JOURNAL OF NUMBER THEORY **28**, 306–346 (1988)

Combinatorial Sieves of Dimension Exceeding One*

H. DIAMOND AND H. HALBERSTAM

Mathematics Department, University of Illinois, 1409 West Green Street, Urbana, Illinois 61801

AND

H.-E. RICHERT

Mathematics Department, University of Ulm, Oberer Eselsberg, 7900 Ulm, West Germany

Communicated by P. T. Bateman

Received April 16, 1987

A general sieve for each dimension $\kappa > 1$ is given which improves the sieve estimates of Ankeny and Onishi. The work depends on a combinatorial identity which is invariant under Buchstab iteration and on the solution of a pair of differential-difference equations with side conditions.

1. INTRODUCTION

Let $\mathscr{A}$ be a finite integer sequence whose members are not necessarily positive or distinct. Let $\mathscr{P}$ be a set of primes, $z \geqslant 2$ a real number, and write

$$P(z) := \prod_{\substack{p < z \\ p \in \mathscr{P}}} p, \quad P(z_1, z) := \prod_{\substack{z_1 \leqslant p < z \\ p \in \mathscr{P}}} p = P(z)/P(z_1) \quad (2 \leqslant z_1 \leqslant z). \tag{1.1}$$

The first and simplest objective of sieve theory is to estimate the *sifting function*

$$S(\mathscr{A}, z) := S(\mathscr{A}, \mathscr{P}, z) := |\{a \in \mathscr{A} : (a, P(z)) = 1\}|, \tag{1.2}$$

the number of elements remaining in $\mathscr{A}$ after the removal from $\mathscr{A}$ of all

* The authors gratefully acknowledge support from the National Science Foundation, Grants DMS 86-40693 and 84-00013.

0022-314X/88 $3.00

multiples of primes $p<z$ that belong to $\mathscr{P}$. Thus $S(\mathscr{A},2)=|\mathscr{A}|$, the cardinality of $\mathscr{A}$; and if

$$\mathscr{A}_d := \{a\in\mathscr{A}: a\equiv 0 \bmod d\}, \qquad d\in\mathbb{N}$$

(so that $\mathscr{A}_1=\mathscr{A}$), we have the "inclusion–exclusion" principles

$$S(\mathscr{A},\mathscr{P},z)=\sum_{d\mid P(z)}\mu(d)\,|\mathscr{A}_d| = \sum_{d\mid P(z_1,z)}\mu(d)\,S(\mathscr{A}_d,\mathscr{P},z_1) \tag{1.3}$$

using the basic property

$$\sum_{d\mid n}\mu(d)=\begin{cases}1, & n=1,\\ 0, & n>1,\end{cases} \tag{1.4}$$

of the Moebius function.

It is evident from the first statement in (1.3) that we cannot take matters further unless we have information about the counting functions $|\mathscr{A}_d|$, that is, unless we know something about the way $\mathscr{A}$ is distributed relative to each of the arithmetic progressions $0 \bmod d$, at least for all those natural numbers d that are squarefree and composed of primes from $\mathscr{P}$. Experience shows (see, e.g., Chapter 1 of "Sieve Methods" [6]) that such information is available (at varying levels of depth) for many of the most interesting sequences $\mathscr{A}$, and takes the following form: there exists an approximation X to $|\mathscr{A}|$ and a non-negative multiplicative arithmetic function $\omega(\cdot)$, equal to 1 at 1 and to 0 at the primes not in $\mathscr{P}$, such that the "remainders"

$$R_d := |\mathscr{A}_d| - \frac{\omega(d)}{d}X \tag{1.5}$$

are small, at least on average (in some sense) over squarefree d's that are made up of primes from $\mathscr{P}$ and are not too large; and such that there exist constants $\kappa>0$ and $A\geqslant 2$ so that

$$0\leqslant\omega(p)<p$$

and

$$\prod_{w_1\leqslant p<w}\left(1-\frac{\omega(p)}{p}\right)^{-1}\leqslant\left(\frac{\log w}{\log w_1}\right)^{\kappa}\left(1+\frac{A}{\log w_1}\right) \qquad \text{if} \quad 2\leqslant w_1\leqslant w.$$

This inequality implies at once that

$$\sum_{w_1\leqslant p<w}\frac{\omega(p)}{p}\leqslant\kappa\log\left(\frac{\log w}{\log w_1}\right)+\frac{A}{\log w_1}, \qquad 2\leqslant w_1\leqslant w, \tag{1.6}$$

which makes more apparent that what we assume here about $\omega(\cdot)$ is no more than that $\omega(p)$ is, in a very weak average sense, at most as large as κ. The (smallest such) number κ has come to be known as the *dimension* of the sieve problem under consideration. ("Sifting density" is an alternative name for κ.)

Let

$$V(w) := \prod_{p<w} \left(1 - \frac{\omega(p)}{p}\right); \tag{1.7}$$

then the product condition above requires that

$$(\Omega(\kappa)) \qquad \frac{V(w_1)}{V(w)} \leqslant \left(\frac{\log w}{\log w_1}\right)^\kappa \left(1 + \frac{A}{\log w_1}\right), \qquad 2 \leqslant w_1 \leqslant w.$$

It is noteworthy that $(\Omega(\kappa))$ is virtually the only arithmetic condition (definitions and notations apart) that we impose throughout this account.

Loosely speaking, $\omega(p)/p$ may be viewed as the "probability" that an element a of $\mathscr{A}$ is divisible by a prime p of $\mathscr{P}$, and therefore one expects $S(\mathscr{A}, \mathscr{P}, z)$ to be estimated in terms of $XV(z)$. Our main theorem below shows the extent to which this expectation can be realized in the case of sieve problems of dimension $\kappa \geqslant 1$; but to state this theorem we have to introduce two functions—$F_\kappa(u)$ and $f_\kappa(u)$—as well as two crucial parameters—α_κ and β_κ—and to assume some basic information from [2] about them.

Let $\sigma_\kappa(u)$ be the continuous solution of the differential-difference problem

$$\begin{cases} u^{-\kappa}\sigma(u) = A_\kappa^{-1}, & 0 < u \leqslant 2,\ A_\kappa := (2e^\gamma)^\kappa\, \Gamma(\kappa+1), \\ (u^{-\kappa}\sigma(u))' = -\kappa u^{-\kappa-1}\sigma(u-2), & 2 < u; \end{cases} \tag{1.8}$$

here γ denotes Euler's constant. The basic information that we shall assume throughout this paper is summarized in the following

THEOREM 0. *Let $\kappa \geqslant 1$ be given. Then there exist numbers $\alpha_\kappa, \beta_\kappa$ satisfying*

$$\alpha_\kappa \geqslant \beta_\kappa \geqslant 2 \tag{1.9}$$

such that the simultaneous differential-difference system

$$\begin{array}{lll} \text{(i)} & F(u) = 1/\sigma_\kappa(u), & 0 < u \leqslant \alpha_\kappa, \\ \text{(ii)} & f(u) = 0, & 0 < u \leqslant \beta_\kappa, \\ \text{(iii)} & (u^\kappa F(u))' = \kappa u^{\kappa-1} f(u-1), & u > \alpha_\kappa, \\ \text{(iv)} & (u^\kappa f(u))' = \kappa u^{\kappa-1} F(u-1), & u > \beta_\kappa, \end{array} \tag{1.10}$$

has continuous solutions $F_\kappa(u)$ and $f_\kappa(u)$ having also the properties

$$F_\kappa(u) = 1 + O(e^{-u}), \qquad f_\kappa(u) = 1 + O(e^{-u}), \tag{1.11}$$

$$F_\kappa(u) \text{ decreases monotonically towards } 1 \text{ as } u \to +\infty, \tag{1.12}$$

and

$$f_\kappa(u) \text{ increases monotonically towards } 1 \text{ as } u \to \infty. \tag{1.13}$$

We shall deal with Theorem 0 in a forthcoming paper. Our object here is to show how higher dimensional sieves are constructed given the analytic information contained in Theorem 0.

We remark that, as a consequence of (1.12), (1.10ii), and (1.13),

$$0 \leqslant f_\kappa(u) < 1 < F_\kappa(u), \qquad u > 0. \tag{1.14}$$

We require later on also the following straightforward consequence of (1.8) through (1.13) (the proof is given in Appendix I):

$$0 \leqslant \begin{Bmatrix} F_\kappa(u_1) - F_\kappa(u_2) \\ f_\kappa(u_2) - f_\kappa(u_1) \end{Bmatrix} \leqslant \frac{u_2 - u_1}{u_1} \kappa A_\kappa \qquad \text{if} \quad 1 \leqslant u_1 < u_2. \tag{1.15}$$

We now state our main result:

THEOREM. *Suppose $\kappa \geqslant 1$ and that condition $(\Omega(\kappa))$ holds. Then we have for any numbers y and z satisfying*

$$y \geqslant z \geqslant 2 \tag{1.16}$$

that

$$S(\mathscr{A}, \mathscr{P}, z) \leqslant XV(z) \left\{ F_\kappa\left(\frac{\log y}{\log z}\right) + O\left(\frac{\log\log y}{(\log y)^{1/(2\kappa+2)}}\right)\right\} + \sum_{\substack{m \mid P(z) \\ m < y}} c^+(m) R_m \tag{1.17}$$

and

$$S(\mathscr{A}, \mathscr{P}, z) \geqslant XV(z) \left\{ f_\kappa\left(\frac{\log y}{\log z}\right) + O\left(\frac{\log\log y}{(\log y)^{1/(2\kappa+2)}}\right)\right\} - \sum_{\substack{m \mid P(z) \\ m < y}} c^-(m) R_m, \tag{1.18}$$

where the constants implied by the O-notation depend at most on κ and A

(from $(\Omega(\kappa)))$ *and the coefficients* $c^{\pm}(m)$ *in the remainder sums satisfy*[1] $|c^{\pm}(m)| \leqslant 1 + 4^{\nu(m)}$.

The classical case $\kappa = 1$ (when $\alpha_1 = \beta_1 = 2$) of the so called "linear" sieve is known, of course (from Jurkat and Richert [8] and, in general form, from [6], Chapter 8, also from Iwaniec [7]), and is included in this theorem only for completeness and, as it were, for calibration. For $\kappa > 1$, Appendix III gives instances of pairs of values of $\alpha_\kappa, \beta_\kappa$. Observe that (1.18) becomes trivial if $y \leqslant z^{\beta_\kappa}$, that is, if z is too large. We therefore refer to β_κ as the *sieving limit.* In (1.17), if z is large in the sense that $y \leqslant z^{\alpha_\kappa}$, $F_\kappa(\log y/\log z)$ coincides with $1/\sigma_\kappa(\log y/\log z)$ and (1.17) is then, essentially, the known upper bound Selberg sieve estimate of Ankeny and Onishi [1] (see also [6], Chapter 6); the theorem improves on [1] for $z < y^{1/\alpha_\kappa}$. Ankeny and Onishi [1] (see also [6], Chapter 7) give also a result[2] of type (1.18), but here our lower bound is always superior, both in the value of the lower sieving limit and the size of f.

Our method rests on a combinatorial identity (see Lemma 2.2 below) which appears to embody infinitely many iterations of Buchstab's identity, and an "initial" use of Selberg's upper bound sieve. In both these respects it may be viewed as a natural development, long delayed, of the approach in [8], and as having also points of similarity with Rawsthorne [11]. On the other hand, we make no direct use of [8] or [11]; on the contrary, our use of Lemma 2.2—we call it here, as we have done elsewhere [5], the Fundamental Sieve Identity—and of other combinatorial ideas (some deriving from Motohashi [9] and Halberstam [4]) leads to significant simplification of standard sieve techniques; so much so that this approach can be used also in the Buchstab–Rosser–Iwaniec sieve for $\frac{1}{2} < \kappa \leqslant 1$ to give a much simpler account of that theory.

As Iwaniec has been at pains to point out, the Buchstab–Rosser–Iwaniec sieve for $\kappa > 1$, given by him in [7] for the sake of completeness and for its intrinsic analytic interest, is inferior for those κ's to Ankeny and Onishi [1] and *a fortiori* to our theorem.

Careful comparison between Ankeny and Onishi [1] and the theorem of this paper shows again how good [1] is, and suggests even that, as $\kappa \to \infty$, the theorem is asymptotic to [1]. For κ of intermediate size the improvement of the theorem over [1], modest as it will seem, may nevertheless prove significant in terms of applications; for one has to remember that, in sieve applications, estimations of $S(\mathscr{A}, \mathscr{P}, z)$ are most effective when used in conjunction with weighting procedures such as are described in Chapters 9 and 10 of [6].

[1] Here and elsewhere in this paper, $\nu(m)$ stands for the numbers of prime factors of m.

[2] They use a single application of the Buchstab identity. For an improvement of [1] using a second iteration of this identity, see Porter [10].

There is one respect (at least) in which our theorem is not optimal: because Selberg's sieve is used, the remainder sums are not in Iwaniec's flexible bilinear form. We pose the problem of finding an account of Selberg's sieve which removes this defect.

2. Combinatorial Preliminaries

Let $n>1$ be a squarefree integer. Throughout this paper we shall write the canonical prime decomposition of n in the form

$$n = p_1 \cdots p_r (p_1 > \cdots > p_r). \tag{2.1}$$

It is convenient to have available the notations $p(n)=p_r$ and $q(n)=p_1$ for the least and largest prime factors of n; for the sake of completeness we put $p(1)=\infty$ and $q(1)=1$.

Our main result in this section is Lemma 2.2 below, what we call the Fundamental Sieve Identity.

Lemma 2.1 (The Fundamental Sieve Identity; [5]). *Let $\chi(\cdot)$ be an arithmetic function satisfying $\chi(1)=1$, and associate with $\chi(\cdot)$ the function $\bar{\chi}(\cdot)$ given by*

$$\bar{\chi}(1) := 0, \bar{\chi}(d) := \chi\left(\frac{d}{p(d)}\right) - \chi(d) \qquad \text{if} \quad d>1. \tag{2.2}$$

Then, for any arithmetic function $h(\cdot)$ and any $w \geqslant 2$ we have

$$\sum_{d \mid P(w)} \mu(d)\, h(d) = \sum_{d \mid P(w)} \mu(d)\, \chi(d)\, h(d) + \sum_{d \mid P(w)} \mu(d)\, \bar{\chi}(d) \sum_{t \mid P(p(d))} \mu(t)\, h(dt). \tag{2.3}$$

Corollary 2.1.1. *We have*

$$S(\mathscr{A}, \mathscr{P}, w) = \sum_{d \mid P(w)} \mu(d)\, \chi(d)\, |\mathscr{A}_d| + \sum_{d \mid P(w)} \mu(d)\, \bar{\chi}(d)\, S(\mathscr{A}_d, \mathscr{P}, p(d)). \tag{2.4}$$

Proof. Take $h(d)=|\mathscr{A}_d|$ in (2.3). Then the sum on the left of (2.3) is $S(\mathscr{A}, \mathscr{P}, w)$ by (1.3), while the inner sum of the second expression on the right is, again by (1.3), equal to $S(\mathscr{A}_d, \mathscr{P}, p(d))$. This proves the corollary.

COROLLARY 2.1.2. *We have*

$$S(\mathscr{A}, \mathscr{P}, z) = \sum_{d \mid P(z_1, z)} \mu(d)\, \chi(d)\, S(\mathscr{A}_d, \mathscr{P}, z_1) + \sum_{d \mid P(z_1, z)} \mu(d)\, \bar{\chi}(d)\, S(\mathscr{A}_d, \mathscr{P}, p(d)). \tag{2.5}$$

Proof. In (2.3) with $w = z$ take $h(d)$ to be 0 when $(d, P(z_1)) > 1$, and when $(d, P(z_1)) = 1$ take $h(d)$ to be $S(\mathscr{A}_d, \mathscr{P}, z_1)$. Then the sum on the left of (2.3) becomes

$$\sum_{d \mid P(z_1, z)} \mu(d)\, S(\mathscr{A}_d, \mathscr{P}, z_1) = S(\mathscr{A}, \mathscr{P}, z)$$

by (1.3); and the expression on the right of (2.3) becomes

$$\sum_{d \mid P(z_1, z)} \mu(d)\, \chi(d)\, S(\mathscr{A}_d, \mathscr{P}, z_1) + \sum_{d \mid P(z_1, z)} \mu(d)\, \bar{\chi}(d) \sum_{t \mid P(z_1, p(d))} \mu(t)\, S(\mathscr{A}_{dt}, \mathscr{P}, z_1).$$

This proves (2.5), since

$$\sum_{t \mid P(z_1, p(d))} \mu(t)\, S(\mathscr{A}_{dt}, \mathscr{P}, z_1) = S(\mathscr{A}_d, \mathscr{P}, p(d))$$

by (1.3) with $\mathscr{A}_d$ in place of $\mathscr{A}$.

Some general comments on these two corollaries are in order. First of all, (2.4) and (2.5) are no more than rearrangements of the "inclusion–exclusion" principles (1.3), and (2.4) is just the special case $z_1 = 2$, $z = w$ of (2.5). Nevertheless, (2.4) and (2.5) serve, implicitly or explicitly, as starting points of all known (small) sieves. Combinatorial sieves, starting with Brun's, correspond to assigning to the function $\chi(d)$ only the values 0 or 1 in accordance with a procedure that will be described below; thus $\chi(d)$ may be viewed (in (2.4), say) as the characteristic function of some sub-set of divisors of $P(w)$.

The function $\chi(d)$ will be required also to be divisor-closed in the sense that whenever $\chi(d) = 1$, then, for all $t \mid d$, $\chi(t) = 1$ too. It follows at once that $\bar{\chi}(d)$ also assumes only the values 0 and 1. With these remarks we are ready to begin describing a procedure for the choice of χ^- for a lower bound for $S(\mathscr{A}, \mathscr{P}, z)$: Let $\chi = \chi^-$ be a divisor-closed arithmetic function so that

$\chi^-(1)=1$ and $\chi^-(d)=1$ or 0 when $d>1$, $d \mid P(z_1, z)$. The second sum on the right of (2.5) is

$$\sum_{\substack{1<d \mid P(z_1,z) \\ \nu(d) \text{ even}}} \bar{\chi}^-(d)\, S(\mathscr{A}_d, p(d)) - \sum_{\substack{d \mid P(z_1,z) \\ \nu(d) \text{ odd}}} \bar{\chi}^-(d)\, S(\mathscr{A}_d, p(d)), \qquad (2.6)$$

and the S-functions are, of course, non-negative. This expression is greater than or equal to

$$-\sum_{\substack{d \mid P(z_1,z) \\ \nu(d) \text{ odd}}} \bar{\chi}^-(d)\, S(\mathscr{A}_d, p(d)). \qquad (2.7)$$

It is a characteristic feature of lower bound sieves of dimension $\kappa > \frac{1}{2}$, embodied here in (1.18) and (1.10ii) that there is no better lower estimate than the trivial one $S(\mathscr{A}, \mathscr{P}, z) \geqslant 0$ whenever $\log y/\log z \leqslant \beta_\kappa$. In this case we evidentally lose nothing by choosing $\bar{\chi}^-(d)=1$ when $\mu(d)=1$ and dropping the first sum in (2.6) to obtain (2.7). Here z translates into $p(d)$ and, as will soon be clear, y into y_1/d (where $y_1 < y$). This may be assured by requiring that

$$(y_1/d) \leqslant p(d)^{\beta_\kappa} \text{ when } \bar{\chi}^-(d)=1 \text{ and } \mu(d)=1,\ d \mid P(z_1, z). \qquad (2.8)$$

This leaves (2.7). For this we shall require χ^- to be such that if $\mu(d)=-1$, $d \mid P(z_1, z)$ and $\bar{\chi}^-(d)=1$ then $S(\mathscr{A}_d, p(d))$ may be estimated from above using the Selberg–Ankeny–Onishi sieve (1.17) with $F_\kappa = 1/\sigma_\kappa$ (see (1.10i)) and $u=\log(y_1/d)/\log p(d) \leqslant \alpha_\kappa$. In other words, we require of χ^- that

$$y_1/d \leqslant p(d)^{\alpha_\kappa} \text{ when } \bar{\chi}^-(d)=1 \text{ and } \mu(d)=-1,\ d \mid P(z_1, z). \qquad (2.9)$$

Let us now clarify the implications of (2.8) and (2.9) in the light of (2.2). These requirements virtually determine χ^- uniquely. If χ^- is given what one might call the Buchstab–Rosser structure: with

$$d = p_1 \cdots p_r (p_1 > \cdots > p_r,\ r \geqslant 1).$$

let

$$\chi_{y_1}^-(d) = \chi_{y_1}^-(d; \alpha_k, \beta_k)$$
$$= \eta_{y_1}^-(p_1; \alpha_k, \beta_k)\, \eta_{y_1}^-(p_1 p_2; \alpha_k, \beta_k) \cdots \eta_{y_1}^-(p_1 \cdots p_r; \alpha_k, \beta_k), \qquad (2.10)$$

where $\eta_{y_1}^-(\cdot)$ assumes only the values 0 and 1; then

$$\bar{\chi}^-(d) = \chi^-\left(\frac{d}{p(d)}\right)(1-\eta^-(d)), \qquad (2.11)$$

and now (2.8) and (2.9) are seen to hold if

$$\eta_{y_1}^{-}(n, \alpha_\kappa, \beta_\kappa) = \begin{cases} 1, & \mu(n) = 1 \text{ and } p(n)^{\beta_\kappa} n < y_1, \\ 1, & \mu(n) = -1 \text{ and } p(n)^{\alpha_\kappa} n < y_1, \\ 0, & \text{otherwise;} \end{cases} \tag{2.12}$$

so that $\chi_{y_1}^{-}(d) = 1$ if and only if d satisfies the Buchstab–Rosser inequalities

$$\begin{aligned} p_1^{\alpha_\kappa + 1} &< y_1 \\ p_2^{\beta_\kappa + 1} p_1 &< y_1 \\ p_3^{\alpha_\kappa + 1} p_2 p_1 &< y_1 \\ &\cdots\cdots\cdots\cdots, \end{aligned} \tag{2.13}$$

and $\chi_{y_1}^{-}(d)$ is otherwise zero. With this choice of χ, (2.5) through (2.7) yield the lower estimate

$$S(\mathscr{A}, \mathscr{P}, z) \geqslant \sum_{d \mid P(z_1, z)} \mu(d)\, \chi_{y_1}^{-}(d; \alpha_\kappa, \beta_\kappa)\, S(\mathscr{A}_d, \mathscr{P}, z_1) - \sum_{\substack{d \mid P(z_1, z) \\ \nu(d) \text{ odd}}} \bar{\chi}_{y_1}^{-}(d; \alpha_\kappa, \beta_\kappa)\, S(\mathscr{A}_d, \mathscr{P}, p(d)). \tag{2.14}$$

In similar fashion we require of χ^+ that (cf. (2.8) and (2.9))

$$y_1/d \leqslant p(d)^{\alpha_\kappa} \text{ when } \bar{\chi}_{y_1}^{+}(d) = 1 \text{ and } \mu(d) = 1,\ d \mid P(z_1, z) \tag{2.15}$$

and

$$y_1/d \leqslant p(d)^{\beta_\kappa} \text{ when } \bar{\chi}_{y_1}^{+}(d) = 1 \text{ and } \mu(d) = -1,\ d \mid P(z_1, z); \tag{2.16}$$

and we derive from (2.5)

$$S(\mathscr{A}, \mathscr{P}, z) \leqslant \sum_{d \mid P(z_1, z)} \mu(d)\, \chi_{y_1}^{+}(d; \alpha_\kappa, \beta_\kappa)\, S(\mathscr{A}_d, \mathscr{P}, z_1) + \sum_{\substack{1 < d \mid P(z_1, z) \\ \nu(d) \text{ even}}} \bar{\chi}_{y_1}^{+}(d; \alpha_\kappa, \beta_\kappa)\, S(\mathscr{A}_d, \mathscr{P}, p(d)), \tag{2.17}$$

where, if $d = p_1 \cdots p_r (p_1 > \cdots > p_r; r \geqslant 1)$,

$$\chi_{y_1}^{+}(d) = \chi_{y_1}^{+}(d; \alpha_\kappa, \beta_\kappa) = \eta_{y_1}^{+}(p_1; \alpha_\kappa, \beta_\kappa) \cdots \eta_{y_1}^{+}(p_1 \cdots p_r; \alpha_\kappa, \beta_\kappa) \tag{2.18}$$

with

$$\eta_{y_1}^{+}(n;\alpha_\kappa,\beta_\kappa)=\begin{cases}1, & \mu(n)=-1 \text{ and } p(n)^{\beta_\kappa}n<y_1,\\ 1, & \mu(n)=1 \text{ and } p(n)^{\alpha_\kappa}n<y_1,\\ 0, & \text{otherwise;}\end{cases} \tag{2.19}$$

so that $\chi_{y_1}^{+}(d)=1$ if and only id d satisfies the Buchstab–Rosser inequalities

$$\begin{aligned} &p_1^{\beta_\kappa+1}<y_1\\ &p_2^{\alpha_\kappa+1}p_1<y_1\\ &p_3^{\beta_\kappa+1}p_2p_1<y_1\\ &\dots\dots\dots\dots, \end{aligned} \tag{2.20}$$

and $\chi_{y_1}^{+}(d)$ is otherwise 0.

We conclude this section with an observation concerning the Buchstab–Rosser inequalities (2.13) and (2.20). If

$$\alpha_\kappa \geqslant \beta_\kappa+1, \tag{2.21}$$

then the second, fourth, etc., in other words, the *even* inequalities in (2.13), are implied by the preceding odd ones and are therefore redundant; similarly all the *odd* term inequalities in (2.20) *except the first* are superfluous. The distinction between the cases (2.21) and $\beta_\kappa \leqslant \alpha_\kappa < \beta_\kappa+1$ exists also in the analysis of the differential-difference configuration described by (1.8) through (1.13), although seemingly for quite different reasons.

3. Fundamental Lemma

A fundamental lemma is a result which states that $S(\mathscr{A},\mathscr{P},z)$ is, essentially, asymptotic to $XV(z)$ if z is smaller than any positive power of y. A characteristic feature of a fundamental lemma is that it holds under a condition weaker then $(\Omega(\kappa))$. We quote a version of it from Friedlander and Iwaniec [3]:

Fundamental Lemma. *Assume that there exist constants $C \geqslant 1$ and $\kappa>0$ such that*

$$\frac{V(w_1)}{V(w)} \leqslant C\left(\frac{\log w}{\log w_1}\right)^{\kappa}, \qquad 2\leqslant w_1 \leqslant w. \tag{$\Omega_0(\kappa)$}$$

For any given numbers $L \geqslant 2$, $z_0 \geqslant 2$ and squarefree natural number q coprime with $P(z_0)$, there exist systems of coefficients $\gamma_m^{\pm} = 0$ or 1 such that

$$- \sum_{\substack{m \mid P(z_0) \\ m < z_0^L}} \mu(m)\, \gamma_m^- R_{qm} \leqslant S(\mathscr{A}_q, \mathscr{P}, z_0) - XV(z_0) \frac{\omega(q)}{q} \{1 + O(e^{-L})\}$$

$$\leqslant \sum_{\substack{m \mid P(z_0) \\ m < z_0^L}} \mu(m)\, \gamma_m^+ R_{qm},$$

where the 0*-constant depends at most on C and κ.*

There is one application of the Fundamental Lemma we can make at once: we shall prove our main theorem for small z. In the Fundamental Lemma let $z_0 = z$, $q = 1$ and $L = \log\log y$; clearly $L \geqslant 2$ if y is large enough, as we may suppose. Then, provided only that

$$z \leqslant \exp\left(\frac{\log y}{\log\log y}\right), \tag{3.1}$$

we have, since obviously $(\Omega_0(\kappa))$ is implied by $(\Omega(\kappa))$,

$$XV(z)\left\{1 + O\left(\frac{1}{\log y}\right)\right\} - \sum_{\substack{m \mid P(z) \\ m < y}} \mu(m)\, \gamma_m R_m \leqslant S(\mathscr{A}, \mathscr{P}, z)$$

$$\leqslant XV(z)\left\{1 + O\left(\frac{1}{\log y}\right)\right\} + \sum_{\substack{m \mid P(z) \\ m < y}} \mu(m)\, \gamma_m^+ R_m,$$

and these immediately yield (1.17) and (1.18) in view of (1.11).

4. The Basic Inequalities

In this section we return to the inequalities (2.14) and (2.17), which we now write in the form

$$S(\mathscr{A}, \mathscr{P}, z) \geqslant \Sigma_1^- - \Sigma_2^- \tag{4.1}$$

and

$$S(\mathscr{A}, \mathscr{P}, z) \leqslant \Sigma_1^+ + \Sigma_2^+, \tag{4.2}$$

respectively; here

$$\Sigma_1^{\pm} = \sum_{d \mid P(z_1, z)} \mu(d)\, \chi_{y_1}^{\pm}(d)\, S(\mathscr{A}_d, \mathscr{P}, z_1) \tag{4.3}$$

and

$$\Sigma_2^{\pm} = \sum_{\substack{d \mid P(z_1,z) \\ \mu(d) = \pm 1}} \bar{\chi}_{y_1}^{\pm}(d)\, S(\mathscr{A}_d, \mathscr{P}, p(d)). \tag{4.4}$$

Note that we have written $\chi_{y_1}^{\pm}(d)$ in place of $\chi_{y_1}^{\pm}(d; \alpha_\kappa, \beta_\kappa)$ for the sake of brevity, and we shall maintain this contracted notation for the rest of the paper. We shall estimate Σ_1^- from below, and Σ_1^+ from above, by means of the Fundamental Lemma; and we shall estimate $\Sigma_2^{\pm}$ from above by Selberg's upper bound sieve.

From now on we take z_1 to be given by

$$\log z_1 = \left(\frac{\log^{2\kappa+1} y}{\log\log y}\right)^{1/(2\kappa+2)}. \tag{4.5}$$

In view of the closing remarks of the preceding section, our main theorem has already been proved for $2 \leqslant z \leqslant z_1$ (cf. (3.1)), so that we may assume henceforward that

$$z_1 < z \leqslant y_1. \tag{4.6}$$

Begin with the sums $\Sigma_1^{\pm}$, where we apply to each form the Fundamental Lemma with $z_0 = z_1$, $q = d$, and $L = \log\log y$. We take y_1 in (4.3) (and in (4.4)) to be defined by

$$y_1 z_1^L = y, \text{ so that } y_1 = y\exp(-(\log y \log\log y)^{(2\kappa+1)/(2\kappa+2)}). \tag{4.7}$$

Then

$$\begin{aligned}\Sigma_1^- \geqslant XV(z_1) \sum_{d \mid P(z_1,z)} \mu(d)\, \chi_{y_1}^-(d) \frac{\omega(d)}{d} + O\left(XV(z_1) \frac{1}{\log y} \sum_{d \mid P(z_1,z)} \frac{\omega(d)}{d}\right) \\ - \sum_{d \mid P(z_1,z)} \mu(d)\, \chi_{y_1}^-(d) \sum_{\substack{m \mid P(z_1) \\ m < z_1^L}} \mu(m)\, \gamma_m^{(-)^{\nu(d)+1}} R_{dm}.\end{aligned} \tag{4.8}$$

Now

$$\sum_{d \mid P(z_1,z)} \frac{\omega(d)}{d} = \prod_{z_1 \leqslant p < z} \left(1 + \frac{\omega(p)}{p}\right) \leqslant \frac{V(z_1)}{V(z)}, \tag{4.9}$$

so that the second expression on the right of (4.8) is, by $(\Omega(\kappa))$, (4.5), and (4.6),

$$\begin{aligned}&\ll XV(z)\left(\frac{V(z_1)}{V(z)}\right)^2 \frac{1}{\log y} \ll XV(z)\left(\frac{\log z}{\log z_1}\right)^{2\kappa} \frac{1}{\log y} \\ &\ll XV(z) \frac{\log\log y}{(\log y)^{1/(\kappa+1)}}.\end{aligned}$$

In the third expression on the right of (4.8) write $dm = n$; since $d \mid P(z_1, z)$ and $m \mid P(z_1)$, any divisor n of $P(z)$ has a unique decomposition $n = dm$ of this kind, with d and m coprime. Also, whenever $\chi_{y_1}^-(d) = 1$ we have $d < y_1$, so that $n < y_1 z_1^L = y$ by (4.7). Hence the third expression may be written

$$-\sum_{\substack{n \mid P(z) \\ n < y}} \mu(n)\, b_n^- R_n,$$

where

$$b_n^- := b_{dm}^- := \chi_{y_1}^-(d)\, \gamma_m^{(-)^{\nu(d)+1}} \qquad (d \mid P(z_1, z), m \mid P(z_1)) \tag{4.10}$$

and therefore b_n^- takes only the values 0 or 1. Hence (4.8) takes the form

$$\Sigma_1^- \geqslant XV(z_1) \sum_{d \mid P(z_1, z)} \mu(d)\, \chi_{y_1}^-(d) \frac{\omega(d)}{d} + O\left(XV(z) \frac{\log\log y}{(\log y)^{1/(\kappa+1)}} \right)$$
$$- \sum_{\substack{n \mid P(z) \\ n < y}} \mu(n)\, b_n^- R_n.$$

It is convenient at this point to introduce the notation

$$\phi^+(u) = F_\kappa(u), \qquad \phi^-(u) = f_\kappa(u), \tag{4.11}$$

where F_k and f_κ are defined in Section 1. By (1.14) we have

$$\mu(d)\, \phi^{(-)^{\nu(d)+1}}(u) < \mu(d) < \mu(d)\, \phi^{(-)^{\nu(d)}}(u) \tag{4.12}$$

for any $u > 0$, so that

$$\Sigma_1^- \geqslant XV(z_1) \sum_{d \mid P(z_1, z)} \mu(d)\, \chi_{y_1}^-(d) \frac{\omega(d)}{d}\, \phi^{(-)^{\nu(d)+1}} \left(\frac{\log(y_1/d)}{\log z_1} \right)$$
$$+ O\left(XV(z) \frac{\log\log y}{(\log y)^{1/(\kappa+1)}} \right) - \sum_{\substack{n \mid P(z) \\ n < y}} \mu(n)\, b_n^- R_n. \tag{4.13}$$

The same sort of argument leads, without any new difficulty, to

$$\Sigma_1^+ \leqslant XV(z_1) \sum_{d \mid P(z_1, z)} \mu(d)\, \chi_{y_1}^+(d) \frac{\omega(d)}{d}\, \phi^{(-)^{\nu(d)}} \left(\frac{\log(y_1/d)}{\log z_1} \right)$$
$$+ O\left(XV(z) \frac{\log\log y}{(\log y)^{1/(\kappa+1)}} \right) + \sum_{\substack{n \mid P(z) \\ n < y}} \mu(n)\, b_n^+ R_n, \tag{4.14}$$

where (cf. (4.10))

$$b_n^+ := b_{dm}^+ := \chi_{y_1}^+(d)\, \gamma_m^{(-)^{\nu(d)}} = 0 \text{ or } 1 \ (d \mid P(z_1, z), m \mid P(z_1)). \tag{4.15}$$

We come now to the sums $\Sigma_2^{\pm}$, which we estimate from above by Selberg's λ-sieve. We quote from [6], Theorems 6.1 and 6.3 (with $\xi^2 = Y$, $L \ll 1$):

If q is squarefree and coprime with $P(w)$, and if

$$\tau = \frac{\log Y}{\log w} > 0, \tag{4.16}$$

then

$$S(\mathscr{A}_q, \mathscr{P}, w) \leqslant \frac{\omega(q)}{q} XV(w) \left\{ \frac{1}{\sigma_\kappa(\tau)} + O\left(\frac{1}{\log w} (\tau^{-\kappa-1} + \tau^{2\kappa+1}) \right) \right\}$$
$$+ \sum_{\substack{n \mid P(w) \\ n < Y}} r_n R_{nq}; \tag{4.17}$$

here

$$r_n = \sum_{\substack{d_\nu \mid P(w), d_\nu < Y^{1/2} (\nu = 1,2) \\ LCM(d_1, d_2) = n}} \lambda_{d_1} \lambda_{d_2} \qquad (n \mid P(w), n < Y),$$

where

$$\lambda_t = \frac{\mu(t)}{\prod_{p \mid t} (1 - \omega(p)/p)} \Bigg(\sum_{\substack{m \mid P(w) \\ m < Y^{1/2}/t \\ (m,t) = 1}} g(m) \Bigg) \Bigg(\sum_{\substack{m \mid P(w) \\ m < Y^{1/2}}} g(m) \Bigg)^{-1}, \qquad t \mid P(w),$$

with

$$g(m) = \frac{\omega(m)}{m} \prod_{p \mid m} \left(1 - \frac{\omega(p)}{p} \right)^{-1}.$$

We have $\lambda_t = 0$ if $t \geqslant Y^{1/2}$, $|\lambda_t| \leqslant 1$ (by a well-known argument, for example, [6, pp. 190–191]) and therefore

$$|r_n| \leqslant 3^{\nu(n)}. \tag{4.18}$$

Actually (4.17) is proved in [6, Theorem 6.3] under a stronger condition than $(\Omega(\kappa))$; but we shall show in Appendix II that (4.17) holds even subject to $(\Omega(\kappa))$.

We substitute (4.17) (with $q = d$, $w = p(d)$, and $Y = y_1/d$) in (4.4); we do so, of course, only when $\bar{\chi}_{y_1}^{\pm}(d) = 1$, $\mu(d) = \pm 1$, and $d \mid P(z_1, z)$. In these circumstances, by (2.9) and (2.15), $p(d)^{\alpha_\kappa} d \geqslant y_1$, so that

$$\tau = \tau_d = \frac{\log(y_1/d)}{\log p(d)} \leqslant \alpha_\kappa. \tag{4.19}$$

Moreover, in case (2.15), $\chi_{y_1}^+(d/p(d))=1$ and $v(d)\geqslant 2$, so that, writing $v(d)=2r$ and $d=p_1\cdots p_{2r}$, we have $p_{2r-1}^{\beta_\kappa+1}\cdots p_1<y_1$ by (2.20); hence $p(d)^{\beta_\kappa-1}\,d<y_1$ and, *a fortiori*,

$$\tau_d^{-1}=\frac{\log p(d)}{\log(y_1/d)}<\frac{1}{\beta_\kappa-1}\leqslant 1. \tag{4.20}$$

In case (2.9), if $v(d)>1$, a similar argument based on (2.13) also yields (4.20). There remains the case (2.9) with $v(d)=1$, that is, with $d=p$; and we want an upper bound for $\tau_d^{-1}=\tau_p^{-1}=\log p/\log(y_1/p)$ subject to $p<z$. This case occurs only with Σ_2^-, and Σ_2^- appears only in (4.1), when we seek a lower bound for $S(\mathscr{A}, z)$. A glance at (1.18) (the lower bound to be proved) and (1.10ii) shows that for the purpose of proving (1.18) we may as well suppose that $\log y/\log z>\beta_\kappa\geqslant 2$. But then $\log(y/z)/\log z>\beta_\kappa-1$, or

$$\frac{\log z}{\log(y/z)}<\frac{1}{\beta_\kappa-1};$$

since $\tau_p^{-1}<\log z/\log(y_1/z)$ we may conclude that in this case too (4.20) holds, at least in the less precise form

$$\tau_d^{-1}\ll 1. \tag{4.21}$$

To sum up this discussion, for the purpose of each application in (4.4), (4.17) implies that

$$S(\mathscr{A}_d,\mathscr{P},p(d))\leqslant\frac{\omega(d)}{d}XV(p(d))\left\{\frac{1}{\sigma_\kappa(\log(y_1/d)/\log p(d))}+O\left(\frac{1}{\log p(d)}\right)\right\}+\sum_{\substack{n\mid P(p(d))\\ n<y_1/d}} r_n R_{dn}. \tag{4.22}$$

Hence, by (4.4),

$$\Sigma_2^+\leqslant X\sum_{\substack{d\mid P(z_1,z)\\ \mu(d)=1}}\bar{\chi}_{y_1}^+(d)\frac{\omega(d)}{d}V(p(d))\frac{1}{\sigma_\kappa(\log(y_1/d)/\log p(d))}+O\left(X\sum_{d\mid P(z_1,z)}\frac{\omega(d)}{d}\frac{V(p(d))}{\log p(d)}\right)+\sum_{\substack{m\mid P(z)\\ m<y_1}}B_m^+R_m, \tag{4.23}$$

where

$$B_m^+:=\sum_{\substack{dn=m\\ d\mid P(z_1,z),\,\mu(d)=1\\ n\mid P(p(d))}}\bar{\chi}_{y_1}^+(d)\,r_n,\qquad m\mid P(z),\, m<y_1, \tag{4.24}$$

so that, by (4.18),

$$|B_m^+| \leqslant 4^{\nu(m)}. \tag{4.25}$$

The O-term in (4.23) is, using ($\Omega(\kappa)$) and (4.9), at most of order

$$\begin{aligned}
&XV(z)\frac{1}{\log y}\sum_{d \mid P(z_1,z)} \frac{V(p(d))}{V(z)}\cdot\frac{\log y}{\log p(d)}\cdot\frac{\omega(d)}{d}\\
&\ll XV(z)\frac{1}{\log y}\sum_{d \mid P(z_1,z)}\left(\frac{\log y}{\log p(d)}\right)^{\kappa+1}\frac{\omega(d)}{d}\\
&\ll XV(z)\frac{1}{\log y}\left(\frac{\log y}{\log z_1}\right)^{\kappa+1}\sum_{d \mid P(z_1,z)}\frac{\omega(d)}{d}\\
&\ll \frac{XV(z)}{\log y}\left(\frac{\log y}{\log z_1}\right)^{2\kappa+1} \ll XV(z)\frac{\log\log y}{(\log y)^{1/(2\kappa+2)}}
\end{aligned}$$

by (4.5). Hence, by (4.23),

$$\begin{aligned}
\Sigma_2^+ \leqslant X &\sum_{\substack{d \mid P(z_1,z)\\ \mu(d)=1}} \bar{\chi}_{y_1}^+(d)\frac{\omega(d)}{d}V(p(d))\frac{1}{\sigma_\kappa(\log(y_1/d)/\log p(d))}\\
&+O\left(XV(z)\frac{\log\log y}{(\log y)^{1/(2\kappa+2)}}\right)+\sum_{\substack{m \mid P(z)\\ m<y_1}} B_m^+ R_m,
\end{aligned} \tag{4.26}$$

where the coefficients B_m^+ are given by (4.24) and estimated in (4.25).

We deal with Σ_2^- in exactly the same way on the basis of (4.22); we have only to remember that here we may assume that $z < y^{1/\beta_\kappa} \leqslant y^{1/2}$ so that (4.21) holds and (4.22) is indeed available. We obtain, subject to

$$z < y^{1/\beta_\kappa}, \tag{4.27}$$

$$\begin{aligned}
\Sigma_2^- \leqslant X &\sum_{\substack{d \mid P(z_1,z)\\ \mu(d)=-1}} \bar{\chi}_{y_1}^-(d)\frac{\omega(d)}{d}V(p(d))\frac{1}{\sigma_\kappa(\log(y_1/d)/\log p(d))}\\
&+O\left(XV(z)\frac{\log\log y}{(\log y)^{1/(2\kappa+2)}}\right)+\sum_{\substack{m \mid P(z)\\ m<y_1}} B_m^- R_m,
\end{aligned} \tag{4.28}$$

where

$$B_m^- := \sum_{\substack{dn=m\\ d \mid P(z_1,z),\mu(d)=-1\\ n \mid P(p(d))}} \bar{\chi}_{y_1}^-(d)\, r_n, \qquad m \mid P(z), m<y_1, \tag{4.29}$$

so that

$$|B_m^-| \leqslant 4^{\nu(m)}. \tag{4.30}$$

We sum up the results of this section. By (4.1), (4.13), (4.28) and by (4.2), (4.14), (4.26) we have for $z_1 < z \leqslant y_1$,

$$XE^- + O\left(XV(z)\frac{\log\log y}{(\log y)^{1/(2\kappa+2)}}\right) - \sum_{\substack{m \mid P(z) \\ m < y}} c^-(m)\, R_m \leqslant S(\mathscr{A}, \mathscr{P}, z)$$

$$\leqslant XE^+ + O\left(XV(z)\frac{\log\log y}{(\log y)^{1/(2\kappa+2)}}\right) + \sum_{\substack{m \mid P(z) \\ m < y}} c^+(m)\, R_m, \tag{4.31}$$

where

$$E^+ := V(z_1) \sum_{d \mid P(z_1, z)} \mu(d)\, \chi_{y_1}^+(d) \frac{\omega(d)}{d} \phi^{(-)^{\nu(d)}}\left(\frac{\log(y_1/d)}{\log z_1}\right)$$

$$+ \sum_{\substack{d \mid P(z_1, z) \\ \mu(d) = 1}} \bar{\chi}_{y_1}^+(d) \frac{\omega(d)}{d} V(p(d)) \frac{1}{\sigma_\kappa(\log(y_1/d)/\log p(d))}, \tag{4.32}$$

$$E^- := V(z_1) \sum_{d \mid P(z_1, z)} \mu(d)\, \chi_{y_1}^-(d) \frac{\omega(d)}{d} \phi^{(-)^{\nu(d)+1}}\left(\frac{\log(y_1/d)}{\log z_1}\right)$$

$$- \sum_{\substack{d \mid P(z_1, z) \\ \mu(d) = -1}} \bar{\chi}_{y_1}^-(d) \frac{\omega(d)}{d} V(p(d)) \frac{1}{\sigma_\kappa(\log(y_1/d)/\log p(d))}, \quad z < y^{1/\beta_\kappa},$$

$$= 0, \qquad \text{otherwise}, \tag{4.33}$$

$$c^{\pm}(m) := \mu(m)\, b_m^{\pm} + B_m^{\pm}, \qquad m \mid P(z), m < y, \tag{4.34}$$

where $b_m^{\pm}$ are given by (4.10), (4.15) and[3] $B_m^{\pm}$ by (4.24), (4.29); and obviously, by (4.25) and (4.30), $|c^{\pm}(m)| \leqslant 1 + 4^{\nu(m)}$. Thus the proof of our main theorem requires only that we show that

$$E^+ \leqslant V(z)\left\{F_\kappa\left(\frac{\log y}{\log z}\right) + O\left(\frac{\log\log y}{(\log y)^{1/(2\kappa+2)}}\right)\right\}, \qquad z_1 < z \leqslant y_1, \tag{4.35}$$

and that

$$E^- \geqslant V(z)\left\{f_\kappa\left(\frac{\log y}{\log z}\right) + O\left(\frac{\log\log y}{(\log y)^{1/(2\kappa+2)}}\right)\right\}, \qquad z_1 < z < y^{1/\beta_\kappa}. \tag{4.36}$$

We shall deal with the short gap $y_1 < z \leqslant y$ at the end of Section 6.

[3] $B_m^{\pm}$ is defined as 0 for $y_1 \leqslant m < y$.

5. Technical Preparation

We have, since $\omega(\cdot)$ is multiplicative,

$$V(w)=\sum_{d \mid P(w)} \mu(d)\frac{\omega(d)}{d}=1+\sum_{p<w}\sum_{\substack{d \mid P(w)\\ q(d)=p}} \mu(d)\frac{\omega(d)}{d}$$

$$=1-\sum_{p<w}\frac{\omega(p)}{p}\sum_{t \mid P(p)} \mu(t)\frac{\omega(t)}{t},$$

so that

$$V(w)=1-\sum_{p<w}\frac{\omega(p)}{p}V(p),$$

and by subtraction, that

$$V(z_1)-V(w)=\sum_{z_1 \leqslant p<w}\frac{\omega(p)}{p}V(p), \qquad z_1 \leqslant w. \tag{5.1}$$

Lemma 5.1. *Suppose that $z_1 \leqslant w$, and that $B(t)$ is a non-negative, continuous, and increasing function on $[z_1, w]$. Then*

$$\sum_{z_1 \leqslant p<w}\frac{\omega(p)}{p}V(p)\,B(p) \leqslant V(w)(\log w)^\kappa\left\{\kappa\int_{z_1}^{w}\frac{B(t)}{t(\log t)^{\kappa+1}}\,dt + \frac{AB(w)}{(\log z_1)^{\kappa+1}}\right\} \tag{5.2}$$

provided only that $(\Omega(\kappa))$ holds.

Proof. By (5.1) and $(\Omega(\kappa))$ we have

$$\sum_{z_1 \leqslant p<w}\frac{\omega(p)}{p}\frac{V(p)}{V(w)}B(p)=\sum_{z_1 \leqslant p<w}\frac{\omega(p)}{p}\frac{V(p)}{V(w)}\left\{B(z_1)+\int_{z_1}^{p}dB(t)\right\}$$

$$=B(z_1)\sum_{z_1 \leqslant p<w}\frac{\omega(p)}{p}\frac{V(p)}{V(w)}$$

$$+\int_{z_1}^{w}\sum_{t \leqslant p<w}\frac{\omega(p)}{p}\frac{V(p)}{V(w)}\cdot dB(t)$$

$$=B(z_1)\left(\frac{V(z_1)}{V(w)}-1\right)+\int_{z_1}^{w}\left(\frac{V(t)}{V(w)}-1\right)dB(t)$$

$$\leqslant B(z_1)\left\{\left(\frac{\log w}{\log z_1}\right)^\kappa\left(1+\frac{A}{\log z_1}\right)-1\right\}$$

$$+\int_{z_1}^{w}\left\{\left(\frac{\log w}{\log t}\right)^\kappa\left(1+\frac{A}{\log t}\right)-1\right\}dB(t)$$

$$=B(w)\frac{A}{\log w}-\int_{z_1}^{w}B(t)\,d\left\{\left(\frac{\log w}{\log t}\right)^\kappa\left(1+\frac{A}{\log t}\right)\right\}$$

$$\leqslant \kappa(\log w)^\kappa\int_{z_1}^{w}\frac{B(t)}{t(\log t)^{\kappa+1}}\,dt$$

$$+A\frac{B(w)(\log w)^\kappa}{(\log z_1)^{\kappa+1}},$$

and this completes the proof of the lemma.

By (1.12), (1.13), and (4.11) we may apply Lemma 5.1 with

$$B(t)=(-1)^\nu\left(1-\phi^{(-)^{\nu+1}}\left(\frac{\log(x/t)}{\log t}\right)\right),\qquad z_1\leqslant t<x, \tag{5.3}$$

and $\nu=0$ and 1 in turn; and we obtain

LEMMA 5.2. *Supose that $z_1\leqslant w$. If $x\geqslant w^{\beta_\kappa}$, we have*

$$\sum_{z_1\leqslant p<w}\frac{\omega(p)}{p}V(p)\,F_\kappa\left(\frac{\log(x/p)}{\log p}\right)\leqslant V(z_1)\,f_\kappa\left(\frac{\log x}{\log z_1}\right)-V(w)\,f_\kappa\left(\frac{\log x}{\log w}\right)$$
$$+\frac{A}{\sigma_\kappa(1)}\cdot\frac{V(w)}{\log z_1}\left(\frac{\log w}{\log z_1}\right)^\kappa; \tag{5.4}$$

and if $x\geqslant w^{\alpha_\kappa}$, we have

$$\sum_{z_1\leqslant p<w}\frac{\omega(p)}{p}V(p)\,f_\kappa\left(\frac{\log(x/p)}{\log p}\right)\geqslant V(z_1)\,F_\kappa\left(\frac{\log x}{\log z_1}\right)-V(w)\,F_\kappa\left(\frac{\log x}{\log w}\right)$$
$$-\frac{A}{\sigma_\kappa(1)}\frac{V(w)}{\log z_1}\left(\frac{\log w}{\log z_1}\right)^\kappa. \tag{5.5}$$

Proof. With $B(t)$ given by (5.3) and $\nu=0$ or 1, we deal first with the integral on the right of (5.2); we put $t=x^{1/\zeta}$ and obtain

$$\kappa \int_{z_1}^{w} \frac{B(t)}{t(\log t)^{\kappa+1}}\, dt$$

$$= (-1)^{\nu} \kappa \int_{z_1}^{w} \left\{1 - \phi^{(-)^{\nu+1}}\left(\frac{\log(x/t)}{\log t}\right)\right\} \frac{dt}{t(\log t)^{\kappa+1}}$$

$$= \frac{(-1)^{\nu} \kappa}{(\log x)^{\kappa}} \int_{\log x/\log w}^{\log x/\log z_1} \{1 - \phi^{(-)^{\nu+1}}(\zeta - 1)\}\, \zeta^{k-1} d\zeta$$

$$= (-1)^{\nu} \left\{\frac{1}{(\log z_1)^{\kappa}}\left(1 - \phi^{(-)^{\nu}}\left(\frac{\log x}{\log z_1}\right)\right)\right.$$

$$\left. - \frac{1}{(\log w)^{\kappa}}\left(1 - \phi^{(-)^{\nu}}\left(\frac{\log x}{\log w}\right)\right)\right\}$$

using (1.10iii) and (1.10iv) at the last step, as we may do since we require $\log x/\log w \geqslant \alpha_{\kappa}$, i.e., $x \geqslant w^{\alpha_{\kappa}}$, when $\nu = 0$, and $\log x/\log w \geqslant \beta_{\kappa}$, i.e., $x \geqslant w^{\beta_{\kappa}}$, when $\nu = 1$. Hence, by Lemma 5.1 and (5.1), and subject to the specified restrictions on $\log x/\log w$,

$$(-1)^{\nu} \left\{V(z_1) - V(w) - \sum_{z_1 \leqslant p < w} \frac{\omega(p)}{p} V(p)\, \phi^{(-)^{\nu+1}}\left(\frac{\log(x/p)}{\log p}\right)\right\}$$

$$\leqslant (-1)^{\nu} \left\{V(w)\left(\frac{\log w}{\log z_1}\right)^{\kappa} - V(w) - V(w)\left(\frac{\log w}{\log z_1}\right)^{\kappa}\right.$$

$$\times \phi^{(-)^{\nu}}\left(\frac{\log x}{\log z_1}\right) + V(w)\, \phi^{(-)^{\nu}}\left(\frac{\log x}{\log w}\right)$$

$$\left. + A \frac{V(w)}{\log z_1}\left(\frac{\log w}{\log z_1}\right)^{\kappa}\left(1 - \phi^{(-)^{\nu+1}}\left(\frac{\log (x/w)}{\log w}\right)\right)\right\}$$

or, after rearrangement,

$$(-1)^{\nu} \left\{V(z_1)\, \phi^{(-)^{\nu}}\left(\frac{\log x}{\log z_1}\right) - V(w)\, \phi^{(-)^{\nu}}\left(\frac{\log x}{\log w}\right)\right.$$

$$\left. - \sum_{z_1 \leqslant p < w} \frac{\omega(p)}{p} V(p)\, \phi^{(-)^{\nu+1}}\left(\frac{\log(x/p)}{\log p}\right)\right\}$$

$$\leqslant (-1)^{\nu} V(w)\left(\frac{\log w}{\log z_1}\right)^{\kappa} \left\{\frac{V(z_1)}{V(w)}\left(\frac{\log z_1}{\log w}\right)^{\kappa}\right.$$

$$\times \left(\phi^{(-)^{\nu}}\left(\frac{\log x}{\log z_1}\right) - 1\right) - (\phi^{(-)^{\nu}}\left(\frac{\log x}{\log z_1}\right) - 1\Big)$$

$$\left. + \frac{A}{\log z_1}\left(1 - \phi^{(-)^{\nu+1}}\left(\frac{\log(x/w)}{\log w}\right)\right)\right\}$$

$$\leqslant A \frac{V(w)}{\log z_1}\left(\frac{\log w}{\log z_1}\right)^{\kappa} (-1)^{\nu} \left\{\phi^{(-)^{\nu}}\left(\frac{\log x}{\log z_1}\right)\right.$$

$$\left. - \phi^{(-)^{\nu+1}}\left(\frac{\log x}{\log w} - 1\right)\right\}$$

$$\leqslant A \frac{V(w)}{\log z_1}\left(\frac{\log w}{\log z_1}\right)^{\kappa} F_k(\beta_\kappa - 1),$$

since

$$\frac{\log x}{\log z_1} \geqslant \frac{\log x}{\log w} > \frac{\log x}{\log w} - 1 \geqslant \beta_\kappa - 1$$

in both cases. Since $\beta_\kappa - 1 \geqslant 1$ and F_κ is decreasing (cf. (1.12)), both results follow from (1.10i).

In the next section we shall derive (4.35) and (4.36) from Lemma 5.2.

6. Proof of the Main Theorem

We shall prove inequalities (4.35) with $y_1 > z^{\alpha_\kappa}$ and (4.36) with $y_1 > z^{\beta_\kappa}$, and we refer the reader to the definitions of E^+ and E^-, namely (4.32) and (4.33), respectively. It is important to recall definitions (4.7) and (4.11). Begin with (4.35) subject to

$$y_1 \geqslant z^{\alpha_\kappa}. \tag{6.1}$$

We introduce the expression

$$E_r^+ := V(z_1) \sum_{\substack{d \mid P(z_1, z) \\ \nu(d) < r}} \mu(d)\, \chi_{y_1}^+(d) \frac{\omega(d)}{d} \phi^{(-)^{\nu(d)}}\left(\frac{\log(y_1/d)}{\log z_1}\right)$$

$$+ \sum_{\substack{d \mid P(z_1, z) \\ \mu(d) = 1 \\ \nu(d) < r}} \bar{\chi}_{y_1}^+(d) \frac{\omega(d)}{d} V(p(d)) \frac{1}{\sigma_\kappa(\log(y_1/d)/\log p(d))}$$

$$+ (-1)^r \sum_{\substack{d \mid P(z_1, z) \\ \nu(d) = r}} \chi_{y_1}^+(d) \frac{\omega(d)}{d} V(p(d))\, \phi^{(-)^r}\left(\frac{\log(y_1/d)}{\log p(d)}\right)$$

$$- V(z) F_\kappa\left(\frac{\log y_1}{\log z}\right), \qquad r = 1, 2, 3, \ldots. \tag{6.2}$$

We begin with the observation that

$$E_1^+ = V(z_1)\, F_k\left(\frac{\log y_1}{\log z_1}\right) - \sum_{z_1 \leqslant p < z} \chi_{y_1}^+(p)\, \frac{\omega(p)}{p}\, V(p)\, f_\kappa\left(\frac{\log(y_1/p)}{\log p}\right)$$
$$- V(z)\, F_\kappa\left(\frac{\log y_1}{\log z}\right).$$

By (2.28), $\chi_{y_1}^+(p) = 1$ if and only if $p < y_1^{1/(\beta_\kappa+1)}$, so that if $z \leqslant y_1^{1/(\beta_k+1)}$ the factor $\chi_{y_1}^+(p)$ may be replaced by 1 in the sum on the right. By (6.1) $z < y_1^{1/\alpha_\kappa} \leqslant y_1^{1/(\beta_k+1)}$ if $\alpha_\kappa \geqslant \beta_\kappa + 1$. Hence, when $\alpha_\kappa \geqslant \beta_\kappa + 1$, or when $\alpha_\kappa < \beta_\kappa + 1$ but $z \leqslant y_1^{1/(\beta_\kappa+1)}$, we have

$$\begin{aligned} E_1^+ &= V(z_1)\, F_\kappa\left(\frac{\log y_1}{\log z_1}\right) - V(z)\, F_\kappa\left(\frac{\log y_1}{\log z}\right) \\ &\quad - \sum_{z_1 \leqslant p < z} \frac{\omega(p)}{p}\, V(p)\, f_\kappa\left(\frac{\log(y_1/p)}{\log p}\right) \\ &\leqslant \frac{A}{\sigma_\kappa(1)}\, \frac{V(z)}{\log z_1}\left(\frac{\log z}{\log z_1}\right)^\kappa \end{aligned} \tag{6.3}$$

by (5.5) with $w = z$ and $x = y$, this part of the lemma being applicable in view of (6.1).

Suppose we are in the case of $\alpha_\kappa < \beta_\kappa + 1$, and that

$$y_1^{1/(\beta_\kappa+1)} < z < y_1^{1/\alpha_k}.$$

Here, again by (5.5),

$$\begin{aligned} E_1^+ &= V(z_1)\, F_\kappa\left(\frac{\log y_1}{\log z_1}\right) - V(z)\, F_\kappa\left(\frac{\log y_1}{\log z}\right) \\ &\quad - \sum_{z_1 \leqslant p < y_1} 1/(\beta_\kappa + 1)\, \frac{\omega(p)}{p}\, V(p)\, f_\kappa\left(\frac{\log(y_1/p)}{\log p}\right) \\ &\leqslant V(y_1^{1/(\beta_\kappa+1)})\, F_\kappa(\beta_\kappa + 1) - V(z)\, F_\kappa\left(\frac{\log y_1}{\log z}\right) \\ &\quad + \frac{A}{\sigma_\kappa(1)}\, \frac{V(y_1^{1/(\beta_\kappa+1)})}{\log z_1}\left(\frac{\log y_1^{1/(\beta_\kappa+1)}}{\log z_1}\right)^\kappa. \end{aligned}$$

The first two terms on the right contribute, by $(\Omega(\kappa))$,

$$V(z)\left\{\frac{V(y_1^{1/(\beta_\kappa+1)})}{V(z)}F_\kappa(\beta_\kappa+1)-F_\kappa\left(\frac{\log y_1}{\log z}\right)\right\}$$

$$\leqslant V(z)\left\{\left(\frac{\log z}{\log y_1^{1/(\beta_\kappa+1)}}\right)^\kappa\left(1+\frac{A}{\log y_1^{1/(\beta_\kappa+1)}}\right)F_\kappa(\beta_\kappa+1)-F_\kappa\left(\frac{\log y_1}{\log z}\right)\right\}$$

$$=A\frac{V(z)}{\log y_1^{1/(\beta_\kappa+1)}}\left(\frac{\log z}{\log y_1^{1/(\beta_\kappa+1)}}\right)^\kappa F_\kappa(\beta_\kappa+1)$$

$$+V(z)\left(\frac{\log z}{\log y_1}\right)^\kappa\left\{(\beta_\kappa+1)^\kappa F_\kappa(\beta_\kappa+1)-\left(\frac{\log y_1}{\log z}\right)^\kappa F_\kappa\left(\frac{\log y_1}{\log z}\right)\right\}$$

$$=AF_\kappa(\beta_\kappa+1)\frac{V(z)}{\log y_1^{1/(\beta_\kappa+1)}}\left(\frac{\log z}{\log y_1^{1/(\beta_\kappa+1)}}\right)^\kappa$$

since $u^\kappa F_\kappa(u)$ is constant when $\alpha_\kappa<u\leqslant\beta_\kappa+1$, by (1.10iii) and (1.10ii). Since $y_1^{1/(1+\beta_\kappa)}$ is much larger than z_1 we obtain finally, with one further application of ($\Omega(\kappa)$),

$$E_1^+\leqslant AF_\kappa(\beta_\kappa+1)\frac{V(z)}{\log z_1}\left(\frac{\log z}{\log z_1}\right)^\kappa$$

$$+\frac{A}{\sigma_\kappa(1)}\frac{V(z)}{\log z_1}\left(\frac{\log z}{\log z_1}\right)^\kappa\left(1+\frac{A}{\log z_1}\right)$$

$$\leqslant\frac{A(2+A)}{\sigma_\kappa(1)}\frac{V(z)}{\log z_1}\left(\frac{\log z}{\log z_1}\right)^\kappa, \tag{6.4}$$

since $F_\kappa(\beta_\kappa+1)\leqslant F_\kappa(1)=1/\sigma_\kappa(1)$ by (1.12) and (1.10i). From (6.3) and (6.4) we have in all cases

$$E_1^+\leqslant\frac{3A^2}{\sigma_\kappa(1)}\frac{V(z)}{\log z_1}\left(\frac{\log z}{\log z_1}\right)^\kappa,\qquad y_1\geqslant z^{\alpha_\kappa}. \tag{6.5}$$

For any integer $s\geqslant 1$, consider

$$E_{2s+1}^+-E_{2s-1}^+=V(z_1)\sum_{\substack{d\mid P(z_1,z)\\ \nu(d)=2s}}\chi_{y_1}^+(d)\frac{\omega(d)}{d}F_\kappa\left(\frac{\log(y_1/d)}{\log z_1}\right)$$

$$-V(z_1)\sum_{\substack{d\mid P(z_1,z)\\ \nu(d)=2s-1}}\chi_{y_1}^+(d)\frac{\omega(d)}{d}f_\kappa\left(\frac{\log(y_1/d)}{\log z_1}\right)$$

$$+\sum_{\substack{d\mid P(z_1,z)\\ \nu(d)=2s}} \bar{\chi}_{y_1}^{+}(d)\frac{\omega(d)}{d}V(p(d))\frac{1}{\sigma_\kappa((\log y_1/d)/\log p(d))}$$

$$-\sum_{\substack{d\mid P(z_1,z)\\ \nu(d)=2s+1}} \chi_{y_1}^{+}(d)\frac{\omega(d)}{d}V(p(d))\,f_\kappa\left(\frac{\log(y_1/d)}{\log p(d)}\right)$$

$$+\sum_{\substack{d\mid P(z_1,z)\\ \nu(d)=2s-1}} \chi_{y_1}^{+}(d)\frac{\omega(d)}{d}V(p(d))\,f_\kappa\left(\frac{\log(y_1/d)}{\log p(d)}\right)$$

$$=H_1-H_2+H_3-H_4+H_5, \tag{6.6}$$

say. We introduce the sum

$$H_6=\sum_{\substack{d\mid P(z_1,z)\\ \nu(d)=2s}} \chi_{y_1}^{+}(d)\frac{\omega(d)}{d}V(p(d))\,F_\kappa\left(\frac{\log(y_1/d)}{\log p(d)}\right) \tag{6.7}$$

and write

$$E_{2s+1}^{+}-E_{2s-1}^{+}=(H_1-H_6-H_4)+(H_6+H_3-H_2+H_5). \tag{6.8}$$

It is easy to see that

$$H_1-H_6-H_4=\sum_{\substack{d\mid P(z_1,z)\\ \nu(d)=2s}}\left\{V(z_1)\,\chi_{y_1}^{+}(d)\frac{\omega(d)}{d}F_\kappa\left(\frac{\log(y_1/d)}{\log z_1}\right)\right.$$

$$-\chi_{y_1(d)}^{+}\frac{\omega(d)}{d}V(p(d))\,F_\kappa\left(\frac{\log(y_1/d)}{\log p(d)}\right)$$

$$\left.-\sum_{z_1\leqslant p<p(d)}\chi_{y_1}^{+}(pd)\frac{\omega(pd)}{pd}V(p)\,f_\kappa\left(\frac{\log(y_1/dp)}{\log p}\right)\right\}$$

$$=\sum_{\substack{d\mid P(z_1,z)\\ \nu(d)=2s}}\chi_{y_1}^{+}(d)\frac{\omega(d)}{d}\left\{V(z_1)\,F_\kappa\left(\frac{\log(y_1/d)}{\log z_1}\right)\right.$$

$$-V(p(d))\,F_\kappa\left(\frac{\log(y_1/d)}{\log p(d)}\right)$$

$$\left.-\sum_{z_1\leqslant p<p(d)}\eta_{y_1}^{+}(dp)\frac{\omega(p)}{p}V(p)\,f_\kappa\left(\frac{\log(y_1/dp)}{\log p}\right)\right\};$$

but when $\nu(dp)=2s+1$ is odd, $\eta_{y_1}^{+}(dp)=1$ when, by (2.19), $p^{\beta_\kappa+1}d<y_1$,

that is, when $u=\log(y_1/dp)/\log p>\beta_\kappa$—and otherwise $\eta^+_{y_1}(dp)=0$. But then $u<\beta_\kappa$ and $f_\kappa(u)=0$ (cf. (1.10ii)) anyway, so that we may write

$$H_1-H_6-H_4=\sum_{\substack{d\mid P(z_1,z)\\ \nu(d)=2s}}\chi^+_{y_1}(d)\frac{\omega(d)}{d}\left\{V(z_1)\,F_\kappa\left(\frac{\log(y_1/d)}{\log z_1}\right)\right.$$
$$-V(p(d))\,F_\kappa\left(\frac{\log(y_1/d)}{\log p(d)}\right)$$
$$\left.-\sum_{z_1\leqslant p<p(d)}\frac{\omega(p)}{p}V(p)\,f_\kappa\left(\frac{\log(y_1/dp)}{\log p}\right)\right\}. \qquad (6.9)$$

Similarly we have

$$H_6+H_3-H_2+H_5$$
$$=\sum_{\substack{d\mid P(z_1,z)\\ \nu(d)=2s-1}}\left\{\sum_{z_1\leqslant p<p(d)}\frac{\omega(dp)}{d}V(p)\left(\chi^+_{y_1}(dp)\,F_\kappa\left(\frac{\log(y_1/dp)}{\log p}\right)\right.\right.$$
$$\left.+\bar{\chi}^+_{y_1}(dp)\frac{1}{\sigma_\kappa(\log(y_1/dp)/\log p)}\right)$$
$$-V(z_1)\,\chi^+_{y_1}(d)\frac{\omega(d)}{d}f_\kappa\left(\frac{\log(y_1/d)}{\log z_1}\right)$$
$$\left.+\chi^+_{y_1}(d)\frac{\omega(d)}{d}V(p(d))\,f_\kappa\left(\frac{\log(y_1/d)}{\log p(d)}\right)\right\}.$$

In the inner sum of the latter expression, the form in parentheses is (cf. (2.11))

$$\chi^+_{y_1}(d)\left(\eta^+_{y_1}(dp)\,F_\kappa\left(\frac{\log(y_1/dp)}{\log p}\right)+(1-\eta^+_{y_1}(dp))\,\sigma_\kappa^{-1}\left(\frac{\log(y_1/dp)}{\log p}\right)\right).$$

When $\eta^+_{y_1}(dp)=1$, this equals

$$\chi^+_{y_1}(d)\,F_\kappa\left(\frac{\log(y_1/dp)}{\log p}\right); \qquad (6.10)$$

when $\eta^+_{y_1}(dp)=0$, it equals

$$\chi^+_{y_1}(d)\,\sigma_\kappa^{-1}\left(\frac{\log(y_1/dp)}{\log p}\right).$$

But when $\nu(dp)=2s$ and $\eta^+_{y_1}(dp)=0$, (2.19) tells us that $p^{\alpha_\kappa+1}d\geqslant y_1$, so that $u=\log(y_1/dp)/\log p\leqslant\alpha_\kappa$ and consequently $F_\kappa(u)=1/\sigma_\kappa(u)$

(cf. (1.10i)). Hence the term in parentheses is given by (6.10) in either case, and we have

$$H_6 + H_3 - H_2 + H_5$$
$$= \sum_{\substack{d \mid P(z_1,z) \\ \nu(d) = 2s-1}} \chi_{y_1}^{+}(d) \frac{\omega(d)}{d} \left\{ \sum_{z_1 \leqslant p < p(d)} \frac{\omega(p)}{p} V(p) F_\kappa\left(\frac{\log(y_1/dp)}{\log p}\right) - V(z_1) f_\kappa\left(\frac{\log(y_1/d)}{\log z_1}\right) + V(p(d)) f_\kappa\left(\frac{\log(y_1/d)}{\log p(d)}\right)\right\}. \tag{6.11}$$

To estimate the expressions (6.9) and (6.11) we turn to Lemma 5.2, and apply it with $x = y_1/d$ and $w = p(d)$. The expressions in parentheses on the right of (6.9) and (6.11) are each at most

$$\frac{A}{\sigma_\kappa(1)} \frac{V(p(d))}{\log z_1} \left(\frac{\log p(d)}{\log z_1}\right)^\kappa$$

provided that $y_1 \geqslant p(d)^{\alpha_\kappa} d$ in the sum on the right of (6.9) and $y_1 \geqslant p(d)^{\beta_\kappa} d$ on the right of (6.11). But this is indeed the case, for in (6.9), $\nu(d)$ even and $\chi_{y_1}^{+}(d) = 1$ imply that $p(d)^{\alpha_\kappa} d < y_1$, and in (6.11), $\nu(d)$ odd and $\chi_{y_1}^{+}(d) = 1$ imply that $p(d)^{\beta_\kappa} d < y_1$ (cf. (2.19) and (2.18)). Hence, by (6.8), (6.9), and (6.11) we have

$$E_{2s+1}^{+} - E_{2s-1}^{+}$$
$$\leqslant \frac{A}{\sigma_\kappa(1)} \frac{1}{\log z_1} \sum_{\substack{d \mid P(z_1,z) \\ \nu(d) = 2s-1,2s}} \chi_{y_1}^{+}(d) \frac{\omega(d)}{d} V(p(d)) \left(\frac{\log p(d)}{\log z_1}\right)^\kappa;$$

it follows from addition that,

$$E_{2s+1}^{+} - E_1^{+}$$
$$\leqslant \frac{A}{\sigma_\kappa(1)} \frac{1}{\log z_1} \sum_{\substack{d \mid P(z_1,z) \\ \nu(d) \leqslant 2s}} \chi_{y_1}^{+}(d) \frac{\omega(d)}{d} V(p(d)) \left(\frac{\log p(d)}{\log z_1}\right)^\kappa,$$

so that, by (6.5), if $y_1 \geqslant z^{\alpha_\kappa}$,

$$E_{2s+1}^{+} \leqslant \frac{3A^2}{\sigma_\kappa(1)} \frac{V(z)}{\log z_1} \left(\frac{\log z}{\log z_1}\right)^\kappa$$
$$\times \left\{1 + \sum_{\substack{d \mid P(z_1,z) \\ \nu(d) \leqslant 2s}} \chi_{y_1}^{+}(d) \frac{\omega(d)}{d} \frac{V(p(d))}{V(z)} \left(\frac{\log p(d)}{\log z}\right)^\kappa\right\}.$$

It is evident from (4.32) and (6.2) that $E^+ = \lim_{s\to\infty} E^+_{2s+1} + V(z)\, F_\kappa(\log y_1/\log z)$, whence, by $(\Omega(\kappa))$,

$$E^+ \leqslant V(z)\, F_\kappa\left(\frac{\log y_1}{\log z}\right) + \frac{3A^2}{\sigma_\kappa(1)} \frac{V(z)}{\log z_1} \left(\frac{\log z}{\log z_1}\right)^\kappa \left(1 + \frac{A}{\log z_1}\right)$$
$$\times \left\{1 + \sum_{r=1}^{\infty} \sum_{\substack{d \mid P(z_1, z) \\ \nu(d) = r}} \frac{\omega(d)}{d}\right\}.$$

But

$$1 + \sum_{r=1}^{\infty} \sum_{\substack{d \mid P(z_1, z) \\ \nu(d) = r}} \frac{\omega(d)}{d} \leqslant 1 + \sum_{r=1}^{\infty} \frac{1}{r!} \left(\sum_{z_1 \leqslant p < z} \frac{\omega(p)}{p}\right)^r$$
$$= \exp\left(\sum_{z_1 \leqslant p < z} \frac{\omega(p)}{p}\right) \leqslant \frac{V(z_1)}{V(z)}$$
$$\leqslant \left(\frac{\log z}{\log z_1}\right)^\kappa \left(1 + \frac{A}{\log z_1}\right),$$

using $(\Omega(\kappa))$ once again; hence

$$E^+ \leqslant V(z)\, F_\kappa\left(\frac{\log y_1}{\log z}\right) + \frac{3A^2}{\sigma_\kappa(1)} \left(1 + \frac{A}{\log z_1}\right)^2 \frac{V(z)}{\log z_1} \left(\frac{\log z}{\log z_1}\right)^{2\kappa}, \qquad y_1 \geqslant z^{\alpha_\kappa},$$

so that, by (4.5)

$$E^+ \leqslant V(z)\, F_\kappa\left(\frac{\log y_1}{\log z}\right) + \frac{3A^2}{\sigma_\kappa(1)} \left(1 + \frac{A}{\log 2}\right)^2 \left(\frac{\log y}{\log z_1}\right)^{2\kappa+1} \frac{V(z)}{\log y}$$
$$\leqslant V(z) \left\{F_\kappa\left(\frac{\log y_1}{\log z}\right) + O\left(\frac{\log\log y}{(\log y)^{1/(2\kappa+2)}}\right)\right\}, \qquad y_1 \geqslant z^{\alpha_\kappa}. \tag{6.12}$$

It follows from (4.31) that

$$S(\mathscr{A}, \mathscr{P}, z) \leqslant XV(z) \left\{F_\kappa\left(\frac{\log y_1}{\log z}\right) + O\left(\frac{\log\log y}{(\log y)^{1/(2\kappa+2)}}\right)\right\}$$
$$+ \sum_{\substack{m \mid P(z) \\ m < y}} c^+(m)\, R_m, \qquad z_1 \leqslant z < y_1^{1/\alpha_\kappa}.$$

If $y_1^{1/\alpha_\kappa} \leqslant z \leqslant y_1$ there is (4.17) at our disposal, now to be applied with $q = 1$,

$w = z$ and (cf. (4.16)) with $Y = y_1$—so that $1 \leqslant \tau \leqslant \alpha_\kappa$—in view of (1.10i), we have immediately

$$S(\mathscr{A}, \mathscr{P}, z) \leqslant XV(z)\left\{F_\kappa\left(\frac{\log y_1}{\log z}\right) + O\left(\frac{\log\log y}{(\log y)^{1/(2\kappa+2)}}\right)\right\}$$
$$+ \sum_{\substack{m \mid P(z)\\ m < y}} c_0^+(m)\, R_m, \qquad z_1 \leqslant z \leqslant y_1, \tag{6.13}$$

where $c_0^+(m) = c^+(m) + r_m$. In view of the remarks at the conclusion of Section 3, we may drop the condition $z_1 \leqslant z$ in this inequality, and replace it by $2 \leqslant z$.

By (1.15) with $u_1 = \log y_1/\log z$ and $u_2 = \log y/\log z$,

$$F_\kappa\left(\frac{\log y_1}{\log z}\right) - F_\kappa\left(\frac{\log y}{\log z}\right) \leqslant \frac{\log(y/y_1)}{\log y_1}\,\kappa A_\kappa \ll \frac{\log\log y}{(\log y)^{1/(2\kappa+2)}}$$

by (4.7). Hence $F_\kappa(\log y_1/\log z)$ may be replaced by $F_\kappa(\log y/\log z)$ on the right of (6.13). This all but proves the upper bound in our main theorem. All that remains to do is to bridge the gap

$$y_1 < z \leqslant y,$$

but this is straightforward. We have only to observe that, initially,

$$S(\mathscr{A}, \mathscr{P}, z) < S(\mathscr{A}, \mathscr{P}, y_1) \qquad \text{if} \quad y_1 < z,$$

and to apply (6.13), with $z = y_1$, to $S(\mathscr{A}, \mathscr{P}, y_1)$. Only the first term on the right requires examination. Here

$$F_\kappa(1) \leqslant F_\kappa\left(\frac{\log y_1}{\log z}\right) \leqslant F_\kappa\left(\frac{\log y}{\log z}\right) + \frac{\log(y/y_1)}{\log y_1}\,\kappa A_\kappa$$
$$\leqslant F_\kappa\left(\frac{\log y}{\log z}\right) + O\left(\frac{\log\log y}{(\log y)^{1/(2\kappa+2)}}\right)$$

as above, and, by $(\Omega(\kappa))$,

$$V(y_1) = V(z)\frac{V(y_1)}{V(z)} \leqslant V(z)\left(\frac{\log z}{\log y_1}\right)^\kappa \left(1 + \frac{A}{\log y_1}\right)$$
$$\leqslant V(z)\left(\frac{\log y}{\log y_1}\right)^\kappa \left(1 + \frac{A}{\log y_1}\right)$$
$$\leqslant V(z)\left(1 + O\left(\frac{\log\log y}{(\log y)^{1/(2\kappa+2)}}\right)\right)$$

by (4.7). Hence even when $y_1 < z \leqslant y$ we have

$$S(\mathscr{A}, \mathscr{P}, z) \leqslant S(\mathscr{A}, \mathscr{P}, y_1) \leqslant XV(z)\left\{1 + O\left(\frac{\log\log y}{(\log y)^{1/(2\kappa+2)}}\right)\right\}\left\{F_\kappa\left(\frac{\log y}{\log z}\right) + O\left(\frac{\log\log y}{(\log y)^{1/(2\kappa+2)}}\right)\right\} + \sum_{\substack{m \mid P(y_1) \\ m < y}} c_0^+(m) R_m$$
$$\leqslant XV(z)\left\{F_\kappa\left(\frac{\log y}{\log z}\right) + O\left(\frac{\log\log y}{(\log y)^{1/(2\kappa+2)}}\right)\right\} + \sum_{\substack{m \mid P(z) \\ m < y}} c_0^+(m) R_m,$$

where $c_0^+(m) = 0$ if $m \mid P(z)$ but $m \nmid P(y_1)$. This proves the upper bound part of our main theorem.

We turn to the estimation of E^-, as given by (4.33), and aim for (4.36). Accordingly, we assume that

$$z_1 < z \leqslant y^{1/\beta}, \qquad \beta = \beta_\kappa \geqslant 2. \tag{6.14}$$

The procedure we follow is similar to that used in the discussion of E^+, but we give it in detail for the sake of completeness.

Define, for $r \geqslant 1$,

$$E_r^- := V(z) f_\kappa\left(\frac{\log y_1}{\log z}\right) - V(z_1) \sum_{\substack{d \mid P(z_1, z) \\ \nu(d) < r}} \mu(d) \chi_{y_1}^-(d) \times \frac{\omega(d)}{d} \phi_\kappa^{(-)^{\nu(d)+1}}\left(\frac{\log y_1/d}{\log z_1}\right)$$
$$+ \sum_{\substack{d \mid P(z_1, z) \\ \mu(d) = -1 \\ \nu(d) < r}} \bar{\chi}_{y_1}^-(d) \frac{\omega(d)}{d} V(p(d)) \frac{1}{\sigma_\kappa((\log y_1/d)/\log p(d))}$$
$$- (-1)^r \sum_{\substack{d \mid P(z_1, z) \\ \nu(d) = r}} \chi_{y_1}^-(d) \frac{\omega(d)}{d} V(p(d)) \phi_\kappa^{(-)^{r+1}}\left(\frac{\log y_1/d}{\log p(d)}\right), \tag{6.15}$$

so that

$$E^- = V(z) f_\kappa\left(\frac{\log y_1}{\log z}\right) - \lim_{r \to \infty} E_r^-; \tag{6.16}$$

we require an upper bound for E_r^-. We have

$$E_1^- = V(z) f_\kappa\left(\frac{\log y_1}{\log z}\right) - V(z_1) f_\kappa\left(\frac{\log y_1}{\log z_1}\right) + \sum_{z_1 \leqslant p < z} \chi_{y_1}^-(p)\frac{\omega(p)}{p} V(p) F_\kappa\left(\frac{\log y_1/p}{\log p}\right).$$

If we apply Lemma 5.2, (5.5), with $w = z$ and $x = y_1$ we obtain at once (since $\chi_{y_1}^-(p) = 1 - \bar{\chi}_{y_1}^-(p)$ by (2.2) with $d = p$)

$$E_1^- \leqslant \frac{A}{\sigma_\kappa(1)}\frac{V(z)}{\log z_1}\left(\frac{\log z}{\log z_1}\right)^\kappa - \sum_{z_1 \leqslant p < z} \bar{\chi}_1^-(p)\frac{\omega(p)}{p} V(p) F_\kappa\left(\frac{\log y_1/p}{\log p}\right). \tag{6.17}$$

Next,

$$\begin{aligned}
E_2^- &= E_1^- + (E_2^- - E_1^-) \\
&= E_1^- + V(z_1) \sum_{z_1 \leqslant p < z} \chi_{y_1}^-(p)\frac{\omega(p)}{p} F_\kappa\left(\frac{\log y_1/p}{\log z_1}\right) \\
&\quad + \sum_{z_1 \leqslant p < z} \bar{\chi}_{y_1}^-(p)\frac{\omega(p)}{p} V(p)\frac{1}{\sigma_\kappa((\log y_1/p)/\log p)} \\
&\quad - \sum_{z_1 \leqslant p < z} \chi_{y_1}^-(p)\frac{\omega(p)}{p} V(p) F_\kappa\left(\frac{\log y_1/p}{\log p}\right) \\
&\quad - \sum_{z_1 \leqslant p < z} \frac{\omega(p)}{p} \sum_{z_1 \leqslant p < p} \chi_{y_1}^-(pp_1)\frac{\omega(p_1)}{p_1} V(p_1) f_\kappa\left(\frac{\log(y_1/pp_1)}{\log p_1}\right) \\
&= E_1^- + \sum_{z_1 \leqslant p < z} \chi_{y_1}^-(p)\frac{\omega(p)}{p}\left\{V(z_1) F_\kappa\left(\frac{\log y_1/p}{\log z_1}\right) - V(p) F_\kappa\left(\frac{\log y_1/p}{\log p}\right)\right. \\
&\quad \left. - \sum_{z_1 \leqslant p < p} \eta_{y_1}^-(pp_1)\frac{\omega(p_1)}{p_1} V(p_1) f_\kappa\left(\frac{\log(y_1/pp_1)}{\log p_1}\right)\right\} \\
&\quad + \sum_{z_1 \leqslant p < z} \bar{\chi}_{y_1}^-(p)\frac{\omega(p)}{p} V(p)\frac{1}{\sigma_\kappa((\log y_1/p)/\log p)}.
\end{aligned}$$

In the inner sum over p_1, $\eta_{y_1}^-(pp_1)$ may be replaced by 1, for, by (2.12), it equals 1 if $p_1^{\beta_\kappa+1}p < y_1$, and if $p_1^{\beta_\kappa+1} \geqslant y_1$ it is zero, but then so is $f_\kappa(\log(y_1/pp_1)/\log p_1)$, by (1.10ii). This inner sum therefore is, by (5.5) (with $w = p$ and $x = y_1/p$), at most

$$\frac{A}{\sigma_\kappa(1)}\frac{V(p)}{\log z_1}\left(\frac{\log p}{\log z_1}\right)^\kappa \qquad \text{if} \quad y_1 \geqslant p^{\alpha_\kappa+1}.$$

But $\chi_{y_1}^{-}(p) = 1$ if and only if $y_1 > p^{\alpha_\kappa + 1}$ (cf. (2.13)), and is otherwise zero. Hence

$$\begin{aligned} E_2^- &\leqslant E_1^- + \frac{A}{\sigma_\kappa(1)} \frac{1}{\log z_1} \sum_{\substack{z_1 \leqslant p < z \\ p^{\alpha_\kappa+1} < y_1}} \frac{\omega(p)}{p} V(p) \left(\frac{\log p}{\log z_1}\right)^\kappa \\ &\quad + \sum_{z_1 \leqslant p < z} \bar{\chi}_{y_1}^{-}(p) \frac{\omega(p)}{p} V(p) \frac{1}{\sigma_\kappa((\log y_1/p)/\log p)} \\ &\leqslant \frac{A}{\sigma_\kappa(1)} \frac{V(z)}{\log z_1} \left(\frac{\log z}{\log z_1}\right)^\kappa \left\{1 + \sum_{z_1 \leqslant p < z} \frac{\omega(p)}{p} \frac{V(p)}{V(z)} \left(\frac{\log p}{\log z}\right)^\kappa\right\} \end{aligned} \tag{6.18}$$

by (6.17), since $1/\sigma_\kappa(u) = F_\kappa(u)$ when $u \leqslant \alpha_\kappa$ and

$$\frac{\log(y_1/p)}{\log p} \leqslant \alpha_\kappa$$

when $y_1 \leqslant p^{\alpha_\kappa + 1}$, i.e., precisely when $\bar{\chi}_{y_1}^{-}(p) = 1$.

For $s \geqslant 1$ we now consider $E_{2s+2}^- - E_{2s}^-$. By (6.15) we have

$$\begin{aligned} E_{2s+2}^- - E_{2s}^- &= V(z_1) \sum_{\substack{d \mid P(z_1,z) \\ \nu(d) = 2s+1}} \chi_{y_1}^{-}(d) \frac{\omega(d)}{d} F_\kappa\left(\frac{\log y_1/d}{\log z_1}\right) \\ &\quad - V(z_1) \sum_{\substack{d \mid P(z_1,z) \\ \nu(d) = 2s}} \chi_{y_1}^{-}(d) \frac{\omega(d)}{d} f_\kappa\left(\frac{\log y_1/d}{\log z_1}\right) \\ &\quad + \sum_{\substack{d \mid P(z_1,z) \\ \nu(d) = 2s+1}} \bar{\chi}_{y_1}^{-}(d) \frac{\omega(d)}{d} V(p(d)) \frac{1}{\sigma_\kappa((\log y_1/d)/\log p(d))} \\ &\quad - \sum_{\substack{d \mid P(z_1,z) \\ \nu(d) = 2s+2}} \chi_{y_1}^{-}(d) \frac{\omega(d)}{d} V(p(d)) f_\kappa\left(\frac{\log y_1/d}{\log p(d)}\right) \\ &\quad + \sum_{\substack{d \mid P(z_1,z) \\ \nu(d) = 2s}} \chi_{y_1}^{-}(d) \frac{\omega(d)}{d} V(p(d)) f_\kappa\left(\frac{\log y_1/d}{\log p(d)}\right). \end{aligned}$$

In the third sum on the right, the presence of $\bar{\chi}_{y_1}^{-}(d)$ implies that we may take $\eta_{y_1}^{-}(d) = 0$; since $\nu(d)$ is odd this means (cf. (2.12)) that $p(d)^{\alpha_\kappa} \geqslant y_1/d$ so that we may write $F_\kappa((\log y_1/d)/\log p(d))$ in place of $1/\sigma_\kappa((\log y_1/d)/\log p(d))$. Having done that, replace $\bar{\chi}_{y_1}^{-}(d)$ in the third sum by $\chi_{y_1}^{-}(d/p(d)) - \chi_{y_1}^{-}(d)$ (cf. (2.2))—in other words, write the third sum as

$$\sum_{\substack{d \mid P(z_1,z) \\ \nu(d)=2s+1}} \chi_{y_1}^{-}\left(\frac{d}{p(d)}\right) \frac{\omega(d)}{d} V(p(d)) F_\kappa\left(\frac{\log y_1/d}{\log p(d)}\right)$$

$$- \sum_{\substack{d \mid P(z_1,z) \\ \nu(d)=2s+1}} \chi_{y_1}^{-}(d) \frac{\omega(d)}{d} V(p(d)) F_\kappa\left(\frac{\log y_1/d}{\log p(d)}\right).$$

Then

$$\begin{aligned} &E_{2s+2}^{-} - E_{2s}^{-} \\ &= \sum_{\substack{d \mid P(z_1,z) \\ \nu(d)=2s+1}} \chi_{y_1}^{-}(d) \frac{\omega(d)}{d} \left\{ V(z_1) F_\kappa\left(\frac{\log y_1/d}{\log z_1}\right) - V(p(d)) F_\kappa\left(\frac{\log y_1/d}{\log p(d)}\right) \right. \\ &\quad \left. - \sum_{z_1 \leqslant p < p(d)} \eta_{y_1}^{-}(dp) \frac{\omega(p)}{p} V(p) f_\kappa\left(\frac{\log(y_1/dp)}{\log p}\right) \right\} \\ &\quad + \sum_{\substack{d \mid P(z_1,z) \\ \nu(d)=2s}} \chi_{y_1}^{-}(d) \frac{\omega(d)}{d} \left\{ \sum_{z_1 \leqslant p < p(d)} \frac{\omega(p)}{p} V(p) F_\kappa\left(\frac{\log(y_1/dp)}{\log p}\right) \right. \\ &\quad \left. - V(z_1) f_\kappa\left(\frac{\log y_1/d}{\log z_1}\right) + V(p(d)) f_\kappa\left(\frac{\log y_1/d}{\log p(d)}\right) \right\}. \end{aligned}$$

In the first sum on the right, the factor $\eta_{y_1}^{-}(dp)$ may be replaced by 1, for $\eta_{y_1}^{-}(dp)$ vanishes precisely when the f-term does (remember that $f_\kappa(u)=0$ when $u \leqslant \beta_\kappa$). In the first sum also, $\chi_{y_1}^{-}(d)=1$ implies that $p(d)^{\alpha_\kappa} < y_1/d$, and in the second sum $\chi_{y_1}^{-}(d)=1$ implies that $p(d)^{\beta_\kappa} < y_1/d$. Applying Lemma 5.2 in the two sums, with $x = y_1/d$ and $w = p(d)$, we obtain

$$E_{2s+2}^{-} - E_{2s}^{-} \leqslant \frac{A}{\sigma_\kappa(1)} \frac{V(z)}{\log z_1} \left(\frac{\log z}{\log z_1}\right)^\kappa \sum_{\substack{d \mid P(z_1,z) \\ \nu(d)=2s,2s+1}} \frac{\omega(d)}{d} \times \frac{V(p(d))}{V(z)} \left(\frac{\log p(d)}{\log z}\right)^\kappa.$$

Hence, by (6.18) and then the identical argument leading up to (6.12),

$$\begin{aligned} E_{2s+2}^{-} &\leqslant \frac{A}{\sigma_\kappa(1)} \frac{V(z)}{\log z_1} \left(\frac{\log z}{\log z_1}\right)^\kappa \\ &\quad \times \left\{ 1 + \sum_{r=1}^{\infty} \sum_{\substack{d \mid P(z_1,z) \\ \nu(d)=r}} \frac{\omega(d)}{d} \frac{V(p(d))}{V(z)} \left(\frac{\log p(d)}{\log z}\right)^\kappa \right\} \\ &\leqslant \frac{A}{\sigma_\kappa(1)} \left(1 + \frac{A}{\log z_1}\right)^2 \frac{V(z)}{\log z_1} \left(\frac{\log z}{\log z_1}\right)^{2\kappa} \ll V(z) \frac{\log\log y}{(\log y)^{1/(2\kappa+2)}}. \end{aligned}$$

It follows from (6.16) that, subject to (6.14),

$$E^- \geqslant V(z)\left\{f_\kappa\left(\frac{\log y_1}{\log z}\right) + O\left(\frac{\log\log y}{(\log y)^{1/(2\kappa+2)}}\right)\right\}, \tag{6.19}$$

and consequently, by (4.31), that

$$S(\mathscr{A}, \mathscr{P}, z) \geqslant XV(z)\left\{f_\kappa\left(\frac{\log y_1}{\log z}\right) + O\left(\frac{\log\log y}{(\log y)^{1/(2\kappa+2)}}\right)\right\}$$

$$-\sum_{\substack{m \mid P(z)\\ m<y}} c^-(m)\, R_m, \qquad z_1 < z \leqslant y_1^{1/\beta_\kappa}.$$

The rest is cosmetics. Equation (1.15) in conjunction with (4.7) permits us as before to replace $f_\kappa(\log y_1/\log z)$ on the right by $f_\kappa(\log y/\log z)$. Moreover, our main theorem gives only the trivial lower bound $S(\mathscr{A}, \mathscr{P}, z) \geqslant 0$ when $y_1^{1/\beta_\kappa} \leqslant z \leqslant y^{1/\beta_\kappa}$; for with such a z,

$$f_\kappa\left(\frac{\log y}{\log z}\right) \leqslant f_\kappa\left(\beta_\kappa \frac{\log y}{\log y_1}\right) = f_\kappa\left(\beta_\kappa + \beta_\kappa \frac{\log(y/y_1)}{\log y_1}\right) \leqslant \kappa A_\kappa \frac{\log(y/y_1)}{\log y_1}$$

$$\ll \frac{\log\log y}{(\log y)^{1/(2\kappa+2)}}$$

by (1.15) and (4.7). Thus the proof of our theorem is, in fact, complete.

Appendix I

Proof of (1.15). By the mean-value theorem

$$F_\kappa(u_1) - F_\kappa(u_2) = -F_\kappa'(u_0)(u_2 - u_1), \qquad u_1 < u_0 < u_2,$$

and

$$f_\kappa(u_2) - f_\kappa(u_1) = f_\kappa'(u^*)(u_2 - u_1), \qquad u_1 < u^* < u_2.$$

If $u_0 > \alpha_\kappa$, (1.10iii) implies that

$$-F_\kappa'(u_0) = \frac{\kappa}{u_0}\left(F_\kappa(u_0) - f_\kappa(u_0 - 1)\right) \leqslant \frac{\kappa}{u_0} F_\kappa(u_0) \leqslant \frac{1}{u_1} \kappa F_\kappa(1).$$

If $u_0 \leqslant \alpha_\kappa$, $F_\kappa(u_0) = 1/(\sigma_\kappa(u_0))$ and (1.8) applies: if $u_0 \leqslant 2$ even,

$$-F_\kappa'(u_0) = \frac{\kappa}{u_0} \frac{1}{\sigma_\kappa(u_0)} \leqslant \frac{\kappa}{u_1} \frac{1}{\sigma_\kappa(u_1)} \leqslant \frac{1}{u_1} \kappa F_\kappa(1),$$

and if $2 < u_0 \leqslant \alpha_\kappa$, the second statement in (1.8) implies that

$$-F'_\kappa(u_0) \leqslant \frac{\kappa}{u_0} \frac{1}{\sigma_\kappa(u_0)} \leqslant \frac{1}{u_1} \kappa F_\kappa(1).$$

Since $F_\kappa(1) = 1/\sigma_\kappa(1) = A_\kappa$, the first statement in the lemma follows.

Now for the second statement. We may as well suppose that $u^* > \beta_\kappa$, and then, by (1.10iv),

$$f'_\kappa(u^*) = \frac{\kappa}{u^*}(F_\kappa(u^* - 1) - f_\kappa(u^*)) \leqslant \frac{1}{u_1} \kappa F_\kappa(u^* - 1) \leqslant \frac{1}{u_1} \kappa F_\kappa(1)$$

since $\beta_\kappa \geqslant 2$. This leads at once to the second statement of (1.15).

Appendix II

Inequality (4.17) was proved in [6] under the two-sided condition

$$-L \leqslant \sum_{w_1 \leqslant p < w} \frac{\omega(p) \log p}{p} - \kappa \log \frac{w}{w_1} \leqslant A, \qquad 2 \leqslant w_1 \leqslant w, \qquad (*)$$

where $A \geqslant 1$ and $L \geqslant 1$ are independent of w_1 and w. In [12], Rawsthorne shows in an ingenious way that the right hand inequality alone suffices. The right hand inequality in (*) is, however, not a consequence of our $(\Omega(\kappa))$. This is readily seen from the example

$$\omega(p) = \kappa\left(1 + \frac{1}{\log p}\right),$$

which satisfies $(\Omega(\kappa))$ but not the upper inequality in (*).

We follow the procedure in [12] (first suggested by Jurkat in a lecture) of showing that, for the purpose at hand, the values of $\omega(\cdot)$ may be "topped up" so as to satisfy a two-sided inequality (see (5) below)). The details are somewhat more complicated than in [12].

From Mertens prime number theory we know that

$$\prod_{w_1 \leqslant p < w_2} \left(1 - \frac{1}{p}\right)^{-1} = \frac{\log w_2}{\log w_1}\left(1 + O\left(\frac{1}{\log w_1}\right)\right), \qquad 2 \leqslant w_1 < w_2. \quad (1)$$

Let $g(\cdot)$ be a non-negative function defined on the primes such that

$$\prod_{w_1 \leqslant p < w_2} (1 + g(p)) \leqslant \left(\frac{\log w_2}{\log w_1}\right)^\kappa \left(1 + \frac{A}{\log w_1}\right), \qquad 2 \leqslant w_1 < w_2, \quad (2)$$

where $\kappa \geqslant 1$ and $A \geqslant 1$ are constants. In view of (1) we may write (2) in the form

$$\prod_{w_1 \leqslant p < w_2} (1+g(p))\left(1-\frac{1}{p}\right)^{\kappa} \leqslant 1+\frac{A_1}{\log w_1}, \qquad 2 \leqslant w_1 < w_2,$$

where $A_1 \geqslant 1$ is a constant. Let us even weaken this condition on g a little to

$$\prod_{w_1 \leqslant p < w_2} (1+g(p))\left(1-\frac{1}{p}\right)^{\kappa} \leqslant \exp\left(\frac{A_1}{\log w_1}\right), \qquad 2 \leqslant w_1 < w_2. \tag{3}$$

LEMMA. *Let $g(\cdot)$ be a non-negative arithmetic function whose values at the primes satisfy* (3). *Then there exists a function $g'(\cdot)$ defined on the primes such that*

$$g'(p) \geqslant g(p) \tag{4}$$

for every prime and

$$\exp\left(-\frac{8A_1}{\log u}\right) \leqslant \prod_{u \leqslant p \leqslant v} (1+g'(p))\left(1-\frac{1}{p}\right)^{\kappa} \leqslant \exp\left(\frac{4A_1}{\log u}\right) \tag{5}$$

for all pairs of integers u, v satisfying $2 \leqslant u < v$.

By (1) it follows directly from (5) that there exists a constant $B \geqslant 1$ such that

$$\left(\frac{\log w_2}{\log w_1}\right)^{\kappa} \exp\left(-\frac{B}{\log w_1}\right) \leqslant \prod_{w_1 \leqslant p < w_2} (1+g'(p))$$
$$\leqslant \left(\frac{\log w_2}{\log w_1}\right)^{\kappa} \exp\left(\frac{B}{\log w_1}\right), \qquad 2 \leqslant w_1 < w_2. \tag{6}$$

Proof of the Lemma. Let

$$b_p := \log(1+g(p))\left(1-\frac{1}{p}\right)^{\kappa} \tag{7}$$

for all primes p. Then, by (3),

$$\sum_{w_1 \leqslant p < w_2} b_p \leqslant \frac{A_1}{\log w_1}, \qquad 2 \leqslant w_1 < w_2. \tag{8}$$

Let q be the *least* prime such that $b_q > 0$. If $q > 2$, define

$$b'_p = 0 \geqslant b_p, \qquad p < q. \tag{9}$$

Suppose that for every integer r, $q \leqslant r \leqslant n$,

$$\sum_{q \leqslant p \leqslant r} b_p > 0,$$

but that

$$\sum_{q \leqslant p \leqslant n+1} b_p \leqslant 0. \tag{10}$$

The integer n here defined may be very large or even infinite. But if it is finite then $n+1$ is a prime and $b_{n+1} < 0$. We break up the argument into two cases.

Case I. $n \leqslant q^2$. Define

$$\begin{aligned} b'_p &= b_p, \qquad q \leqslant p \leqslant n, \\ b'_{n+1} &= -\sum_{q \leqslant p \leqslant n} b_p (\geqslant b_{n+1}). \end{aligned} \tag{11}$$

Suppose that $[u, v] \subset [q, n+1]$. When $v \leqslant n$, we deduce at once from (8) that

$$\sum_{u \leqslant p \leqslant v} b'_p \leqslant \frac{A_1}{\log u},$$

and since $b'_{n+1} < 0$ this inequality is all the more true when $v = n+1$. To estimate this sum from below we suppose first that $v \leqslant n$. Then, by (8) and (11),

$$0 < \sum_{q \leqslant p \leqslant v} b_p = \sum_{q \leqslant p \leqslant u-1} b_p + \sum_{u \leqslant p \leqslant v} b_p \leqslant \frac{A_1}{\log q} + \sum_{u \leqslant p \leqslant v} b'_p$$

so that

$$\sum_{u \leqslant p \leqslant v} b'_p > -\frac{A_1}{\log q} \geqslant -\frac{2A_1}{\log u}$$

(because $u \leqslant v \leqslant n \leqslant q^2$). Next, admit the possibility that $v = n+1$. By (11) and (8), $b'_{n+1} \geqslant -A_1/\log q$, whence, by the preceding argument,

$$\sum_{u \leqslant p \leqslant n+1} b'_p > -\frac{A_1}{\log q} + b'_{n+1} \geqslant -\frac{2A_1}{\log q} \geqslant -\frac{4A_1}{\log u}.$$

To sum up Case I, we have defined in (11) a block $\{b'_p: q \leqslant p \leqslant n+1\}$ of new terms whose sum is 0 and which have the desired property

$$-\frac{4A_1}{\log u} \leqslant \sum_{u \leqslant p \leqslant v} b'_p \leqslant \frac{A_1}{\log u}, \text{ whenever } [u, v] \subset [q, n+1]. \tag{12}$$

We shall refer to the Case I block as a *short block*.

Case II. $n \geqslant q^2+1$. Here we terminate the block at q^2+1; that is, we define

$$b'_p = b_p, \qquad q \leqslant p \leqslant q^2+1, \tag{13}$$

and refer to $\{b'_p: q \leqslant p \leqslant q^2+1\}$ as a *long block*. The sum of elements in a long block is no longer zero, but from (8) we do know that

$$0 < \sum_{q \leqslant p \leqslant q^2+1} b'_p \leqslant \frac{A_1}{\log q}. \tag{14}$$

Suppose that $[u, v] \subset [q, q^2+1]$. By (13) and (8) we see at once that

$$\sum_{u \leqslant p \leqslant v} b'_p \leqslant \frac{A_1}{\log u}.$$

As for a lower bound, we argue as in Case I: we have by (8)

$$\sum_{u \leqslant p \leqslant v} b'_p > -\sum_{q \leqslant p \leqslant u-1} b_p \geqslant -\frac{A_1}{\log q} > -\frac{3A_1}{\log u}$$

since $u \leqslant v \leqslant q^2+1 < q^3$. Thus for a long block we have

$$-\frac{3A_1}{\log u} \leqslant \sum_{u \leqslant p \leqslant v} b'_p \leqslant \frac{A_1}{\log u}, \text{ whenever } [u, v] \subset [q, q^2+1]. \tag{15}$$

With the first block defined, we begin again: we start a new block with the first element $b_{q'}$ that is positive, and define $b'_p = 0$ for the primes $p < q'$ (at which $b_p \leqslant 0$ necessarily) that come after the first block; etc.

We are now ready to complete the proof of the Lemma. Consider any sum

$$\sum_{u \leqslant p \leqslant v} b'_p,$$

where, without any loss of generality,[4] we may suppose that b'_u lies in block $\{b'_p : q \leqslant p \leqslant q'\}$ and b'_v lies in block $\{b'_p : r \leqslant p \leqslant r'\}$. Moreover, the terms of the sum lie in non-overlapping blocks and are otherwise zero. Since the sum of elements in any one complete block is non-negative, we see at once that, by (12) and (15),

$$\sum_{u \leqslant p \leqslant v} b'_p \geqslant \sum_{u \leqslant p \leqslant q'} b'_p + \sum_{r \leqslant p \leqslant v} b'_p \geqslant -4A_1\left(\frac{1}{\log u} + \frac{1}{\log r}\right) > -\frac{8A_1}{\log u} \tag{16}$$

since $u < r$.

Also, again by (12) and (15),

$$\sum_{u \leqslant p \leqslant v} b'_p \leqslant A_1\left(\frac{1}{\log u} + \frac{1}{\log r}\right) + A_1 \sum_q \frac{1}{\log q},$$

where the sum over q extends over the suffices of the first elements of the intervening *long* blocks. The short blocks may be ignored since their sums are zero. Each $q > u$; and if $\{b_p : q_1 \leqslant p \leqslant q_1^2 + 1\}$, $\{b_p : q_2 \leqslant p \leqslant q_2^2 + 1\}$ are two successive long blocks (with $q_1 < q_2$), then in fact $q_2 > q_1^2$ and therefore

$$\frac{1}{\log q_2} < \frac{1}{2}\frac{1}{\log q_1}.$$

Hence

$$\sum_q \frac{1}{\log q} < \frac{1}{\log u}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right) = \frac{2}{\log u}$$

and

$$\sum_{u \leqslant p \leqslant v} b'_p < \frac{4A_1}{\log u}. \tag{17}$$

This completes the proof of the Lemma; for we have only to define g' by means of the relation

$$(1 + g'(p))\left(1 - \frac{1}{p}\right)^{\kappa} = b'_p,$$

and (4) follows from $b'_p \geqslant b_p$ (for all p) in our construction. Then (5) is an immediate consequence of (16) and (17).

[4] The point is that elements b'_p not in a block are 0.

Let

$$G(x, z) = \sum_{\substack{d \mid P(z) \\ d < x}} g(d), \tag{18}$$

where $g(d)$ is multiplicative on the squarefree numbers and satisfies (3) on the sequence of primes. Note that

$$G(x, z) \leqslant \sum_{p < z} (1 + g(p)); \tag{19}$$

also that

$$G(x, z) = G(x) := \sum_{d < x} \mu^2(d)\, g(d) \text{ when } x \leqslant z. \tag{20}$$

Let $G'(x, z)$ be the same summatory function associated with the function g' whose existence we established in the preceding lemma. By an argument of Rawsthorne [12], we have

$$\prod_{p < z} (1 + g'(p))^{-1}\, G'(x, z) \leqslant \prod_{p < z} (1 + g(p))^{-1}\, G(x, z). \tag{21}$$

The proof is so short that we repeat it here, for the sake of completeness. First, it clearly suffices to prove the inequality for the simple case when $g'(p) = g(p)$ for all primes $p < z$ except one, say p_0, when $g'(p_0) > g(p_0)$. Now

$$G(x, z) = \sum_{\substack{d \mid P(z)/p_0 \\ d < x}} g(d) + g(p_0) \sum_{\substack{d \mid P(z)/p_0 \\ d < x/p_0}} g(d) = S_1 + g(p_0)\, S_2,$$

say, where obviously $S_1 \geqslant S_2$; and similarly $G'(x, z) = S_1 + g'(p_0)\, S_2$. Hence

$$\frac{G(x, z)}{1 + g(p_0)} - \frac{G'(x, z)}{1 + g'(p_0)} = \frac{(g'(p_0) - g(p_0))(S_1 - S_2)}{(1 + g(p_0))(1 + g'(p_0))} \geqslant 0,$$

as we claimed. By iterating this procedure, if necessary, the proof of (21) is complete.

On the basis of the two-sided condition (5), asymptotic formulae may be derived for $G'(x)$ and $G'(x, z)$ by the method used in [6] (see Chapter 5, Lemma 5.4 and Chapter 6, Lemma 6.1; or use the alternative procedure indicated by Remark 2 on p. 198). While the condition $\Omega_2(\kappa, L)$ used there is slightly stronger than (5), this causes no new difficulties and, in view of (21), justifies assertion (4.17).

APPENDIX III[5]

κ	α_κ	β_κ	ν_κ
1	2	2	2.06..
1.5	3.9114..	3.1158..	3.22..
2	5.3577..	4.2664..	4.42..
2.5	6.8399..	5.4440..	5.63..
3	8.3719..	6.6408..	6.85..
3.5	9.9388..	7.8514..	8.09..
4	11.5317..	9.0722..	9.32..
4.5	13.1447..	10.3006..	
5	14.7735..	11.5347..	11.80..
5.5	16.4153..	12.7730..	
6	18.0679..	14.0146..	14.28..
6.5	19.7295..	15.2585..	
7	21.3989..	16.5042..	16.77..
7.5	23.0751..	17.7511..	
8	24.7571..	18.9988..	19.25..
8.5	26.4444..	20.2470..	
9	28.1326..	21.4955..	21.74..
9.5	29.8323..	22.7440..	
10	31.5320..	23.9924..	24.22..

Note added in proof. Motohashi has now resolved the problem mentioned at the end of Section 1.

REFERENCES

1. N. C. ANKENY AND H. ONISHI, The general sieve, *Acta Arith.* **10** (1964), 31–62.
2. H. DIAMOND, H. HALBERSTAM, AND H.-E. RICHERT, Combinatorial sieves of dimension exceeding one, II, in preparation.
3. J. FRIEDLANDER AND H. IWANIEC, On Bombieri's asymptotic sieve, *Ann. Scu. Norm. Sup.-Pisa (IV)* **5** (1978), 719–756.
4. H. HALBERSTAM, Lectures on the Linear Sieve: "Topics in Analytic Number Theory," pp. 165–220, Univ. of Texas Press, Austin, 1985.

[5] The last column gives information (quoted from [6]) about the sieving limit ν_κ in the Ankeny–Onishi method [1].

5. H. Halberstam and H.-E. Richert, A Weighted Sieve of Greaves' Type I: "Elementary and Analytic Theory of Numbers," Banach Center Publications, Vol. 17, pp. 155–182, Warsaw, 1985.
6. H. Halberstam and H.-E. Richert, "Sieve Methods," Academic Press, New York/London, 1974.
7. H. Iwaniec, Rosser's sieve, *Acta Arith.* **36** (1980), 171–202.
8. W. B. Jurkat and H.-E. Richert, An improvement of Selberg's sieve method I, *Acta Arith.* **11** (1985), 217–240.
9. Y. Motohashi, "Lectures on the Sieve Method and Prime Number Theory," Tata Institute for Fundamental Research, 1983.
10. J. W. Porter, On the non-linear sieve, *Acta Arith.* **29** (1976), 377–400.
11. D. A. Rawsthorne, "Improvements in the Small Sieve Estimate of Selberg by Iteration," Ph.D. thesis, University of Illinois, 1980.
12. D. A. Rawsthorne, Selberg's sieve estimate with a one sided hypothesis, *Acta Arith.* **41** (1982), 281–289.