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Nonlinear Second Order Boundary Value Problems: Existence and Regions of Uniqueness*

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It is well known that the nonlinear boundary value problem

$$y''(t) + f(t, y(t), y'(t)) = 0 \tag{1}$$

$$y(a) = A \tag{2}$$

$$y(b) = B \tag{3}$$

has a unique solution $y(t)$ for every pair of real numbers A, B if $f(t, y, y')$ is continuous and satisfies a uniform Lipschitz condition, provided only that $b - a$ is small enough. For sufficient restrictions on the size of the interval $[a, b]$ for a variety of classes of functions f, see [1]–[8].

In [9] the best possible result along these lines was obtained for a class of functions which, in addition to a Lipschitz condition, also satisfies a kind of homogeneity condition $f(t, 0, 0) = 0$. (It has since been possible to remove this condition.) That is, existence and uniqueness were established for the problem on every interval $[a, b]$ of length $b - a < M$, where M depends only on the Lipschitz constants and is the best possible such "constant," sometimes $+\infty$ (see also [10]).

For many practical applications, however, these results need to be modified so as to include some functions $f(t, y, y')$ which are not Lipschitzian for *all* y, but only for y in some interval, and so as to permit at least some kinds of singularities. For example, many applied boundary value problems are on an interval $[a, +\infty)$, with the boundary condition (3) being $y(+\infty) = 0$.

Our main result, Theorem 1, is stated in a form that permits a singularity of some sort at the right hand end of the interval (but nowhere else). The singularity could just as well be at the left end instead, of course, but only one singularity is permitted and it must be at an end point.

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Following the theorem we discuss two different examples in some detail in order to give some idea of the theorem's utility and to illustrate the kinds of applications we have in mind.

As in [9], [10], instead of the usual Lipschitz condition we assume a set of one sided conditions which is no more restrictive but is much more useful, as we have observed before. This time, however, the Lipschitz "constants" are permitted to be functions of the independent variable t in order to allow for the singularity. Thus we require the function $f(t, y, y')$ to be continuous and satisfy

$$G_1(t, y - x, y' - x') \leqslant f(t, y, y') - f(t, x, x') \leqslant G_2(t, y - x, y' - x'), \quad (4)$$

where

$$G_2(t, y, y') = \begin{cases} L_2(t)y' + K_2(t)y, & y \geqslant 0, \quad y' \geqslant 0 \\ L_1(t)y' + K_2(t)y, & y \geqslant 0, \quad y' \leqslant 0 \\ L_1(t)y' + K_1(t)y, & y \leqslant 0, \quad y' \leqslant 0 \\ L_2(t)y' + K_1(t)y, & y \leqslant 0, \quad y' \geqslant 0 \end{cases}$$

$$G_1(t, y, y') = \begin{cases} L_1(t)y' + K_1(t)y, & y \geqslant 0, \quad y' \geqslant 0 \\ L_2(t)y' + K_1(t)y, & y \geqslant 0, \quad y' \leqslant 0 \\ L_2(t)y' + K_2(t)y, & y \leqslant 0, \quad y' \leqslant 0 \\ L_1(t)y' + K_2(t)y, & y \leqslant 0, \quad y' \geqslant 0 \end{cases}$$

and $L_i(t)$, $K_i(t)$, $i = 1, 2$ are continuous on $[a, b)$. We shall also require the "homogeneity" condition

$$f(t, 0, 0) = 0. \quad (5)$$

THEOREM 1. *Let $I = [a, b)$, $b = +\infty$ is allowed, $J = [m, M]$ denote intervals of the real line R. For $t \in I$, $y \in J$, $y' \in R$ let $f(t, y, y')$ be continuous and satisfy (4) and (5). If the two problems*

$$\begin{aligned} u_i''(t) + G_i(t, u_i(t), u_i'(t)) &= 0 \\ u_i(a') = A' \qquad u_i(b') &= B' \end{aligned} \quad (6)$$

have unique solutions on every subinterval $[a', b']$ for arbitrary A', B', and if for $a' = a$, $b' = b$, $A' = A$, $B' = B$ the ranges are subsets of J, then the problem

$$y''(t) = f(t, y(t), y'(t)) = 0 \quad (1)$$

$$y(a) = A \quad (2)$$

$$y(b) = B \quad (3)$$

has a unique solution, $y(t)$, which remains in J, and it satisfies

$$u_1(t) \leqslant y(t) \leqslant u_2(t).$$

Note that the assumption of uniqueness for (6) implies, in particular, that no nontrivial solution can have two zeros on $[a, b]$. If $L_i(t)$, $K_i(t)$ are constants, the uniqueness of solutions of (6) is guaranteed whenever the interval is small enough that no nontrivial solution has two zeros on $[a, b]$. (This has been shown in [9].)

For the proof of this theorem we need two lemmas, the first of which is essentially the same as Lemma 2 of [9]. For completeness we include the proof.

LEMMA 1. *Suppose $f(t)$, $g_1(t)$, $g_2(t)$ are piecewise continuous functions on $[a, b)$ and $u(t)$, $v(t)$ are nonnegative functions satisfying*

$$u''(t) + \begin{Bmatrix} g_1(t)\, u'(t) \text{ if } u'(t) \geqslant 0 \\ g_2(t)\, u'(t) \text{ if } u'(t) \leqslant 0 \end{Bmatrix} + f(t)\, u(t) \geqslant 0$$

$$v''(t) + \begin{Bmatrix} g_1(t)\, v'(t) \text{ if } v'(t) \geqslant 0 \\ g_2(t)\, v'(t) \text{ if } v'(t) \leqslant 0 \end{Bmatrix} + f(t)\, v(t) \leqslant 0$$

on (a, b) with

$$u(t_0) = v(t_0)$$
$$u'(t_0) = v'(t_0)$$

for some $t_0 \in [a, b)$. Let I be a maximal interval containing t_0 which does not contain a zero of u or v in its interior. Then

$$u(t) \geqslant v(t)$$

in I.

PROOF. Define $\theta(t)$, $\omega(t)$ on the closure of I by

$$\theta(t) = \frac{\pi}{2} - \tan^{-1} \frac{u'(t)}{u(t)}$$

$$\omega(t) = \frac{\pi}{2} - \tan^{-1} \frac{v'(t)}{v(t)}$$

at interior points and by continuity at end points. Then $\theta(t_0) = \omega(t_0)$. We shall show that

$$0 \leqslant \omega(t) \leqslant \theta(t) \qquad \text{for} \qquad t \leqslant t_0\,,$$
$$\theta(t) \leqslant \omega(t) \leqslant \pi \qquad \text{for} \qquad t \geqslant t_0\,.$$

Differentiating $\theta(t)$, we have

$$\theta'(t) = -\frac{\dfrac{u''(t)}{u(t)} - \left(\dfrac{u'(t)}{u(t)}\right)^2}{1 + \left(\dfrac{u'(t)}{u(t)}\right)^2} \leqslant \frac{\begin{Bmatrix} g_1 \dfrac{u'}{u} \text{ if } u' > 0 \\ g_2 \dfrac{u'}{u} \text{ if } u' < 0 \end{Bmatrix} + f + \left(\dfrac{u'}{u}\right)^2}{1 + \left(\dfrac{u'}{u}\right)^2},$$

or

$$\theta' \leqslant \cos^2\theta + \begin{Bmatrix} g_1 \sin\theta\cos\theta \text{ if } \cos\theta \geqslant 0 \\ g_2 \sin\theta\cos\theta \text{ if } \cos\theta \leqslant 0 \end{Bmatrix} + f\sin^2\theta,$$

and similarly

$$\omega' \geqslant \cos^2\omega + \begin{Bmatrix} g_1 \sin\omega\cos\omega \text{ if } \cos\omega \geqslant 0 \\ g_2 \sin\omega\cos\omega \text{ if } \cos\omega \leqslant 0 \end{Bmatrix} + f\sin^2\omega.$$

Hence

$$(\theta - \omega)' \leqslant \varphi(\theta - \omega),$$

where

$$\varphi = \frac{\begin{Bmatrix} g_1 \sin 2\theta \text{ if } \cos\theta \geqslant 0 \\ g_2 \sin 2\theta \text{ if } \cos\theta \leqslant 0 \end{Bmatrix} - \begin{Bmatrix} g_1 \sin 2\omega \text{ if } \cos\omega \geqslant 0 \\ g_2 \sin 2\omega \text{ if } \cos\omega \leqslant 0 \end{Bmatrix}}{2\theta - 2\omega} + (f - 1)\frac{\sin^2\theta - \sin^2\omega}{\theta - \omega}$$

is clearly bounded and integrable. It follows that

$$\frac{d}{dt}\left((\theta(t) - \omega(t)) \exp\left(-\int_{t_0}^{t} \varphi\right)\right) \leqslant 0,$$

and hence

$$(\theta(t) - \omega(t)) \exp\left(-\int_{t_0}^{t} \varphi\right)$$

is monotone decreasing. This means that for $t \leqslant t_0$

$$(\theta(t) - \omega(t)) \exp\left(-\int_{t_0}^{t} \varphi\right) \geqslant \theta(t_0) - \omega(t_0) = 0,$$

so

$$\theta(t) - \omega(t) \geqslant 0 \quad \text{for} \quad t \leqslant t_0 .$$

Similarly

$$\theta(t) - \omega(t) \leqslant 0 \quad \text{for} \quad t \geqslant t_0 .$$

Now suppose there were some point $t_1 > t_0$ for which $u(t_1) < v(t_1)$. Let $t_2 < t_1$ be the last point to the right of t_0 for which $u(t_2) = v(t_2)$. Since $\pi > \omega(t_2) \geqslant \theta(t_2) \geqslant 0$, and since

$$\tan \omega(t_2) = \frac{v(t_2)}{v'(t_2)}$$

$$\tan \theta(t_2) = \frac{u(t_2)}{u'(t_2)},$$

we can conclude that either $v'(t_2)$, $u'(t_2)$ are both of the same sign with $v'(t_2) < u'(t_2)$, or else that $v'(t_2) \leqslant 0$ and $u'(t_2) \geqslant 0$. But in either case we would have $u(t) > v(t)$ immediately to the right of t_2, which is a contradiction of the assumptions regarding t_2. It follows that $u(t) \geqslant v(t)$ for $t \geqslant t_0$.

Similarly $u(t) \geqslant v(t)$ for $t \leqslant t_0$.

LEMMA 2. *Let $u_i(t)$, $i = 1$ or 2, satisfy (6), (2), (3) and let $y(t)$ satisfy (1), (2). If $y(t) \leqslant u_1(t)$ (or $y(t) \geqslant u_2(t)$) at some point $t = t_0 \in (a, b)$, and $y(t) \in J$ for $a \leqslant t \leqslant t_0$, then the same inequality holds for all $t \in [t_0, b]$, or $y(t)$ leaves J at some point of $[t_0, b]$.*

PROOF. We shall treat only the case $A > 0$, $B < 0$, since none of the other cases is more complicated nor more difficult. Suppose that at $t_0 \in (a, b)$, $u_1(t_0) > y(t_0)$. By definition, $u_1(a) = y(a)$. Let t_1 be the first point to the left of t_0 for which $u_1(t_1) = y(t_1)$ and suppose first that $u_1(t_1) > 0$. Define $\bar{u}_1(t)$ as the solution to (6), $i = 1$, satisfying $\bar{u}_1(t_1) = y(t_1)$, $\bar{u}_1'(t_1) = y'(t_1)$. Then by Lemma 1,

$$\bar{u}_1(t) \geqslant y(t)$$

for $t_1 \leqslant t \leqslant$ the first zero to the right of t_1 (if any) of $y(t)$, or until $y(t)$ leaves J. Now

$$u_1(t_1) = \bar{u}_1(t_1) = y(t_1)$$

and

$$u_1'(t_1) > y'(t_1) = \bar{u}_1'(t_1).$$

By the uniqueness of the solutions to the boundary value problem (6), (2), (3) on all subintervals of $[a, b]$ it follows that

$$u_1(t) \geqslant \bar{u}_1(t) \geqslant y(t)$$

for $t_1 \leqslant t \leqslant$ the first zero of $y(t)$, or until $y(t)$ leaves J.

If $y(t)$ has no zero in (t_1, b), there is nothing more to prove. If $y(t)$ does have a zero (it cannot have more than one), say at t_2, define $\tilde{u}_1(t)$ as the solution to (6) which satisfies $\tilde{u}_1(t_2) = y(t_2) = 0$, $\tilde{u}_1'(t_2) = y'(t_2)$. By Lemma 1

$$\tilde{u}_1(t) \geqslant y(t)$$

for $t_1 \leqslant t \leqslant t_2$. Likewise by Lemma 1 applied to $-y(t)$ and $-\tilde{u}_1(t)$, we get $-y(t) \geqslant -\tilde{u}_1(t)$ or

$$\tilde{u}_1(t) \geqslant y(t)$$

for $t_2 \leqslant t \leqslant b$, or y leaves J on this interval. Thus if $y(t)$ does not leave J on $[t_1, b]$, we have

$$y(t) \leqslant u_1(t), \qquad t_1 \leqslant t \leqslant t_2$$
$$y(t) \leqslant \tilde{u}_1(t), \qquad t_1 \leqslant t \leqslant b.$$

On (t_1, t_2)

$$u_1(t_1) = y(t_1) \leqslant \tilde{u}_1(t_1)$$
$$u_1(t_2) \geqslant y(t_2) = \tilde{u}_1(t_2).$$

Thus there is a $t_3 \in (t_1, t_2)$ for which

$$u_1(t_3) = \tilde{u}_1(t_3).$$

By the uniqueness of solutions to (6), (2), (3) on subintervals of $[a, b]$,

$$u_1(t) \geqslant \tilde{u}_1(t)$$

for $t_3 \leqslant t \leqslant b$. In particular

$$u_1(t) \geqslant \tilde{u}_1(t) \geqslant y(t)$$

for $t_3 \leqslant t_2 \leqslant t \leqslant b$, or $y(t)$ leaves J. This completes the proof for the case $u_1(t_1) > 0$.

If $u_1(t_1) \leqslant 0$, the proof is essentially the same and is therefore omitted.

Proof of the Theorem. We first prove that the boundary value problem (1), (2), (3) cannot have more than one solution. The argument is essentially the same as that in [9]. Let $y_1(t)$, $y_2(t)$ be two distinct solutions, if possible, and put $z(t) = y_1(t) - y_2(t)$. Without loss of generality we may assume b is the first zero of $z(t)$ to the right of a, and that $z'(a) > 0$. Then z satisfies

$$z''(t) + G_2(t, z(t), z'(t)) \geqslant 0$$
$$z(a) = 0 \qquad z(b) = 0$$

and $z(t) > 0$ for $a < t < b$. Let $x(t)$ be the solution of

$$x''(t) + G_2(t, x(t), x'(t)) = 0$$
$$x(a) = 0 \qquad x'(a) = z'(a).$$

By Lemma 1, $x(t)$ has a zero on $(a, b]$, since $z(t)$ does. But this contradicts the hypothesis that no nontrivial solution of (6) has two zeros on $[a, b]$, and the contradiction establishes the stated uniqueness.

We call attention to the fact, which we shall use, that no two distinct solutions of (1), (2) can meet again in $[a, b]$, in view of the uniqueness just proved. We also observe at this time that solutions $y(t)$ of (1) which remain in J are continuous (for any fixed t in (a, b)) with respect to initial conditions, because of the fact that $f(t, y, y')$ satisfies a Lipschitz condition.

Our next step is to show that every solution $y(t)$ of (1) and (2) either can be continued as far as $t = k$, for every $k < b$, or else $y(t)$ leaves J. (Local existence of solutions is already assured.) For $a < k < b$, let

$$L = \sup \{| L_i(t) | : a \leqslant t \leqslant k\},$$
$$K = \sup \{| K_i(t) | : a \leqslant t \leqslant k\}, \qquad W = \max \{| m | , | M |\}.$$

Then so long as $y(t) \in J$ we have the uniform bound

$$| y'(t) | \leqslant | y'(a) | + 2LW + KW(k - a).$$

For since, by (4),

$$\begin{Bmatrix} -Ky - Ly' \text{ if } y \geqslant 0, y' \geqslant 0 \\ -Ky + Ly' \text{ if } y \geqslant 0, y' \leqslant 0 \\ Ky + Ly' \text{ if } y \leqslant 0, y' \leqslant 0 \\ Ky - Ly' \text{ if } y \leqslant 0, y' \geqslant 0 \end{Bmatrix} \leqslant y''(t) \leqslant \begin{Bmatrix} Ky + Ly' \text{ if } y \geqslant 0, y' \geqslant 0 \\ Ky - Ly' \text{ if } y \geqslant 0, y' \leqslant 0 \\ -Ky - Ly' \text{ if } y \leqslant 0, y' \leqslant 0 \\ -Ky + Ly' \text{ if } y \leqslant 0, y' \geqslant 0 \end{Bmatrix},$$

the result follows by integrating over $[a, t]$. It now follows ([11], p. 61, Problem 4) that either $y(t)$ can be continued as far as $t = k$ or else $y(t)$ leaves J.

Let $u_i(t)$, $i = 1$ or 2, be the solutions to the problems

$$u_i''(t) + G_i(t, u_i(t), u_i'(t)) = 0$$
$$u_i(a) = A \qquad u_i(b) = B.$$

We define a sequence of points t_n and two sequences of functions $y_{i,n}(t)$, $i = 1$ or 2, by induction as follows:

Define $y_{i,0}(t)$ as the solutions to the initial value problems (1), (2) and $y'_{i,0}(a) = u_i'(a)$. If both can be continued to $t = k$ for every $k < b$, define $t_1 = \frac{1}{2}(a + b)$, if $b < +\infty$, or $t_1 = a + 1$ if $b = +\infty$. If not, then there is a first point, t_1, for which either $y_{1,0}(t_1) = m$ or $y_{2,0}(t_1) = M$. This defines t_1, and we note that $y_{1,0}(t_1) \leqslant u_1(t_1)$, $y_{2,0}(t_1) \geqslant u_2(t_1)$ by Lemma 2 and the fact that initially $y_{i,0}(t)$ lie outside the interval $[u_1(t), u_2(t)]$. Hence by the continuity of solutions $y(t)$ to (1) with respect to initial conditions, in particular the remarks on p. 23 of [11], there exist initial slopes for which the corresponding solutions $y_{1,1}(t)$, $y_{2,1}(t)$ to (1), (2) with those slopes satisfy

$$y_{1,1}(t_1) = u_1(t_1)$$
$$y_{2,1}(t_1) = u_2(t_1).$$

Assume the points t_n and functions $y_{i,n}(t)$ have been defined for $1 \leqslant n \leqslant N$.

Either both $y_{i,N}(t)$ can be continued to $t = k$ for every $k < b$, in which case we define $t_{N+1} = \frac{1}{2}(t_N + b)$ if $b < +\infty$, or $t_{N+1} = t_N + 1$ if $b = +\infty$, or else there is a first point, t_{N+1}, for which either $y_{1,N}(t_{N+1}) = m$ or $y_{2,N}(t_{N+1}) = M$. Since $y_{1,N}(t_{N+1}) \leqslant u_1(t_{N+1})$ and $y_{2,N}(t_{N+1}) \geqslant u_2(t_{N+1})$, by Lemma 2 and an easy induction on N, it follows by continuity with respect to initial conditions again that there exist (unique) solutions $y_{i,N+1}(t)$ satisfying (1), (2), and $y_{i,N+1}(t_{N+1}) = u_i(t_{N+1})$.

By Lemma 2 and uniqueness, the intervals $[y'_{1,n}(a), y'_{2,n}(a)]$ form a nested sequence of closed intervals, and consequently have a nonempty intersection. Let

$$p \in \bigcap_{n=1}^{\infty} [y'_{1,n}(a), y'_{2,n}(a)].$$

Then the solution $y(t)$ to the initial value problem (1), (2) and $y'(a) = p$ satisfies $y(b) = B$, as desired.

EXAMPLE 1. Our first example illustrates the importance of using the restricted interval for the Lipschitz condition. We consider a problem of Collatz [12], pp. 145-147,

$$y''(t) - \tfrac{3}{2} y^2(t) = 0$$
$$y(0) = 4 \qquad y(1) = 1.$$

(There are two distinct solutions, one of which is expressible in elementary terms while the other involves elliptic functions. The former, namely

$$y(t) = 4(1 + t)^{-2},$$

clearly stays in the interval $[1, 4]$, whereas the latter decreases from 4 to below -10 and then increases to 1.)

In this example our function f is just

$$f(y) = -\tfrac{3}{2} y^2,$$

so that

$$f(y) - f(x) = -\tfrac{3}{2}(y + x)(y - x).$$

Hence if $J = [m, M]$ is an interval containing the subinterval $[1, 4]$, and if $y, x \in J$, then

$$-3M(y - x) \leqslant f(y) - f(x) \leqslant -3m(y - x).$$

Thus the functions G_1, G_2 of the theorem can be taken to be

$$G_1(y, y') = \begin{cases} -3My & \text{if } y \geqslant 0 \\ -3my & \text{if } y \leqslant 0 \end{cases}$$

$$G_2(y, y') = \begin{cases} -3my & \text{if } y \geqslant 0 \\ -3My & \text{if } y \leqslant 0 \end{cases}.$$

Since $M \geqslant 0$ (actually $M \geqslant 4$), the only circumstance under which the problems (6) could fail to have unique solutions is that some solution have two zeros on [0, 1], i.e., if $m \leqslant -\pi^2/3$. Or to put the matter the other way around, the problems (6) have unique solutions on [0, 1] if $m > -\pi^2/3$. And in this case, the solutions to (6) when $A' = A = 4$, $B' = B = 1$, $a = 0$, $b = 1$, are simply the solutions to the linear problems

$$u_1''(t) - 3Mu_1(t) = 0, \qquad u_1(0) = 4, \qquad u_1(t) = 1$$
$$u_2''(t) - 3mu_2(t) = 0, \qquad u_2(0) = 4, \qquad u_2(1) = 1.$$

Hence

$$u_1(t) = 4 \cosh \sqrt{3M}\, t + \left(\frac{1 - 4 \cosh \sqrt{3M}}{\sinh \sqrt{3M}}\right) \sinh \sqrt{3M}\, t$$

$$u_2(t) = 4 \cosh \sqrt{3m}\, t + \left(\frac{1 - 4 \cosh \sqrt{3m}}{\sinh \sqrt{3m}}\right) \sinh \sqrt{3m}\, t,$$

and the ranges of u_1 and u_2 are subsets of J.

Thus the conditions of the theorem are seen to be satisfied. We conclude there is exactly one solution $y(t)$ which remains in J, and we have

$$u_1(t) \leqslant y(t) \leqslant u_2(t).$$

The theorem also tells us that if there is any other solution to the problem (1), (2), (3) it must leave J, which in this case amounts to saying it must drop below $-\pi^2/3$. (There *is* another solution, as we mentioned earlier, and it drops below -10 actually.)

By taking $M = 4$ we get the best lower bound $u_1(t)$, and taking its minimum value for m gives us the best upper bound $u_2(t)$. These bounds furnish quite good approximations to y.

EXAMPLE 2. For our second application we treat a problem on the infinite interval to illustrate the theorem when a singularity is present. We consider R. E. Kidder's [13] similarity solution of the unsteady flow of gas through a semi-infinite porous medium, initially filled with gas at a uniform pressure P_0. At time $t = 0$ the pressure at the outflow face is suddenly reduced from P_0 to P_1 and thereafter maintained at this lower pressure. In terms of a dimension free quantity w, defined by

$$w(z) = \alpha^{-1}\left(1 - \frac{P^2(z)}{P_0^{\,2}}\right), \tag{7}$$

where

$$\alpha = 1 - \frac{P_1^{\,2}}{P_0^{\,2}}, \tag{8}$$

the problem takes the form (see the original paper of Kidder for the details of the physical problem and the reduction of the equation

$$\nabla^2(P^2) = A^2 \frac{\partial P}{\partial t}$$

to an ordinary differential equation)

$$w''(z) + \frac{2z}{\sqrt{1 - \alpha w(z)}} w'(z) = 0,$$

$$w(0) = 1, \qquad w(+\infty) = 0. \tag{9}$$

From (7) and (8) it is clear we should expect $w(z)$ to belong to the interval $J = [0, 1]$, in which case we would have

$$L_1(z) = 2z \leqslant \frac{2z}{\sqrt{1 - \alpha w(z)}} \leqslant 2(1 - \alpha)^{-1/2} z = L_2(z).$$

Hence for comparison equations we take

$$u_1''(z) + \begin{pmatrix} 2zu_1'(z) & \text{if } \; u_1'(z) \geqslant 0 \\ 2(1-\alpha)^{-1/2} zu_1'(z) & \text{if } \; u_1'(z) \leqslant 0 \end{pmatrix} = 0$$

$$u_2''(z) + \begin{pmatrix} 2(1-\alpha)^{-1/2} zu_2'(z) & \text{if } \; u_2'(z) \geqslant 0 \\ 2zu_2'(z) & \text{if } \; u_2'(z) \leqslant 0 \end{pmatrix} = 0.$$

Now neither of these equations has a nontrivial solution with two zeros on $[0, +\infty]$. For example, by integration,

$$u_1'(z) = \begin{Bmatrix} Ce^{-z^2} & \text{if } \; u_1'(z) \geqslant 0 \\ De^{-(1-\alpha)^{-1/2} z^2} & \text{if } \; u_1'(z) \leqslant 0 \end{Bmatrix},$$

which clearly never vanishes (and hence $u_1(z)$ cannot have two zeros) on $(0, +\infty)$ except in the trivial case C or $D = 0$. Thus we readily verify the assumptions of the theorem, and find that

$$u_2(z) = \operatorname{erfc} z = 1 - \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt$$

$$u_1(z) = \operatorname{erfc}\{(1 - \alpha)^{-1/4} z\} = \operatorname{erfc}\left\{ z \sqrt{\frac{P_0}{P_1}} \right\}$$

are the solutions which satisfy the boundary conditions

$$u_i(0) = 1 \qquad u_i(+\infty) = 0.$$

Hence there is exactly one solution $w(z)$ of the problem which lies in $[0, 1]$, and this $w(z)$ satisfies

$$\operatorname{erfc}\left\{z\sqrt{\frac{P_0}{P_1}}\right\} \leqslant w(z) \leqslant \operatorname{erfc} z.$$

The difference between these bounds is

$$\operatorname{erfc} z - \operatorname{erfc} z\sqrt{\frac{P_0}{P_1}} = \frac{2}{\sqrt{\pi}}\int_z^{z\sqrt{P_0/P_1}} e^{-t^2}\,dt \leqslant \frac{2}{\sqrt{\pi}}\, z\left(1 - \sqrt{\frac{P_0}{P_1}}\right) e^{-z^2}.$$

Comparison of these bounds on $w(z)$ with the perturbation solution of (9) in powers of α obtained by Kidder is interesting. He centers attention on his zero order solution as a convenient and moderately accurate solution for all α. This solution is of added interest because it is the exact solution of a certain linearization of the nonlinear partial differential equation. His numerical results suggest that it is everywhere too large, which is what we have just shown above, since his zero order solution is precisely our $u_2(z)$.

Our bounds function as a perturbation solution, since they collapse to the correct one as $\alpha \to 0$. Their use for engineering purposes is facilitated by the extreme ease of computation and the fact that it is easy to assess the accuracy of the approximation.

References

1. E. Picard. Sur l'application des Méthodes d'approximations successives à l'Étude de certaines Équations Différentielles Ordinaires, *J. Math.* **9** (1893), 217-271.
2. E. Picard. "Traité d'Analyse," Gouthier-Villars, Paris, 1929.
3. F. Lettenmeyer. Über die von einem Punkt ausgehenden Integralkurven einer Differentialgleichung 2. Ordnung. *Deutsche Math.* **7** (1944), 56-74.
4. Z. Opial. Sur une Inégalité de C. de la Vallée Poussin dans la Théorie de l'Équation Différentielle du Second Ordre. *Ann. Polon Math.* **6** (1959), 87-91.
5. C. de la Vallée Poussin. Sur l'équation différentielle linéaire du second ordre. *J. Math. Pures Appl.* **8** (1929), 125-144.
6. W. J. Coles and T. L. Sherman. Two-point problems for non-linear second order ordinary differential equations. Math. Research Center, Report 513, 1964.
7. W. Petry. Das Iterationsverfahren zum Lösen von Randwertproblemen gewöhnlicher nichtlinearer Differentialgleichungen zweiter Ordnung. *Math. Z.* **87** (1965), 323-333.
8. J. Schroeder. Neue Fehlerabschätzungen für verschiedene Iterationsverfahren, *Z. Angew. Math. Mech.* **36** (1956), 471-484.
9. P. Bailey and P. Waltman. Existence and uniqueness of solutions to the first boundary value problem for nonlinear second order di × erential equations. *Arch. Rational Mech. Anal.*, to appear.
10. P. Bailey and P. Waltman. On the distance between consecutive zeros for second order differential equations. *J. Math. Anal. Appl.*, to appear.

11. E. A. Coddington and N. Levinson. "Theory of Ordinary Differential Equations," McGraw-Hill, New York, 1955.
12. L. Collatz. "The Numerical Treatment of Differential Equations." Springer Verlag, Berlin, 1960.
13. R. E. Kidder. Unsteady flow of gas through a semi-infinite porous medium. *J. Appl. Mech.* **27** (1957), 329-332.