced counting function of $f_1$ and $f_2$ related to the common 1-point and $T(r) = T(r, f_1) + T(r, f_2), S(r) = o(T(r))$ ($r \to \infty, r \notin E$) only depending on $f_1$ and $f_2$.

3. Proof of Theorem 1

Let $\Gamma = \{F = f - a: f \in \Gamma\}$, for every $F \in \Gamma$, we get

$$F = 0 \Rightarrow F' = a, \quad F'' = a \Rightarrow L(F) \leq M. \hspace{1cm} (3.2)$$

Without loss of generality, we may assume $a_k = 1$. If $\Gamma$ is normal on $D$, we can get $\zeta$ is also normal on $D$. Thus we only need to prove $\Gamma$ is normal on $D$.

We may assume that $D = \Delta$, the unit disc. Suppose that $\Gamma$ is not normal on $\Delta$, then by Lemma 1 we can find $F_n \in \Gamma$, $z_n \in \Delta, |z_n| < r < 1$, and $\rho_n \to 0$, such that $g_n(\xi) = \rho_n^{-1} F_n(z_n + \rho_n \xi)$ converges locally uniformly to a nonconstant entire function $g$ on $\mathbb{C}$ which satisfies $g^{(l)}(\xi) \leq g^{(l)}(0) = |a| + 2$. Moreover, $g$ is of order at most one. Again, we may assume that $z_n \to z_0 \in \Delta$.

We claim

$$g = 0 \Rightarrow g' = a, \quad g' = a \Rightarrow g^{(k)} = 0. \hspace{1cm} (3.2)$$

Suppose that $g'(\zeta_0) = 0$, then by Hurwitz’s theorem, there exist $\zeta_n, \zeta_n \to \zeta_0$, such that (for $n$ sufficiently large)

$$g_n(\zeta_n) = \rho_n^{-1} F_n(z_n + \rho_n \zeta_n) = 0. \hspace{1cm} (3.3)$$

Thus $F_n(z_n + \rho_n \zeta_n) = 0$, since $F_n(z) = 0 \Rightarrow F'_n(z) = a$, we have $g'_n(\zeta_n) = F'_n(z_n + \rho_n \zeta_n) = a$. Hence

$$g'(\zeta_0) = \lim_{n \to \infty} g'_n(\zeta_n) = a.$$  \hspace{1cm} (3.4)

Thus we prove $g = 0 \Rightarrow g' = a$. We know that

$$g^{(l)}_n(\xi) = \rho_n^{l-1} F^{(l)}_n(z_n + \rho_n \xi) \to g^{(l)}(\xi) \quad (l = 1, 2, \ldots),$$

thus

$$\rho_n^{k-1} L(F_n)(z_n + \rho_n \xi) = \rho_n^{k-1} F^{(k)}_n(z_n + \rho_n \xi) + \rho_n^{k-1} a_{k-1} F^{(k-1)}_n(z_n + \rho_n \xi) + \cdots + \rho_n^{k-1} a_1 F'_n(z_n + \rho_n \xi)$$

converges locally uniformly to $g^{(k)}(\xi)$ on $\mathbb{C}$.

Suppose that $g'(\eta_0) = a$, then $g'(\xi) \equiv a$, otherwise $g(\xi) = a(z - z_0)$, where $z_0$ is a constant. A simple calculation shows that

$$g^{(2)}(0) \leq |a| < |a| + 2,$$

which is a contradiction.
Then by Hurwitz’s theorem, there exist \( \eta_n, \eta_n \to \eta_0 \), such that (for \( n \) sufficiently large)
\[
g_n'(\eta_n) = F_n'(z_n + \rho_n \eta_n) = a. \tag{3.5}
\]
From (3.1), we have \( |L(F_n)(z_n + \rho_n \eta_n)| \lesssim M \). Thus we have
\[
g^{(k)}(\eta_0) = \lim_{n \to \infty} \rho_n^{k-1} L(F_n)(z_n + \rho_n \eta_n) = 0.
\]
Therefore, we complete the claim.

By Lemma 8, we get
\[
g(z) = a(z - z_0),
\]
where \( z_0 \) is a constant. Then
\[
g(0) \leq |a| < |a| + 2,
\]
which is a contradiction.

Thus \( \Gamma \) is normal on \( \Delta \) and hence on \( D \). This completes the proof of Theorem 1.

4. Proof of Theorem 2

From the assumption, we see that \( f \) is a transcendental entire function. Let us now show that \( f \) is of exponential type.

Set \( \zeta = \{ f(z + w): w \in C \} \), then \( \zeta \) is a family of holomorphic functions on the unit disc \( \Delta \). By the assumption, for any function \( g(z) = f(z + w) \), we have
\[
g(z) = a \Rightarrow g'(z) = a, \quad g'(z) = a \Rightarrow |g^{(k)}(z)| = |a|,
\]
hence by Theorem 1, \( \zeta \) is normal in \( \Delta \). Thus by Lemma 3, there exists \( M > 0 \) satisfying \( f^{\#}(z) \leq M \) for all \( z \in C \). By Lemma 2, \( f \) is of exponential type. Then \( \rho(f) \leq 1 \),
\[
f(z) = a \Rightarrow f'(z) = a, \quad f'(z) = a \Rightarrow f^{(k)}(z) = a. \tag{4.1}
\]
We divide into two cases.

**Case 1.** \( a \) is a picard value of \( f \). Then we can easily get \( f = Ae^{\lambda z} + a \), thus
\[
f' = A\lambda e^{\lambda z} \quad \text{and} \quad f^{(k)} = A\lambda^k e^{\lambda z}.
\]
From \( f' = a \Rightarrow f^{(k)} = a \), we have \( \lambda^{k-1} = 1 \), thus \( f'(z) = f^{(k)}(z) \).

**Case 2.** \( a \) is not a picard value of \( f \). Set
\[
\phi = \frac{f' - a}{f - a} \quad \text{and} \quad \varphi = \frac{f^{(k)} - f'}{f - a} . \tag{4.2}
\]
Noting that \( \phi \) and \( \varphi \) are two entire functions, we consider two subcases.

**Subcase 2.1.** \( \varphi \equiv 0 \), then \( f' = f^{(k)} \).

**Subcase 2.2.** \( \varphi \neq 0 \), then we have
\[
T(r, \varphi) = m(r, \varphi) = S(r, f) = O(\log r) \quad (r \not\in E).
\]
Hence we can get \( \varphi \) is a polynomial. We get
\[
N_{2}\left(r, \frac{1}{f' - a}\right) \leq N\left(r, \frac{1}{\varphi}\right) \leq T(r, \varphi) = O(\log r) \tag{4.3}
\]
and
\[
N(r, f' = a | f \neq a) \leq N\left(r, \frac{1}{\varphi}\right) \leq T(r, \varphi) = O(\log r).
\tag{4.4}
\]
Suppose that \( \phi \equiv 0 \), then \( f' \equiv a \), a contradiction. Thus we have \( \varphi \neq 0 \).
We get
\[ T(r, \phi) = m(r, \phi) \leq m\left(r, \frac{1}{f}\right) + S(r, f) \leq T(r, f) + S(r, f). \] (4.5)

Thus
\[ \rho(\phi) \leq \rho(f) \leq 1. \] (4.6)

We also have
\[ N\left(r, \frac{f - a}{f' - a}\right) = N\left(r, \frac{1}{\phi}\right) \leq N\left(r, \frac{1}{f'} - a\right) + N(\log r) = O(\log r). \] (4.7)

Then by (4.1), (4.6)–(4.7) and Lemma 7, we get
\[ \frac{f' - a}{f - a} = \lambda, \] (4.8)

where \( \lambda \) is a nonzero constant. By (4.8) we have
\[ f(z) = Ae^{\lambda z} + a - \frac{a}{\lambda}. \]

By the hypothesis of Theorem 2, we can get \( f'(z) = f^{(k)}(z) \).

5. Proof of Theorem 3

From the assumption, we see that \( f \) is a transcendental entire function. Let us now show that \( f \) is of exponential type.

Set \( \zeta = \{ f(z + w) : w \in C \} \), then \( \zeta \) is a family of holomorphic functions on the unit disc \( \Delta \). By the assumption, for any function \( g(z) = f(z + w) \), we have
\[ g(z) = a \Rightarrow g'(z) = a, \quad g'(z) = a \Rightarrow |L(z)| = |a|, \]

hence by Theorem 1, \( \zeta \) is normal in \( \Delta \). Thus by Lemma 3, there exists \( M > 0 \) satisfying \( f^\zeta(z) \leq M \) for all \( z \in C \). By Lemma 2, \( f \) is of exponential type. Then \( \rho(f) \leq 1 \).

Noting that
\[ f = a \Rightarrow f = a. \] (5.1)

We consider two cases.

Case 1. \( a \) is a Picard value of \( f \). Then we can easily get \( f = Ae^{\lambda z} + a \), thus \( f' = A\lambda e^{\lambda z} \) and
\[ L(f) = a_1 f' + a_2 f'' + \cdots + a_k f^{(k)} = A e^{\lambda z}[a_1 + a_2 \lambda + \cdots + a_k \lambda^{k-1}], \] (5.2)
\[ L''(f) = a_1 f''' + a_2 f^{(4)} + \cdots + a_k f^{(k+2)} = A \lambda^3 e^{\lambda z}[a_1 + a_2 \lambda + \cdots + a_k \lambda^{k-1}]. \] (5.3)

From
\[ f' = a \Rightarrow L(z) = L''(z) = a, \]
we get
\[ a_1 + a_2 \lambda + \cdots + a_k \lambda^{k-1} = \lambda^2 = 1. \] (5.4)

Thus we have \( f' = L(f) = L''(f) \) and \( f = Ae^{\lambda z} + a \), where \( \lambda^2 = 1 \).
Case 2. $a$ is not a Picard value of $f$. Let
\[ \phi = \frac{f' - a}{f - a}. \]  
(5.5)

We can easily get $\rho(\phi) \leq 1$ and $\phi \not\equiv 0$. We set
\[ \varphi = \frac{L''(f) - f'}{f - a} \quad \text{and} \quad \psi = \frac{L(f) - f'}{f' - a}. \]  
(5.6)

Obviously, $\varphi$ and $\psi$ are two entire functions. Now we consider four subcases.

Subcase 2.1. $\varphi \equiv \psi \equiv 0$. Then $L''(f) = f'$ and $L(f) = f'$. Thus, we get
\[ f''' = f'. \]  
(5.7)

From (5.7), we obtain
\[ f = Ae^z + Be^{-z} + a_0, \]  
(5.8)
where $A$, $B$ and $a_0$ are constants. By (5.1) and (5.8), we derive that
\[ f = a \implies f' = a. \]  
(5.9)

Thus $N(r, f' = a | f \neq a) = 0$, then by Lemma 7 we get
\[ \frac{f' - a}{f - a} = \lambda, \]  
where $\lambda$ is a nonzero constant. Then, we get
\[ f = Ae^{\lambda z} + a - \frac{a}{\lambda}. \]  
(5.10)

Similarly as Case 1, we get $\lambda^2 = 1$ and $f' = L(f) = L''(f)$.

Subcase 2.2. $\psi \equiv 0, \varphi \not\equiv 0$. We get
\[ T(r, \varphi) = m(r, \varphi) = S(r, f) = O(\log r) \quad (r \notin E). \]  
(5.11)

Thus $\varphi$ is a polynomial and
\[ N(r, f' = a | f \neq a) \leq N\left(r, \frac{1}{\varphi}\right) \leq T(r, \varphi) = S(r, f). \]  
(5.12)

Now we discuss two subcases separately.

Subcase 2.2.1. $N(2(r, \frac{1}{f' - a})) = S(r, f)$. Thus
\[ N\left(r, \frac{f - a}{f' - a}\right) \leq N(2(r, \frac{1}{f' - a})) + N(r, f' = a | f \neq a) = S(r, f). \]  
(5.13)

By (5.12) and the hypothesis of Theorem 3, using the similar way of Theorem E we get
\[ f(z) = Ae^{\lambda z} + a - \frac{a}{\lambda}. \]  
(5.14)

Similarly as Case 1, we get $\lambda^2 = 1$ and $f' = L(f) = L'(f)$. 

Subcase 2.2.2.\( N(2, r, \frac{1}{f-a}) \neq S(r, f) \). We can get \( N(r, \frac{1}{f-a}) \neq S(r, f) \). If not, we have

\[
N(2, r, \frac{1}{f'-a}) \leq N(r, \frac{1}{f'-a}) \leq N(r, f' = a \mid f = a) + N(r, f' = a \mid f \neq a) \\
\leq kN(r, \frac{1}{f-a}) + N(r, f' = a \mid f \neq a) \leq kN(r, \frac{1}{f-a}) + S(r, f),
\]

which is a contradiction.

We have

\[
N(2, r, \frac{1}{f'-a}) \leq N(r, f' = a \mid f \neq a) \leq N(r) + N(r, f' = a \mid f \neq a) \leq N(r) + S(r, f).
\]

Thus, we get

\[
N(r) \neq S(r, f).
\] (5.15)

Then

\[
\exists |z_1| \leq |z_2| \leq \cdots \leq |z_n| \leq \cdots, \quad |z_n| \to \infty \quad (n \to \infty),
\] (5.16)

where \( z_n \) is the \( a \)-point of \( f \) and the multiple \( a \)-point of \( f' \). Let us prove the proposition:

\[
f(z_n) = a \Rightarrow f'(z_n) = a \Rightarrow L''(z_n) = a \Rightarrow |L'''(z_n)| \leq M_1 \quad (n = 1, 2, 3, \ldots).
\] (5.17)

If the inequality (5.17) is not right, we suppose

\[
|L'''(z_n)| = b_n \to \infty \quad (n \to \infty).
\] (5.18)

Let \( g_n(z) = f(z + z_n) \), we have \( \{f(z + w) : w \in \mathbb{C}\} \) is normal on \( \Delta \). We see that

\[
\{g_n\} \subset \{f(z + w) : w \in \mathbb{C}\},
\]

hence \( \{g_n\} \) is normal on \( \Delta \). \( \forall g_n \in \{g_n\} \) we have

\[
g_n(0) = f(z_n) = a,
\]

hence \( \{g_n\} \) is uniformly bounded on compact subsets of \( \Delta \). Yet we have \( \{g^{(l)}_n\} \) \( (l = 1, 2, 3, \ldots) \) is uniformly bounded in \( |z| \leq \frac{1}{2} \). Let

\[
L''_n(z) = L''(g_n) = a_1g''_n + a_2g^{(4)}_n + \cdots + a_kg^{(k+2)}_n.
\]

Then \( \{L''_n\} \) is uniformly bounded in \( |z| \leq \frac{1}{2} \). Thus \( \{L''_n\} \) is normal in \( |z| \leq \frac{1}{2} \). We get

\[
L'''_n(0) = \frac{|L'''_n(0)|}{1 + |L''_n(0)|^2} = \frac{|L'''(z_n)|}{1 + |L''(z_n)|^2} = \frac{b_n}{1 + a^2} \to \infty,
\]

which is a contradiction.

Thus we complete the proof of the proposition.

Let

\[
f(z) = a + a(z - z_n) + A_3(z - z_n)^3 + \cdots \quad (n = 1, 2, 3, \ldots).
\] (5.19)

Then

\[
f'(z) = a + 3A_3(z - z_n)^2 + \cdots \quad (n = 1, 2, 3, \ldots),
\] (5.20)

and

\[
L''(f) = a + L'''(z_n)(z - z_n) + \cdots \quad (n = 1, 2, 3, \ldots).
\] (5.21)
We have
\[ \varphi(z_n) = \left. \frac{L'' - f'}{f - a} \right|_{z=z_n} = \frac{L'''(z_n)}{a}. \quad (5.22) \]

Thus
\[ |\varphi(z_n)| = |L''(z_n)| \leq M_1/a \quad (n = 1, 2, 3, \ldots). \quad (5.23) \]

Note that \( \varphi(z) \) is a polynomial and \( z_n \to \infty \) as \( n \to \infty \), we can get \( \varphi(z) \) is a nonzero constant. Let \( \varphi = c_0 \), we obtain
\[ L'' - f' = c_0(f - a). \quad (5.24) \]

From \( \psi \equiv 0 \), we have
\[ L(f) = f'. \quad (5.25) \]

By (5.24) and (5.25) we get
\[ f''' - f' = c_0(f - a). \quad (5.26) \]

Let \( g = f - a \), then
\[ g''' - g' = c_0g, \quad (5.27) \]

and
\[ g = 0 \implies g' = a. \quad (5.28) \]

We denote by \( N^*(r) \) the counting function of those multiple \( a \)-points of \( g' \), counted according to multiplicity, which are the zeros of \( g \). Then
\[ N^*(r) = N(r) \neq S(r, f) = S(r, g). \quad (5.29) \]

In the following, we discuss Eq. (5.27). We divide into two subcases.

**Subcase 2.2.2.1.** The equation \( \lambda^3 - \lambda = c_0 \) has a multiple zero, then we can get its multiplicity is two. Thus, we get
\[ g = (C_{11} + C_{12}z)e^{\lambda_1z} + C_2e^{\lambda_2z}. \quad (5.30) \]

Suppose \( C_{12} \neq 0 \), then
\[ g'' = (2\lambda_1C_{12} + C_{11}\lambda_2^2 + C_{12}\lambda_1^2)e^{\lambda_1z} + C_2\lambda_2e^{\lambda_2z}. \quad (5.31) \]

Let \( a_n \) be the zero of \( g \) and multiple zeros of \( g' - a \), then \( g(a_n) = 0, g''(a_n) = 0 \). From (5.30) and (5.31), we get
\[ 2\lambda_1C_{12} + C_{11}(\lambda_2^2 - \lambda_1^2) + C_{12}(\lambda_1^2 - \lambda_2^2)a_n = 0. \]

By (5.29) we derive \( \lambda_1^2 = \lambda_2^2 \) and \( C_{12} = 0 \), this is a contradiction.

We assume now \( C_{12} = 0 \). Then
\[ g = C_{11}e^{\lambda_1z} + C_2e^{\lambda_2z}. \]

If \( C_{11}C_2 \neq 0 \), similarly we can get \( \lambda_1 = -\lambda_2 \). Thus we have
\[ g = C_{11}e^{\lambda_1z} + C_2e^{\lambda_2z}. \quad (5.32) \]

By (5.28) and (5.32), we can get
\[ g = 0 \implies g' = a. \]

Thus \( N_2(r, \frac{1}{f - a}) = N_2(r, \frac{1}{g - a}) = 0 \), which is a contradiction.

If \( C_{11}C_2 = 0 \), obviously this is a contradiction.
Subcase 2.2.2.2. The equation $\lambda^3 - \lambda = c_0$ only has simple zeros. From (5.28) we get
\[ g = c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z} + c_3 e^{\lambda_3 z}. \] (5.33)

If $c_1 c_2 c_3 = 0$, similarly as above, we can get a contradiction.

If $c_1 c_2 c_3 \neq 0$, let $a_n$ be the zero of $g$ and multiple zero of $g' - a$, then $g(a_n) = 0$, $g'(a_n) = a$, $g''(a_n) = 0$. We can get
\[ e^{\lambda_j z_n} = D_j \quad (1 \leq j \leq 3), \] (5.34)
where $D_j \neq 0$ $(1 \leq j \leq 3)$. Let
\[ f_j = e^{\lambda_j z}/D_j \quad (1 \leq j \leq 3). \]

From (5.34), we get
\[ N^a(r) \leq N_0(r, 1; f_1, f_2). \]

Therefore
\[ N_0(r, 1; f_1, f_2) \neq S(r), \]
and
\[ N(r, f_i) + N \left( r, \frac{1}{f_i} \right) = S(r), \quad i = 1, 2. \]

Thus by Lemma 9, there exist two integers $s_1, t_1$ $(|s_1| + |t_1| > 0)$ such that
\[ f_1^{s_1} f_2^{t_1} = 1. \]

Then $\lambda_1 s_1 + \lambda_2 t_1 = 0$, $\lambda_2 = -s_1 t_1 \lambda_1$. Similarly, we get $\lambda_3 = -s_2 t_2 \lambda_1$. Let $\lambda_1 = t_1 t_2 \lambda = p_1 \lambda$, then
\[ \lambda_2 = -s_1 t_2 = p_2 \lambda, \quad \lambda_3 = -s_2 t_1 = p_3 \lambda. \]

Thus, we know the equation $\lambda^3 - \lambda = c_0$ only have real roots. We derive
\[ \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 = (p_1 p_2 + p_1 p_3 + p_2 p_3) \lambda^2 = -1, \] (5.35)
where $p_1, p_2, p_3$ are integers.

By (5.35), we know there exists a positive integer in $\{p_1, p_2, p_3\}$ and there exists a negative integer in $\{p_1, p_2, p_3\}$. Without loss of generality, we assume $p_1 > 0$ and $p_2 < 0$. Noting that $p_2 \neq p_3$, we suppose $p_3 > p_2$.

From (5.33), we get
\[ g = c_1 e^{p_1 \lambda z} + c_2 e^{p_2 \lambda z} + c_3 e^{p_3 \lambda z}. \] (5.36)

Let
\[ P(z) = c_1 z^{p_1} + c_2 z^{p_2} + c_3 z^{p_3} \] (5.37)
and
\[ Q(z) = \lambda \left[ c_1 p_1 z^{p_1} + c_2 p_2 z^{p_2} + c_3 p_3 z^{p_3} \right]. \] (5.38)

Then
\[ g = P(e^{\lambda z}) \quad \text{and} \quad g' = Q(e^{\lambda z}). \] (5.39)

By (5.28), we obtain that $g(z)$ only has simple zeros. Thus, if $p_1 > p_3$, from (5.37) we get $P(z)$ has $m$ simple roots and $m = p_1 - p_2$. But by (5.38) we get $Q(z) - a$ at most has $m$ roots. Again by (5.28) and (5.39) we get
\[ g = 0 \quad \Leftrightarrow \quad g' = a. \]

Thus $N_2(1 - \frac{1}{g - a}) = N_2(1 - \frac{1}{g - a}) = 0$, which is a contradiction. If $p_1 < p_3$, similarly, we can obtain a contradiction.
Subcase 2.3. \( \psi \not\equiv 0, \varphi \equiv 0 \). In the same way of Subcase 2.2, we get:

If \( N(2(r, \frac{1}{f-a}) = S(r, f) \), then

\[
f(z) = Ae^{\lambda z} + a - \frac{a}{\lambda},
\]

where \( \lambda^2 = 1 \) and \( f' = L(f) = L'(f) \).

If \( N(2(r, \frac{1}{f-a}) \neq S(r, f) \), we get

\[
L(f) - f' = c_1(f-a) \quad \text{and} \quad L'' = f'.
\] (5.40)

From (5.40) we get

\[
f''' + c_1f'' - f' = 0.
\] (5.41)

Thus in the same way of Subcase 2.2, we get a contradiction.

Subcase 2.4. \( \psi \not\equiv 0, \varphi \not\equiv 0 \). In the same way of Subcase 2.2, we can get:

If \( N(2(r, \frac{1}{f-a}) = S(r, f) \), then

\[
f(z) = Ae^{\lambda z} + a - \frac{a}{\lambda},
\]

where \( \lambda^2 = 1 \) and \( f' = L(f) = L'(f) \).

If \( N(2(r, \frac{1}{f-a}) \neq S(r, f) \), we get

\[
L'' - f' = c_0(f-a)
\]

and

\[
L - f' = c_1(f-a).
\]

Thus we have

\[
f''' + c_1f'' - f' - c_0(f-a) = 0.
\]

Then, in the same way of Subcase 2.2, we can get a contradiction.

Thus, we complete the proof of Theorem 3.

For further study, we propose the following question.

**Question.** Let \( f \) be a nonconstant entire function; let \( a \) (\( \neq 0 \)) be a constant; and let \( n (> 2) \) be a positive integer. If \( f = a \Rightarrow f' = a, \ f' = a \Rightarrow L(f) = L^{(n)}(f) = a \), what can we say?

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**References**


