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Equivalences between cluster categories ☆

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Abstract

Tilting theory in cluster categories of hereditary algebras has been developed in [A. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov, Tilting theory and cluster combinatorics, preprint, arXiv: math.RT/0402075, 2004, Adv. Math., in press; A. Buan, R. Marsh, I. Reiten, Cluster-tilted algebras, preprint, arXiv: math.RT/0402054, 2004; Trans. Amer. Math. Soc., in press]. Some of them are already proved for hereditary abelian categories there. In the present paper, all basic results about tilting theory are generalized to cluster categories of hereditary abelian categories. Furthermore, for any tilting object T in a hereditary abelian category $\mathcal{H}$, we verify that the tilting functor $\mathrm{Hom}_{\mathcal{H}}(T,-)$ induces a triangle equivalence from the cluster category $\mathcal{C}(\mathcal{H})$ to the cluster category $\mathcal{C}(A)$, where A is the quasi-tilted algebra $\mathrm{End}_{\mathcal{H}} T$. Under the condition that one of derived categories of hereditary abelian categories $\mathcal{H}$, $\mathcal{H}'$ is triangle equivalent to the derived category of a hereditary algebra, we prove that the cluster categories $\mathcal{C}(\mathcal{H})$ and $\mathcal{C}(\mathcal{H}')$ are triangle equivalent to each other if and only if $\mathcal{H}$ and $\mathcal{H}'$ are derived equivalent, by using the precise relation between cluster-tilted algebras (by definition, the endomorphism algebras of tilting objects in cluster categories) and the corresponding quasi-tilted algebras proved previously. As an application, we give a realization of "truncated simple reflections" defined by Fomin–Zelevinsky on the set of almost positive roots of the corresponding type [S. Fomin, A. Zelevinsky, Cluster algebras II: Finite type classification, Invent. Math. 154 (1) (2003) 63–121; S. Fomin, A. Zelevinsky, Y-systems and generalized associahedra, Ann. of Math. 158 (3) (2003) 977–1018], by taking $\mathcal{H}$ to be the representation category of a valued Dynkin quiver and T a BGP-tilting object (or APR-tilting, in other words).

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1. Introduction

Given a hereditary abelian category $\mathcal{H}$ with tilting objects, the orbit category of the (bounded) derived category $D^b(\mathcal{H})$ of $\mathcal{H}$ by its automorphism $F = \tau^{-1}[1]$ is again a triangulated category [Ke2], called cluster categories of type $\mathcal{H}$ and denoted simply by $\mathcal{C}(\mathcal{H})$. If $\mathcal{H}$ is the category of representations of a Dynkin quiver Q, the corresponding cluster category $\mathcal{C}(Q)$ has been proved to be useful [BMRRT], see also [CCS] for type A: it provides a natural realization of clusters of corresponding cluster algebras, more precisely, there is an one to one correspondence from isoclasses of indecomposable objects in $\mathcal{C}(Q)$ to cluster variables of the corresponding cluster algebras. Under this correspondence, the basic tilting objects in $\mathcal{C}(Q)$ correspond to clusters. (These results are generalized to all non-simply laced Dynkin types in [Z1], and recently are generalized to all acyclic quivers in [CK].) Clusters and cluster algebras are defined and studied by Fomin and Zelevinsky [FZ1,FZ2,FZ3,FZ4,BFZ]. These algebras are defined so that it designs an algebraic framework for total positivity and canonical bases in semisimple algebraic groups. There are interesting connections to their theory in many directions [FZ1,FZ2,FZ3,FZ4, BFZ,CFZ,GSV], amongst them to representation theory of quivers, in particular, to tilting theory [MRZ,BMRRT,BMR,CFZ,Z1]. Tilting theory in cluster category is really an extension of classical titling theory of module category. The tilting objects in $\mathcal{C}(Q)$ are, on the one hand, corresponding to clusters of corresponding cluster algebras (in simply-laced Dynkin type [BMRRT], in all Dynkin cases [Z1] and in all simply-laced cases [CK]); on the other hand, a generalization of tilting modules over hereditary algebras, for example, the endomorphism algebra of a tilting object in $\mathcal{C}(Q)$ may be self-injective.

The aims of the paper are two-fold: The first one is to generalize Buan–Marsh–Reiten theorem in [BMR] to the setting of hereditary abelian categories with tilting objects. Buan–Marsh–Reiten theorem says that the tilting functor $\mathrm{Hom}_{\mathcal{C}(H)}(T, -)$ gives an equivalence from the quotient $\mathcal{C}(H)/\operatorname{add}\tau T$ of cluster category to the module category of the cluster-tilted algebra of T. We prove that the same is true when $\operatorname{mod} H$ is replaced by any hereditary abelian category. Our proof for the general result is obtained by a triangulated realization of that for Buan–Marsh–Reiten theorem, and simplifies the original proof.

The second aim is to study the triangle equivalences between cluster categories. We prove a "Morita type" theorem for cluster categories. We verify the fact that any standard equivalence between two derived categories of hereditary abelian categories induces a triangle equivalence between the two corresponding cluster categories and prove that the inverse also holds provided one of the hereditary abelian categories is derived equivalent to a hereditary algebra. This first part is used in the rest of paper and it was used in literatures, for examples: [BMRRT,BMR], and it is proved also in the updated version of [Ke2]. The reason why we verify the fact is that, the explicit expression of the triangle functor between triangulated orbit categories is very interesting and useful, in particular, in the special case of the triangle equivalences when it is induced by a Bernstein–Gelfand–Ponomarev reflection tilting module or APR tilting. It provides a realization of the "truncated simple reflections" on the set of almost positive roots [Z1] (note that it is proved in [Z2] that these triangle functors induce isomorphisms of cluster algebras which are useful). By using this realization, one can simplify some essential part of the quiver-theoretic interpretation [Z1] for generalized associahedra in the sense of Fomin–Zelevinsky [FZ4,CFZ,MRZ]. In our proof of "Morita type" theorem for cluster categories, we use the precise relation between cluster-tilted algebras and the corresponding quasi-tilted algebras. Starting from a basic tilting object T in $\mathcal{H}$ (note that "basic" means "multiplicity free," we will assume that tilting objects are basic in the rest of the paper), T is a tilting object in cluster category $\mathcal{C}(\mathcal{H})$ (note that any tilting object in $\mathcal{C}(\mathcal{H})$ can be obtained from a tilting object in a hereditary abelian category $\mathcal{H}'$, derived equivalent to $\mathcal{H}$, so we do not loss the generality if we start from these tilting objects), then the cluster-tilted algebra $\mathrm{End}_{\mathcal{C}(\mathcal{H})} T$ of T is the trivial extension of the quasi-tilted algebra $A = \mathrm{End}_{\mathcal{H}} T$ of T with the A-bimodule $\mathrm{Hom}_{\mathcal{H}}(T, \tau^2 T)$. Some consequences follow: the relations on Gabriel quivers and Auslander–Reiten quivers between these two algebras become clear [BMR]. When $\mathcal{H}$ is the module category of a hereditary algebra $H = KQ$, where K is a field, an important feather on the cluster category $\mathcal{C}(H)$ is that the generalization of APR-tilting at any vertex is allowed, i.e., if $T'(i)$ is a projective module with all indecomposable projective modules but one, say, $P(i)$, as its direct summands. Then $T(i) = T'(i) \oplus \tau^{-1}E(i)$ is a tilting object in $\mathcal{C}(H)$. We assume $T(i) \in \mathrm{mod}\, H$, since $\tau^{-1}E(i) \notin \mathrm{mod}\, H$ if and only if $E(i)$ is injective, and in this case, we can replace H by another hereditary algebra H', derived equivalent to H, with $\tau^{-1}E'(i) \in \mathrm{mod}\, H'$. It follows from our result that the cluster-tilted algebra $\mathrm{End}_{\mathcal{C}(H)} T$ isomorphic to $\mathrm{End}_H(T) \ltimes D\,\mathrm{Hom}_H(T, \tau E(i))$. When i is a sink or a source in Q, the cluster-tilted algebra goes back to the tilted algebra of the same tilting module.

This paper is organized as follows: in Section 2, some notions which will be needed later on are recalled. Some basic properties of orbit categories and cluster categories are given. In particular, the result that any standard equivalence between two derived categories of hereditary abelian categories induces a triangle equivalence between the corresponding orbit categories is verified; it is proved that any almost complete tilting object in $\mathcal{C}(\mathcal{H})$ can be completed to a tilting object in exactly two ways. In Section 3, we prove that for a tilting object T in a hereditary abelian category $\mathcal{H}$, the cluster-tilted algebra $\mathrm{End}_{\mathcal{C}(\mathcal{H})} T$ is a trivial extension of a quasi-tilted algebra with a bimodule. We explain through examples how and what the Gabriel quivers or Auslander–Reiten quivers of the two algebras are related. We also prove the generalization of Buan–Marsh–Reiten theorem that the tilting functor $\mathrm{Hom}_{\mathcal{C}(\mathcal{H})}(T, -)$ induces an equivalence from the quotient category $\mathcal{C}(H)/\mathrm{add}\,\tau T$ of $\mathcal{C}(\mathcal{H})$ to the module category of cluster-tilted algebra. Our proof simplifies the original one in [BMR]. In the final section, under the condition that one of the hereditary abelian categories is derived equivalent to a hereditary algebra, we prove that two cluster categories of hereditary abelian categories are triangle equivalent each other if and only if the two derived categories of hereditary abelian categories are triangle equivalent each other. As applications, we give a quiver realization of "truncated simple reflections" on the set of almost positive roots in all Dynkin types (simply laced or non-simply laced) and also give a quiver realization of Weyl generators of Weyl group of any Kac–Moody Lie algebra.

2. Basics on orbit categories and cluster categories

Let $\mathcal{H}$ be a hereditary abelian category with tilting objects and with finite-dimensional Hom-spaces and Ext-spaces over a field K, and denote by $\mathcal{D} = D^b(\mathcal{H})$ the bounded derived category of $\mathcal{H}$ with shift functor [1]. For any category $\mathcal{E}$, we will denote by $\mathrm{ind}\,\mathcal{E}$ the subcategory of the representatives of isomorphism classes of indecomposable objects in $\mathcal{E}$; depending on the context we shall also use the same notation to denote the set of isomorphism classes of indecomposable objects in $\mathcal{E}$. We write $\mathcal{D} = D^b(\mathcal{H})$. Throughout the rest of paper, D denotes the usual duality $\mathrm{Hom}_K(-, K)$ [ARS,Rin].

Let $G : \mathcal{D} \to \mathcal{D}'$ be a standard equivalence, i.e., G is isomorphic to the derived tensor product

$$X \otimes_A - : D^b(A) \to D^b(A')$$

for some complex X of A'–A-bimodules.

Following [Ke2], we also assume G satisfies the following properties:

(g1) For each U in $\operatorname{ind} H$, only a finite number of objects $G^n U$, where $n \in \mathbf{Z}$, lie in $\operatorname{ind} H$.
(g2) There is some $N \in \mathbf{N}$ such that $\{U[n] \mid U \in \operatorname{ind} H,\ n \in [-N, N]\}$ contains a system of representatives of the orbits of G on $\operatorname{ind} \mathcal{D}$.

We denote by $\mathcal{D}/G$ the corresponding factor category. The objects are by definition the G-orbits of objects in $\mathcal{D}$, and the morphisms are given by

$$\operatorname{Hom}_{\mathcal{D}/G}(\tilde{X}, \tilde{Y}) = \bigoplus_{i \in \mathbf{Z}} \operatorname{Hom}_{\mathcal{D}}\bigl(X, G^i Y\bigr).$$

Here X and Y are objects in $\mathcal{D}$, and $\tilde{X}$ and $\tilde{Y}$ are the corresponding objects in $\mathcal{D}/G$ (although we shall sometimes write such objects simply as X and Y). The composition is defined in the natural way: if $\tilde{f} : \tilde{X} \to \tilde{Y}$ and $\tilde{g} : \tilde{Y} \to \tilde{Z}$ with $f : X \to G^n Y$ and $g : Y \to G^m Z$, then $\tilde{f} \circ \tilde{g}$ is defined to be $\widetilde{fG^n g}$, the image of composition of maps f and $G^n g$ in $\mathcal{D}$. The factor category $\mathcal{D}/G$ is Krull–Schmidt [BMRRT] and is a triangulated [Ke2]. The canonical functor $\pi : \mathcal{D} \to \mathcal{D}/G : X \mapsto \tilde{X}$ is a covering functor of triangulated categories [XZ2]. It sends triangles to triangles. We remark that not all the triangles in $\mathcal{D}/G$ are obtained as images of triangles in $\mathcal{D}$ under π. The shift in $\mathcal{D}/G$ is induced by the shift in $\mathcal{D}$, and is also denoted by [1]. In both cases we write as usual $\operatorname{Hom}(U, V[1]) = \operatorname{Ext}^1(U, V)$. We then have

$$\operatorname{Ext}^1_{\mathcal{D}/G}(\tilde{X}, \tilde{Y}) = \bigoplus_{i \in \mathbf{Z}} \operatorname{Ext}^1_{\mathcal{D}}\bigl(X, G^i Y\bigr),$$

where X, Y are objects in $\mathcal{D}$ and $\tilde{X}, \tilde{Y}$ are the corresponding objects in $\mathcal{D}/G$. We shall mainly be concerned with two special choices of functor $F = \tau^{-1}[1]$ or $F = [2]$ where τ is the Auslander–Reiten translation in $\mathcal{D}$. In the first case, the factor category $\mathcal{D}/\tau^{-1}[1]$ is called the cluster category of type $\mathcal{H}$, which is denoted by $\mathcal{C}(\mathcal{H})$ (compare [BMRRT]). If $\mathcal{H}$ is the module category of a hereditary algebra H or equivalently the category of representations of a valued quiver Q, we denote the corresponding cluster category by $\mathcal{C}(H)$ or $\mathcal{C}(Q)$, respectively. In the second case the factor category $\mathcal{D}/[2]$ is called the root category of type $\mathcal{H}$, and we denote it by $\mathcal{R}(\mathcal{H})$ (compare [H1,H3,XZZ]). When $\mathcal{H}$ is the module category of a hereditary algebra H or a valued quiver Q, we denote the corresponding root category by $\mathcal{R}(H)$ or $\mathcal{R}(Q)$, respectively.

Throughout the paper, $\mathcal{H}$ is assumed to be a hereditary abelian category with titling objects. In this case, the Grothendieck group $K_o(\mathcal{H})$ is a free abelian group of finite rank. We recall that an object T in $\mathcal{H}$ is called a tilting object if $\operatorname{Ext}^1_{\mathcal{H}}(T, T) = 0$ and any object X with $\operatorname{Ext}^1_{\mathcal{H}}(T, X) = \operatorname{Hom}_{\mathcal{H}}(T, X) = 0$ must be zero [HRS]. If T is a tilting object in $\mathcal{H}$, then the endomorphism algebra $A = \operatorname{End}_{\mathcal{H}}(T)$ is called a quasi-tilted algebra [HRS]. There are associated torsion pairs $(\mathcal{T}, \mathcal{F})$ in $\mathcal{H}$, and $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{A} = \operatorname{mod} A$, such that there are equivalences of categories $\operatorname{Hom}_{\mathcal{H}}(T, -) : \mathcal{T} \to \mathcal{Y}$ and $\operatorname{Ext}^1_{\mathcal{H}}(T, -) : \mathcal{F} \to \mathcal{X}$. In addition there is an induced equivalence of derived categories

$$\operatorname{RHom}(T, -) : D^b(\mathcal{H}) \to D^b(\mathcal{A}),$$

which is simply denoted by $R(T)$ (compare [KZ,H1]).

We recall the notation of Grothendieck groups of triangulated categories $\mathcal{A}$ from [H1]. Let $K(\mathcal{A})$ be the free abelian group generated by representatives of the isomorphism classes of objects in $\mathcal{A}$. The Grothendieck group $K_0(\mathcal{A})$ of $\mathcal{A}$ is the factor group of K modulo the subgroup generated by elements of the forms: $[A]+[C]-[B]$ corresponding to triangles $A \to B \to C \to A[1]$.

Proposition 2.1. *Let $\mathcal{H}$ be a hereditary abelian category with tilting objects and G a triangle equivalence satisfying* (g1), (g2). *Then $K_0(\mathcal{D}/G) \cong \mathbf{Z}^n/H$, where H is a subgroup of $\mathbf{Z}^n$.*

Proof. We have a covering functor $\pi : \mathcal{D} \to \mathcal{D}/G$, which induces a surjective group morphism $\pi_1 : K(\mathcal{D}) \to K(\mathcal{D}/G) : [X] \mapsto [\tilde{X}]$, since π is a triangle functor, π_1 induces a surjection from the Grothendieck group $K_0(\mathcal{D})$ to the Grothendieck group $K_0(\mathcal{D}/G)$. It follows that $K_0(\mathcal{D}/G) \simeq K_0(\mathcal{D})/H$ and is isomorphic to $\mathbf{Z}^n/H$ since from Theorem 4.6 in Chapter I in [HRS] $K_0(\mathcal{D}) \cong \mathbf{Z}^n$ where n is a positive integer. The proof is finished. $\square$

Remark 2.2. We note that for some non-trivial orbit triangulated categories, its Grothendieck groups might equal to zero. For example $K_0(\mathcal{C}(Q)) = 0$ when Q is a Dynkin quivers of type A_{2n}. In contrary to the root category, the Grothendieck group $K_0(\mathcal{R}(Q))$ is $\mathbf{Z}^n$ for any Dynkin type, where n is the number of vertices of the quiver Q. For further consideration of Grothendieck groups of triangulated categories, we refer to [XZ1].

We recall the notation of exceptional set and of tilting set in $\mathcal{C}(\mathcal{H})$, from [BMRRT]. A subset B of $\operatorname{ind}\mathcal{C}(\mathcal{H})$ is called exceptional if $\operatorname{Ext}^1_{\mathcal{C}(\mathcal{H})}(X,Y) = 0$ for any $X, Y \in B$; An exceptional set B is a tilting set if it is maximal with respect to this property. An object T in $\mathcal{C}(\mathcal{H})$ is called tilting object if $\operatorname{Ext}^1_{\mathcal{C}(\mathcal{H})}(T,T) = 0$ and T has a maximal number of non-isomorphic direct summands. An object M is called an almost complete tilting object if it is not a tilting object and there is an indecomposable object X such that $M \oplus X$ is a tilting object. A subset B of $\mathcal{C}(\mathcal{H})$ is a tilting set if and only if the direct sum of all objects in B is a basic tilting object. For a tilting object T in the cluster category of $\mathcal{H}$, The endomorphism ring $\operatorname{End}_{\mathcal{C}(\mathcal{H})} T$ is called the cluster-tilted algebra of T.

Let $G : \mathcal{D} \to \mathcal{D}$ be a standard equivalence, which is assumed to satisfy the properties (g1), (g2). Let $\Phi : \mathcal{D} \to \mathcal{D}'$ be a standard triangle equivalence. We set $G' = \Phi G \Phi^{-1}$. Then G' is a standard equivalence of $\mathcal{D}'$, it also satisfies the properties (g1) and (g2). For most applications, we set $G = \tau^m[n]$ for some $m, n \in \mathbf{Z}$, and then G' also equals to $\tau^m[n]$.

Definition 2.1. We define the functor Φ_G from $\mathcal{D}/G$ to $\mathcal{D}'/G'$ as follows: for $\tilde{X} \in \mathcal{D}/G$ with $X \in \mathcal{D}$, we set $\Phi_G(\tilde{X}) = \widetilde{\Phi(X)}$. For morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$, we set $\Phi_G(\tilde{f}) : \Phi_G(\tilde{X}) \to \Phi_G(\tilde{Y})$ to be the map $\widetilde{\Phi(f)}$.

The following result is proved in the updated version of [Ke2], by using dg set-up. We will give some applications of it.

Proposition 2.3. *Let Φ and G be as above. Then Φ_G is a triangle equivalence from $\mathcal{D}/G$ to $\mathcal{D}'/G'$.*

Proof. First of all, we verify the definition is well-defined: let $\tilde{X} = \tilde{Y} \in \mathcal{D}/G$ with $X, Y \in \mathcal{D}$. Then we have that $Y = G^i(X)$ for some integer i. It follows that $\Phi(Y) = \Phi G^i(X) \cong G'^i \Phi(X)$.

Hence $\Phi_G(\tilde{X}) = \Phi_G(\tilde{Y})$. The action of Φ_G on morphisms is induced by Φ on morphisms in $\mathcal{D}$, and we have the commutative diagram as follows:

$$\begin{tikzcd}
\bigoplus_{i\in\mathbf{Z}} \mathrm{Hom}_{\mathcal{D}}(X, G^i Y) \arrow[r, "\Phi"] \arrow[d, "\wr"'] & \bigoplus_{i\in\mathbf{Z}} \mathrm{Hom}_{\mathcal{D}'}(\Phi(X), G'^i \Phi(Y)) \arrow[d, "\wr"] \\
\mathrm{Hom}_{\mathcal{D}/G}(\tilde{X}, \tilde{Y}) \arrow[r, "\Phi_G"] & \mathrm{Hom}_{\mathcal{D}'/G'}(\Phi(\tilde{X}), \Phi(\tilde{Y}))
\end{tikzcd}$$

Therefore Φ_G is faithful and full. It is easy to see it is dense since Φ is equivalent. Then Φ_G is an equivalence from $\mathcal{D}/G$ to $\mathcal{D}'/G'$. Combining with that Φ_G is a triangle functor in [Ke2, Section 9.4], we have that Φ_G is a triangle equivalence from $\mathcal{D}/G$ to $\mathcal{D}'/G'$. The proof is finished. □

Under this triangle equivalence, tilting objects correspond to tilting objects, and they have isomorphic endomorphism rings.

Corollary 2.4. *Let $\mathcal{H}$ (or $\mathcal{H}'$) be a hereditary abelian category, $\Phi : D^b(\mathcal{H}) \to D^b(\mathcal{H}')$ a standard triangle equivalence and G (or G') be as in Proposition 2.3. Let T be an object in $\mathcal{C}(\mathcal{H})$. Then T is a tilting object in $\mathcal{C}(\mathcal{H})$ if and only so is $\Phi_G(T)$ in $\mathcal{C}(\mathcal{H}')$. Moreover,* $\mathrm{End}_{\mathcal{C}(\mathcal{H}')}(\Phi_G(T)) \cong \mathrm{End}_{\mathcal{C}(\mathcal{H})}(T)$.

If Φ is induced by a tilting object T in $\mathcal{H}$, i.e.,

$$\Phi = \mathrm{RHom}(T, -) : D^b(\mathcal{H}) \to D^b(A),$$

which is simply denoted by $R(T)$, where A is the endomorphism algebra of T, then we have the following consequence:

Corollary 2.5. *Let T be a tilting object in $\mathcal{H}$. Then $R_G(T)$ is a triangle equivalence from $D^b(\mathcal{H})/G$ to $D^b(\mathcal{A})/G'$.*

We will prove the converse of Proposition 2.3 when the orbit categories are cluster categories and give some applications of Corollary 2.4 in Section 4.

In the rest of this section, we will prove some basic properties on tilting objects in a cluster category $C(\mathcal{H})$, where $\mathcal{H}$ is assumed a hereditary abelian category with tilting objects and with Grothendieck group $\mathbf{Z}^n$. These properties were proved in [BMRRT] when $\mathcal{H}$ is a module category of a finite-dimensional algebra, and hold in the general case, which we will show in the following.

Proposition 2.6.

(a) *Let T be a basic tilting object in $\mathcal{C}(\mathcal{H})$, where $\mathcal{H}$ is a hereditary abelian category with Grothendieck group $\mathbf{Z}^n$. Then*:
  (i) *T is induced by a basic tilting object in a hereditary abelian category $\mathcal{H}'$, derived equivalent to $\mathcal{H}$.*
  (ii) *T has n indecomposable direct summands.*

(b) *Any basic tilting objects in $\mathcal{H}$ induces a basic tilting objects for $\mathcal{C}(\mathcal{H})$.*

Proof. The proof for statement (b) follows from the definitions of tilting objects in various categories, namely, for a tilting object T in $\mathcal{H}$, we have that $\operatorname{Ext}^1_{\mathcal{C}(\mathcal{H})}(T,T) \cong \operatorname{Ext}^1_{\mathcal{H}}(T,T) \oplus D\operatorname{Ext}^1_{\mathcal{H}}(T,T) = 0$ and T has n non-isomorphic indecomposable summands in $\mathcal{C}(\mathcal{H})$. Then T is a tilting object in $\mathcal{C}(\mathcal{H})$. For the proof of (a), we note that any hereditary abelian category with tilting objects is derived equivalent to a module category of a hereditary algebra or to a category of coherent sheaves over a weighted projective space [H2]. By Proposition 2.3 and Corollary 2.4, we can shift the proof to the hereditary algebra case and the coherent sheaves case. For the first case, all the statements were proved in [BMRRT]. For the second case (compare [BMRRT]), through a suitable derived equivalence, we may assume the hereditary abelian category $\mathcal{H}$ has no projective or injective objects. Then the tilting objects in $\mathcal{C}(\mathcal{H})$ and in $\mathcal{H}$ are 1-1 corresponding. Therefore the statements in (a) hold. The proof is finished. □

Proposition 2.7. *Any exceptional object $\bar{T}$ in $\mathcal{C}(\mathcal{H})$, where $\mathcal{H}$ is a hereditary abelian category with Grothendieck group $\mathbf{Z}^n$, can be extended to a tilting object. If $\bar{T}$ is an almost complete basic tilting object, then $\bar{T}$ can be completed to a basic tilting object in $\mathcal{C}(\mathcal{H})$ in exactly two different ways.*

Proof. We use the same strategy as Proposition 2.6 to prove it. Since any hereditary abelian category with tilting objects is derived equivalent to a module category of hereditary algebra or to a category of coherent sheaves over a weighted projective space. By the Proposition 2.3 and Corollary 2.4, we shift the statement to the hereditary algebra case and the coherent sheaves case. For the first case, all the statements are proved in [BMRRT]. For the second case, through derived equivalence, we may assume the hereditary abelian category $\mathcal{H}$ has no projective or injective objects. Then any almost complete tilting object in $\mathcal{C}(\mathcal{H})$ are induced from $\mathcal{H}$. By a result of Happel and Unger [HU, Section 3], there are exactly two complements of $\bar{T}$ in $\mathcal{H}$. The proof is finished. □

3. Cluster-tilted algebras

Since any tilting object T in cluster category $\mathcal{C}(\mathcal{H})$ can be obtained from a tilting object in a hereditary abelian category $\mathcal{H}'$, derived equivalent to $\mathcal{H}$, we may assume that, without loss the generality (compare Proposition 2.3 and Corollary 2.4), T is a tilting object in $\mathcal{H}$, and then it is a tilting object in $\mathcal{C}(\mathcal{H})$. We have the quasi-tilted algebras $A = \operatorname{End}_{\mathcal{H}} T$ and the cluster-tilted algebra $\Lambda = \operatorname{End}_{\mathcal{C}(\mathcal{H})} T$. We will use $\mathcal{H}[k]$ to denote the full subcategory of $D^b(\mathcal{H})$ consisting of objects $X[k]$ with $X \in \mathcal{H}$.

It is easy to see that the quasi-tilted algebras are factor algebras of cluster-tilted algebras from the definition of cluster categories. In the following result, we explain cluster-tilted algebras as the trivial-extensions of quasi-tilted algebras. This explain is helpful for us to understand the relations on combinatorics between these two algebras (compare [BMR]). We also need this result to prove the Morita type theorem for cluster categories. We remind that D denotes the usual duality $\operatorname{Hom}_K(-,K)$ in the following.

Proposition 3.1. *Let $\mathcal{H}$ and $\mathcal{C}(\mathcal{H})$ be as above. Denoted by $A = \operatorname{End}_{\mathcal{H}} T$ and $\Lambda = \operatorname{End}_{\mathcal{C}(\mathcal{H})} T$. Then $\Lambda = A \ltimes D\operatorname{Hom}_{\mathcal{H}}(T, \tau^2 T)$, i.e., Λ is a trivial-extension of A by the A-bimodule $D\operatorname{Hom}_{\mathcal{H}}(T, \tau^2 T)$.*

Before we give the proof, we recall the notation of trivial extensions (compare [ARS]). Given an algebra A and an A-bimodule M, we define the algebra $A \ltimes M$ as follows, the elements are pair (a,m) with $a \in A$ and $m \in M$, addition is componentwise and multiplication is given by $(a,m)(b,n) = (ab, an + mb)$. It is easy to see if A and M are finite-dimensional, then $A \ltimes M$ is a finite-dimensional algebra with $M^2 = 0$.

Proof of Proposition 3.1. From the definition of cluster-tilted algebras, we have that $\Lambda = \operatorname{End}_{\mathcal{C}(\mathcal{H})} T = \operatorname{Hom}_{D^b(\mathcal{H})}(T,T) \oplus \operatorname{Hom}_{D^b(\mathcal{H})}(T, \tau^{-1}T[1]) = A \oplus \operatorname{Hom}_{D^b(\mathcal{H})}(T, \tau^{-1}T[1])$. Where $\operatorname{Hom}_{D^b(\mathcal{H})}(T, \tau^{-1}T[1])$ is a natural left A-right $\operatorname{End}_{D^b(\mathcal{H})}(\tau^{-1}T[1])$-module. Since $\tau^{-1}[1]$ is an automorphism of derived category $D^b(\mathcal{H})$, $\operatorname{End}_{D^b(\mathcal{H})}(\tau^{-1}T[1]) \cong A$, and $\operatorname{Hom}_{D^b(\mathcal{H})}(T, \tau^{-1}T[1])$ is an A-bimodule. It follows from the composition rule of morphisms in orbit category $\mathcal{C}(\mathcal{H})$ (compare [BG]) that Λ is a trivial extension of A with the A-bimodule $\operatorname{Hom}_{D^b(\mathcal{H})}(T, \tau^{-1}T[1])$. The remaining thing is to show $\operatorname{Hom}_{D^b(\mathcal{H})}(T, \tau^{-1}T[1]) \cong D\operatorname{Hom}_{\mathcal{H}}(T, \tau^2 T)$ as A-bimodules. The first, we can view $\operatorname{Hom}_{\mathcal{H}}(T, \tau^2 T)$ is a natural A-bimodule as follows. It is a natural left A-right $\operatorname{End}_{\mathcal{H}}(\tau^2 T)$-bimodule. We assume $T = T_1 \oplus T_2$ with $\tau^2 T_2 = 0$ and assume T_2 is maximal with respect to this property. Then in derived category $D^b(\mathcal{H})$, object $\tau^2 T_2$ lies in the part $\mathcal{H}[-1]$ of degree -1, and $\tau^2 T_1$ lies in the part $\mathcal{H}[0]$ of degree 0, hence $\operatorname{Hom}_{D^b(\mathcal{H})}(\tau^2 T_1, \tau^2 T_2) = 0$. Therefore we have the isomorphisms as follows:

$$\begin{aligned}
\operatorname{End}_{\mathcal{H}}(T) &\cong \operatorname{End}_{D^b(\mathcal{H})}\bigl(\tau^2 T_1 \oplus \tau^2 T_2\bigr) \\
&\cong \begin{pmatrix} \operatorname{End}_{D^b(\mathcal{H})} \tau^2 T_1 & \operatorname{Hom}_{D^b(\mathcal{H})}(\tau^2 T_1, \tau^2 T_2) \\ \operatorname{Hom}_{D^b(\mathcal{H})}(\tau^2 T_2, \tau^2 T_1) & \operatorname{End}_{D^b(\mathcal{H})} \tau^2 T_2 \end{pmatrix} \\
&\cong \begin{pmatrix} \operatorname{End}_{\mathcal{H}} T_1 & 0 \\ \operatorname{Hom}_{\mathcal{H}}(T_2, T_1) & \operatorname{End}_{\mathcal{H}} T_2 \end{pmatrix}.
\end{aligned}$$

Under these isomorphisms, any right $\operatorname{End}_{\mathcal{H}} \tau^2 T$ (which is $\operatorname{End}_{\mathcal{H}} \tau^2 T_1$)-module is a right $\operatorname{End}_{\mathcal{H}} T$-module. In the following, we will prove that, as an A-bimodule, $\operatorname{Hom}_{D^b(\mathcal{H})}(T, \tau^{-1}T[1])$ is isomorphic to $D\operatorname{Hom}_{\mathcal{H}}(T, \tau^2 T)$.

$$\begin{aligned}
\operatorname{Hom}_{D^b(\mathcal{H})}\bigl(T, \tau^{-1}T[1]\bigr) &\cong \operatorname{Ext}^1_{D^b(\mathcal{H})}\bigl(T, \tau^{-1}T\bigr) \\
&\cong D\operatorname{Hom}_{D^b(\mathcal{H})}\bigl(\tau^{-1}T, \tau^{-1}T\bigr) \\
&\cong D\operatorname{Hom}_{D^b(\mathcal{H})}\bigl(T, \tau^2 T\bigr) \\
&\cong D\operatorname{Hom}_{\mathcal{H}}\bigl(T, \tau^2 T\bigr).
\end{aligned}$$

The proof is finished. □

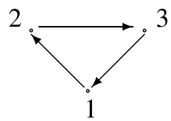
We give some examples to illustrate the proposition.

Example 3.2. Let Q be the quiver

$$1 \to 2 \to 3$$

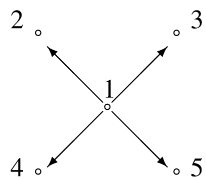
and let $H = KQ$ be the path algebra, where K is a field.

Let T be the tilting module $T = T_1 \oplus T_2 \oplus T_3 = E_1 \oplus P_1 \oplus P_3$, let $A = \mathrm{End}_H(T)$ be the corresponding tilted algebra and $\Lambda = \mathrm{End}_{\mathcal{C}(H)}(T)$ the cluster-tilted algebra. We notice that the tilted algebra A is given by the quiver Q also with $\underline{r}^2 = 0$. Since $\tau^2 T \cong P_3$ and $D\,\mathrm{Hom}_H(T, \tau^2 T) \cong D\,\mathrm{Hom}_H(P_3, P_3)$ is an one-dimensional space over K, it should contribute one arrow in the quiver of Λ. It is easy to check that there is an arrow from vertex 3 to 1 in the quiver of Λ since the right A-action on $\mathrm{Hom}_H(T, \tau^2 T_1)$ is provided by $\mathrm{End}_H(T_1)$ via the isomorphisms indicated in the proof of Theorem 3.1. Therefore the quiver of Λ is the following:



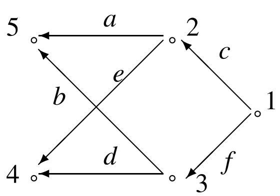
with relations $\underline{r}^2 = 0$.

Example 3.3. Let Q be the quiver:

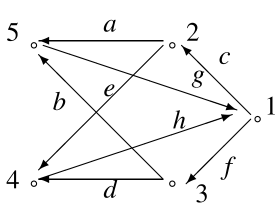


and $H = KQ$ the path algebra of Q.

Let T be the tilting module $T = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5 = R \oplus \tau^{-1} P_2 \oplus \tau^{-1} P_3 \oplus P_4 \oplus P_5$, where R is the regular simple module with composition factors E_1, E_4, E_5. Then the tilted algebra $A = \mathrm{End}_H(T)$ is the path algebra with relations: $ac - bf = 0$, $ec - df = 0$,



Since $\tau^2 T \cong R$ and $D\,\mathrm{Hom}_H(T, \tau^2 T) \cong D\,\mathrm{Hom}_H(T, R)$ is a five-dimensional space over K. Any non-zero map from $\tau^{-1} P_2$ (or $\tau^{-1} P_3$) factors through the identity map of R, any non-zero map from P_4 (or P_5) factors through $\tau^{-1} P_2$ and through $\tau^{-1} P_3$. Then by Proposition 3.1, to get the quiver of the corresponding cluster-tilted algebra Λ, we add two arrows in the quiver of A, one is from 4 to 1, another from 5 to 1, with the additional relations: $ga - he = 0$, $hd - gb = 0$, $cg = fg = ch = fh = 0$, i.e., Λ is the path algebra with relations: $ac - bf = 0$, $ec - df = 0$, $ga - he = 0$, $hd - gb = 0$, $cg = fg = ch = fh = 0$,



Remark 3.4. Since Λ is a trivial extension of A, mod A is embedded in mod Λ as the full subcategory consisting of Λ-modules X which are annihilated by the idea $D\operatorname{Hom}_{\mathcal{H}}(T, \tau^2 T)$. If we consider Auslander–Reiten quivers of these algebras, we can get AR-quiver of A from that of Λ by deleting those vertices from which there are non-zero maps (non-zero path) to $\tau^2 T$ after identifying the AR-quivers between Λ and $\mathcal{C}(\mathcal{H})/(\operatorname{add}\tau T)$ [BMR]. For example, AR-quiver of A in Example 3.2. is obtained from AR-quiver of Λ by deleting P_3, since there is only one indecomposable object, namely, P_3, from which there exist non-zero map to $\tau^2 T \cong P_3$.

Let H (or $H'(k)$) be the tensor algebra of an valued quiver $(\mathbf{M}, \Gamma, \Omega)$ (respectively $(\mathbf{M}, \Gamma, s_k\Omega)$) with indecomposable projective modules P_i, $i \in \Gamma_0$. For any k, let $T'(k) = \bigoplus_{i \neq k} P_j$. It is an almost complete tilting object in $\mathcal{C}(H)$, and any almost complete tilting object in $\mathcal{C}(H)$ can be obtained from an almost complete tilting module over a hereditary algebra H', derived equivalent to H. There are exactly two ways to complete $T'(k)$ into a tilting object: one is to plus P_k, another one is to plus $\tau^{-1}E_k$. The tilting object $T(k) = T'(k) \oplus \tau^{-1}E_k$ is called APR-tilting object in $\mathcal{C}(H)$ at vertex k in [BMR]. In case k is sink in the quiver of H, $T(k)$ is the usual BGP-tilting module, in this case $E_k = P_k$. Denoted by $A(k)$ the tilted algebra of APR-tilting module $T(k)$. The cluster-titled algebras, denoted by $\Lambda(k)$, of APR-tilting objects are described explicitly in the following way:

Corollary 3.5. *Let $T(k)$ be the APR-tilting object in $\mathcal{C}(H)$. Then if k is sink or source in Ω, $\Lambda(k) \cong H'(k)$; otherwise, $\Lambda(k) = A(k) \ltimes D\operatorname{Hom}_H(T'(k), \tau E_k)$.*

Proof. We note that k is sink if and only if P_k is simple module E_k. In this case, $\tau^2 T(k) \cong 0$. It follows from Proposition 3.1 that $\Lambda(k) \cong H'(k)$. We also note that k is source if and only if E_k is injective, hence $\tau^{-1}E_k$ does not exist in H-module, but exists in $\mathcal{C}(H)$. If we consider the reflection at vertex k, we have that the hereditary algebra $H'(k)$ is derived equivalent to H. It follows that $R(S_k^-)(T(k)) = H'(k)$. Then by Corollary 2.4, we have $\Lambda \cong H'(k)$. Suppose k is neither a sink nor a source. Since $\operatorname{Hom}_H(\tau^{-1}E_k, \tau E_k) = 0$, we have that $\operatorname{Hom}_H(T(k), \tau^2(\tau^{-1}E_k)) \cong \operatorname{Hom}_H(T(k), \tau E_k) \cong \operatorname{Hom}_H(T'(k), \tau E_k)$. The proof is finished. □

The cluster-tilted algebras of APR-tilting objects may be quite different from tilted algebras of APR-tilting modules. For example, the tilting object in Example 3.2 is an APR-tilting at vertex 2. The corresponding cluster tilted algebra is self-injective.

From Corollary 3.5, one can easily determine all cluster-tilted algebras of APR-tilting objects in a given cluster category $\mathcal{C}(H)$.

Example 3.6. Let $H = KQ$, where Q is the quiver:

$$\begin{array}{ccccccc} & & & & 5 & & \\ & & & & \downarrow & & \\ 1 & \to & 2 & \to & 3 & \to & 4. \end{array}$$

Now vertices 1 and 5 are sources, then the cluster-tilted algebras $\Lambda(1)$ and $\Lambda(5)$ corresponding to APR-tilting objects are $H'(1)$ or $H'(5)$, respectively, where $H'(1)$ is the quiver algebra $K(s_1Q)$ and $H'(5)$ is the quiver algebra $K(s_5Q)$. Vertex 4 is sink, then $\Lambda(4)$ is the quiver algebra $K(s_4Q)$. We compute other cluster-tilted algebras corresponding to APR-tilting objects at vertices 2 or 3 in $\mathcal{C}(H)$. The APR-tilting module at vertex 2 is $T(2) = P_1 \oplus \tau^{-1}E_2 \oplus P_3 \oplus P_4 \oplus P_5$. The corresponding tilted algebra $A(2)$ is the following:

$$\begin{tikzcd} & & 5 \arrow[d, "\rho"'] & \\ 2 \arrow[r, "\alpha"] & 1 \arrow[r, "\beta"] & 3 \arrow[r, "\gamma"] & 4, \end{tikzcd} \qquad \beta\alpha = 0.$$

Since $\tau E_2 = \tau^{-1}P_1$, $\mathrm{Hom}_H(T(2), \tau E_2) \cong \mathrm{Hom}_H(P_3 \oplus P_5, \tau E_2)$ is two-dimensional, and the non-zero morphism from P_3 to τE_2 factors through P_5. Then to get the quiver of the corresponding cluster-tilted algebra $\Lambda(2)$, we add an arrow δ from vertex 3 to 2 in the quiver of $A(2)$ with the addition relations $\delta\beta = 0$, $\alpha\delta = 0$. That is

$$\begin{tikzcd} & & 5 \arrow[d, "\rho"'] & \\ 2 \arrow[r, "\alpha"] & 1 \arrow[r, "\beta"] & 3 \arrow[r, "\gamma"] \arrow[ll, bend left, "\delta"] & 4, \end{tikzcd} \qquad \beta\alpha = \delta\beta = \alpha\delta = 0.$$

The APR-tilting module at vertex 3 is $T(3) = P_1 \oplus P_2 \oplus \tau^{-1}E_3 \oplus P_4 \oplus P_5$. The corresponding tilted algebra $A(2)$ is the following:

$$\begin{tikzcd} & 3 \arrow[r, "\gamma"] \arrow[d, "\delta"'] & 5 \arrow[d, "\rho"] \\ 2 \arrow[r, "\alpha"] & 1 \arrow[r, "\beta"] & 4, \end{tikzcd} \qquad \beta\delta = \rho\gamma.$$

Since $\tau E_3 = P_4$, $\mathrm{Hom}_H(T(2), \tau E_3) \cong \mathrm{Hom}_H(P_4, \tau E_3)$ is one-dimensional. Then to get the quiver of the corresponding cluster-tilted algebra $\Lambda(3)$, we add an arrow ξ from vertex 4 to 3 in the quiver of $A(3)$ with the addition relations $\xi\beta = \xi\rho = \delta\xi = \gamma\xi = 0$. That is

$$\begin{tikzcd} & 3 \arrow[r, "\gamma"] \arrow[d, "\delta"'] & 5 \arrow[d, "\rho"] \\ 2 \arrow[r, "\alpha"'] & 1 \arrow[r, "\beta"'] & 4, \arrow[ul, "\xi"'] \end{tikzcd} \qquad \beta\delta = \rho\gamma,\ \xi\beta = \xi\rho = \delta\xi = \gamma\xi = 0.$$

One of important results on cluster-tilted algebras is Theorem 2.2 in [BMR] which gives precise relation between the cluster categories and module category over cluster-tilted algebras. We will generalize the result to the general setting, where the module categories over hereditary algebras are replaced by any hereditary abelian categories with tilting objects. Our proof simplifies the proof in [BMR] and uses approximations on triangulated categories.

Let T be a tilting object in $\mathcal{H}$. Then Hom functor $G = \mathrm{Hom}_{\mathcal{H}}(T, -)$ induces a dense and full functor from the cluster category $\mathcal{C}(\mathcal{H})$ to Λ-mod. Where Λ is the cluster-tilted algebra $\mathrm{End}_{\mathcal{C}(\mathcal{H})} T$, the density and fullness of the functor is obtained from [BMR, Proposition 2.1]. (The proof there are not involved using hereditary algebra, so the proof also works in this general case.)

Since $G(\tau T) = \mathrm{Hom}_{\mathcal{C}(\mathcal{H})}(T, \tau T) = 0$, there is an induced functor $\bar{G} : \mathcal{C}(\mathcal{H})/\operatorname{add}(\tau T) \to \operatorname{mod} \Lambda$. We will prove that $\bar{G}$ is faithful in the following. Firstly we recall that there is an embedding which identifies $\mathcal{H}$ with the full subcategory of $D^b(\mathcal{H})$ consisting of complexes which have zero components of any non-zero degree. We remind the reader that F denotes $\tau^{-1}[1]$.

Theorem 3.7. *Let T be a tilting object in $\mathcal{H}$ and $\Lambda = \mathrm{End}_{\mathcal{C}(\mathcal{H})} T$ the cluster-tilted algebra. Then $\bar{G} : \mathcal{C}(\mathcal{H})/\operatorname{add}(\tau T) \to \operatorname{mod} \Lambda$ is an equivalence.*

Proof. We only need to show that $\bar{G}$ is faithful. Let $\bar{f} : \bar{M} \to \bar{N}$ be a map between indecomposable objects in $\mathcal{C}(\mathcal{H})$. We can assume that $\bar{f}$ is induced from a map $f : M \to N$ in $\mathrm{D}^b(\mathcal{H})$ with M, N in $\mathcal{H}$ or $\mathcal{H}[1]$. We will show if $\bar{G}(\bar{f}) = 0$, then f factors through $\operatorname{add}(\tau T)$. It follows from $\bar{G}(\bar{f}) = 0$ that $\mathrm{Hom}_{D^b(\mathcal{H})}(T, f) = 0$ and $\mathrm{Hom}_{D^b(\mathcal{H})}(T, Ff) = 0$.

(I) We consider the minimal right $\operatorname{add} T$-approximation of M, $M_2 \xrightarrow{\alpha_2} M$ with $M_2 \in \operatorname{add} T$ which exists since $\operatorname{add} T$ contains finitely many indecomposable objects. Then we have the triangle:

$$M_2 \xrightarrow{\alpha_2} M \xrightarrow{\alpha_1} M_1 \longrightarrow M_2[1]. \qquad (*)$$

We also have the triangle with $\beta_2 : N \to N_2$ is the minimal left $\operatorname{add} \tau^2 T$-approximation:

$$N_1 \xrightarrow{\beta_1} N \xrightarrow{\beta_2} N_2 \longrightarrow N_1[1], \qquad (**)$$

where $N_2 \in \operatorname{add} \tau^2 T$. It is easy to see $M_1 \in \mathcal{H} \cup \mathcal{H}[1]$ and $N_1 \in \mathcal{H} \cup \mathcal{H}[-1]$.

We will prove that there exists a commutative diagram:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\alpha"'] & N \\ M' \arrow[r, "g_1"] & N' \arrow[u, "\beta"'] \end{tikzcd}$$

with $M_1', N_1' \in \mathcal{H}$ and M_1' (or N_1') is direct summand of M_1 (N_1, respectively):

(1) It follows from $\mathrm{Hom}_{D^b(\mathcal{H})}(T, Ff) = 0$ that $\mathrm{Hom}_{D^b(\mathcal{H})}(\tau^2 T, f) = 0$. It implies that $f\beta_2 = 0$ since $N_2 \in \operatorname{add} \tau^2 T$. Then there is a map $f_1 : M \to N_1$ with $f = f_1\beta_1$.

(2) We prove that $\mathrm{Hom}(T, f_1) = 0$. By applying $\mathrm{Hom}(T, -)$ to the triangle $(**)$, we have the exact sequence:

$$\mathrm{Hom}\bigl(T, N_2[-1]\bigr) \longrightarrow \mathrm{Hom}(T, N_1) \xrightarrow{\mathrm{Hom}(T,\beta_1)} \mathrm{Hom}(T, N) \longrightarrow \mathrm{Hom}(T, N_2).$$

Since $\mathrm{Hom}(T, N_2[-1]) \subseteq \mathrm{Hom}(T, (\tau^2 T[-1])^m) = (D\,\mathrm{Ext}^1(\tau T, T[1]))^m = (D\,\mathrm{Hom}(\tau T, T[2]))^m = 0$, for some positive integer m, the map $\mathrm{Hom}(T, \beta_1)$ is mono. It follows that $\mathrm{Hom}(T, f_1)$ is zero since the its composition with $\mathrm{Hom}(T, \beta_1)$ is $\mathrm{Hom}(T, f) = 0$.

(3) By (2), we have $\alpha_2 f_1 = 0$. It follows that there exists a map $g : M_1 \to N_1$ with $f_1 = \alpha_1 g$.

(4) We write $M_1 = M_1' \oplus M_2'$ and $N_1 = N_1' \oplus N_2'$ with M_1' and N_1' are maximal direct summand in $\mathcal{H}$ of M_1 and N_1, respectively. Then $g = \left(\begin{smallmatrix} g_1, & o \\ 0 & 0 \end{smallmatrix}\right)$. Let α be the component of α_1 on M_1' and β

the component of β_1 on N_1'. Then $f=\alpha g_1\beta$. Then we have proved there exists a commutative diagram which we proposed above. For simplicity, we assume that both M_1 and N_1 are in $\mathcal{H}$.

(II) We will prove that map $g_1 : M_1 \to N_1$ factors through $\operatorname{add} \tau T$.

By applying $\operatorname{Hom}(T,-)$ to the triangle $(*)$, we can get exact sequence $\operatorname{Hom}(T,M_2) \xrightarrow{\operatorname{Hom}(T,\alpha_2)} \operatorname{Hom}(T,M) \to \operatorname{Hom}(T,M_1) \to \operatorname{Hom}(T,M_2[1]) = 0$, where $\operatorname{Hom}(T,\alpha_2)$ is surjective. It follows that $\operatorname{Hom}_{\mathcal{H}}(T,M_1)=0$, with the conditions that $T, M_1 \in \mathcal{H}$. Then there is an embedding of M_1 into τT. Let $0 \to M_1 \xrightarrow{\gamma} E_1 \to E_2 \to 0$ be an exact sequence in $\mathcal{H}$ with γ being the minimal left $\operatorname{add}(\tau T)$-approximation of M_1, where $E_1 \in \operatorname{add}(\tau T)$. Then

$$M_1 \xrightarrow{\gamma} E_1 \longrightarrow E_2 \longrightarrow M_1[1] \qquad (***)$$

is a triangle with γ is the minimal left $\operatorname{add}(\tau T)$-approximation of M_1 in $D^b(\mathcal{H})$. Of course we have that $E_1, E_2 \in \mathcal{H}$:

(1) To prove $\operatorname{Hom}(\tau^{-1}N_1, \tau T) \cong \operatorname{Hom}(N_1, \tau^2 T) = 0$.

By applying $\operatorname{Hom}(-,\tau^2 T)$ to the triangle $(**)$, we have the exact sequence with $\operatorname{Hom}(\beta_2, \tau^2 T)$ being surjective:

$$\operatorname{Hom}\big(N_2,\tau^2 T\big) \xrightarrow{\operatorname{Hom}(\beta_2,\tau^2 T)} \operatorname{Hom}\big(N,\tau^2 T\big) \longrightarrow \operatorname{Hom}\big(N_1,\tau^2 T\big) \longrightarrow \operatorname{Hom}\big(N_2[-1],\tau^2 T\big).$$

But $\operatorname{Hom}(N_2[-1],\tau^2 T) \subseteq \operatorname{Hom}((\tau^2 T[-1])^m, \tau^2 T) = 0$. It follows that $\operatorname{Hom}(N_1,\tau^2 T)=0$.

(2) To prove that $\operatorname{Hom}(T,E_2)=0$.

By applying $\operatorname{Hom}(-,\tau T)$ to the triangle $(***)$, we have the exact sequence with surjective map $\operatorname{Hom}(\gamma,\tau T)$:

$$\operatorname{Hom}(E_1,\tau T) \xrightarrow{\operatorname{Hom}(\gamma,\tau T)} \operatorname{Hom}(M_1,\tau T) \longrightarrow \operatorname{Hom}\big(E_2[-1],\tau T\big) \longrightarrow \operatorname{Hom}\big(E_1[-1],\tau T\big).$$

$\operatorname{Hom}(E_1[-1],\tau T) \cong D\operatorname{Ext}^1(T,E_1[-1]) \subseteq D\operatorname{Ext}^1(T,(\tau T[-1])^n) \cong (D\operatorname{Hom}(T,\tau T))^n$ $\cong (\operatorname{Ext}^1(T,T))^n = 0$. Therefore $\operatorname{Hom}(E_2[-1],\tau T)=0$ and then $\operatorname{Hom}(T,E_2)=0$. Therefore $\operatorname{Hom}_{\mathcal{H}}(T,E_2)=0$.

(3) To prove $\operatorname{Hom}(\tau^{-1}N_1,E_2)=0$.

We have that there is an embedding of E_2 into τT. Let $0 \to E_2 \xrightarrow{\sigma} X_1 \to X_2 \to 0$ be an exact sequence in $\mathcal{H}$ with σ being the minimal left $\operatorname{add}(\tau T)$-approximation of E_2, where $X_1 \in \operatorname{add}(\tau T)$. Then $E_2 \xrightarrow{\sigma} X_1 \to X_2 \to E_2[1]$ is a triangle with σ being the minimal left $\operatorname{add}(\tau T)$-approximation of E_2 in $D^b(\mathcal{H})$, where $X_1, X_2 \in \mathcal{H}$. By applying $\operatorname{Hom}(\tau^{-1}N_1,-)$ to this triangle, we have the exact sequence:

$$\operatorname{Hom}\big(\tau^{-1}N_1,X_2[-1]\big) \longrightarrow \operatorname{Hom}\big(\tau^{-1}N_1,E_2\big) \longrightarrow \operatorname{Hom}\big(\tau^{-1}N_1,X_1\big).$$

Where $\operatorname{Hom}(\tau^{-1}N_1,X_1) \subseteq \operatorname{Hom}(\tau^{-1}N_1,(\tau T)^t) = 0$ (by (1) in II). We also have that $\operatorname{Hom}(\tau^{-1}N_1,X_2[-1]) = \operatorname{Hom}(\tau^{-1}N_1,\tau X_2[-1]) = D\operatorname{Ext}^1(X_2[-1],N_1) = D\operatorname{Ext}^2(X_2,N_1) = D\operatorname{Hom}(X_2,N_1[2]) = 0$ (by $X_2,\ N_1 \in \mathcal{H}$). It follows that $\operatorname{Hom}(\tau^{-1}N_1,E_2)=0$.

(4) By applying $\operatorname{Hom}(-,N_1)$ to triangle $(***)$, we get the exact sequence:

$$\operatorname{Hom}(E_2,N_1) \longrightarrow \operatorname{Hom}(E_1,N_1) \longrightarrow \operatorname{Hom}(M_1,N_1) \longrightarrow \operatorname{Hom}\big(E_2[-1],N_1\big).$$

Where

$$\mathrm{Hom}\big(E_2[-1], N_1\big) \cong \mathrm{Ext}^1(E_2, N_1) \cong D\,\mathrm{Hom}(N_1, \tau E_2) \cong \mathrm{Hom}\big(\tau^{-1} N_1, E_2\big) = 0.$$

Therefore the map g_1 factors through $E_1 \in \mathrm{add}\,\tau T$. The proof is finished. $\square$

4. Equivalences between cluster categories

In Section 2, we verified the fact that any triangle equivalence between derived categories of hereditary categories induces a triangle equivalence between corresponding triangulated orbit categories of derived categories by suitable automorphisms. Then standard equivalences induce triangle equivalences of cluster categories. We will first proved the converse also holds in this section (the problem whether the converse holds is suggested by Professor Steffen Koenig, we thank him very much!), then we give some useful consequences on cluster categories and root categories as applications.

Theorem 4.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be hereditary abelian categories and one of them derived equivalent to module category of a hereditary algebra. Then $\mathcal{C}(\mathcal{H}_1)$ is triangle equivalent to $\mathcal{C}(\mathcal{H}_2)$ if and only if $\mathcal{H}_1$ is derived equivalent to $\mathcal{H}_2$.*

Proof. The sufficiency is the special case of Proposition 2.3 in which $G = \tau^{-1}[1]$ by using the "Morita theorem" on derived categories (compare [Ric,Ke1]). We need to prove the necessity. Let $\alpha : \mathcal{C}(\mathcal{H}_1) \to \mathcal{C}(\mathcal{H}_2)$ be a triangle equivalence. Suppose T is a tilting object in $\mathcal{C}(\mathcal{H}_1)$. We will prove that $\alpha(T)$ is also a tilting object in $\mathcal{C}(\mathcal{H}_2)$. Since $\mathrm{Ext}^1_{\mathcal{C}(\mathcal{H}_2)}(\alpha(T), \alpha(T)) = \mathrm{Ext}^1_{\mathcal{C}(\mathcal{H}_1)}(T, T) = 0$, $\alpha(T)$ is an exceptional object in $\mathcal{C}(\mathcal{H}_2)$. It follows from Theorem 2.7 or Proposition 3.2 in [BMRRT] that there is an object M such that $\alpha(T) \oplus M$ is a tilting object in $\mathcal{C}(\mathcal{H}_2)$. Then $\alpha^{-1}(\alpha(T) \oplus M) \cong T \oplus \alpha^{-1}(M)$ is an exceptional object in $\mathcal{C}(\mathcal{H}_1)$. It follows from that T is a tilting object that $\alpha^{-1}(M) \in \mathrm{add}\, T$. Then $M \in \mathrm{add}\,\alpha(T)$. Hence $\alpha(T)$ is a tilting object in $\mathcal{C}(\mathcal{H}_2)$. Suppose $\mathcal{H}_1$ is derived equivalent to the module category of a hereditary algebra H. Without loss the generality (compare Corollary 2.4), we may assume that $\mathcal{H}_1 = \mathrm{mod}\, H$. Then $\alpha(H)$ is a tilting object in $\mathcal{C}(\mathcal{H}_2)$. By Proposition 2.6, $\alpha(H)$ is induced by a tilting object T' in $\mathcal{H}_2'$, derived equivalent to $\mathcal{H}_2$. Then we have that

$$\begin{aligned} H &\cong \mathrm{End}_{\mathcal{C}(\mathcal{H}_1)}(H) \quad \text{(by Proposition 3.1)} \\ &\cong \mathrm{End}_{\mathcal{C}(\mathcal{H}_2)}\big(\alpha(H)\big) \\ &\cong \mathrm{End}_{\mathcal{C}(\mathcal{H}_2')}(T') \quad \text{(by Corollary 2.4)} \\ &\cong \mathrm{End}_{\mathcal{H}_2'}(T') \ltimes D\,\mathrm{Hom}_{\mathcal{H}_2'}\big(T', \tau^2 T'\big) \quad \text{(by Proposition 3.1).} \end{aligned}$$

This shows $\mathrm{End}_{\mathcal{H}_2'}(T') \ltimes D\,\mathrm{Hom}_{\mathcal{H}_2'}(T', \tau^2 T')$ is hereditary, hence $\mathrm{End}_{\mathcal{H}_2'}(T')$ is hereditary [FGR], which is denoted by A. It is not difficult to see that $D\,\mathrm{Hom}_{\mathcal{H}_2'}(T', \tau^2 T') \cong \mathrm{Ext}^2_A(DA, A)$, hence it is zero. Then $\mathrm{End}_{\mathcal{H}_2'}(T') \cong H$. Therefore $D^b(\mathcal{H}_i)$ is triangle equivalent to $D^b(H)$ for $i = 1, 2$. The proof is finished. $\square$

If we restrict our attention to equivalences induced by tilting objects in $\mathcal{H}$, we can get some applications of Proposition 2.3 and Theorem 4.1. For any tilting object T in $\mathcal{H}$, $\mathrm{Hom}_{\mathcal{H}}(T,-)$ induces a triangle equivalence from $\mathcal{C}(\mathcal{H})$ to the cluster category $\mathcal{C}(A)$, where A is the quasi-tilted algebra of T. When $\mathcal{H}$ is the category of finitely generated left modules over a hereditary algebra, if we set T to be a Bernstein–Gelfand–Ponomarev titling module [BGP] or an APR-tilting module [APR], then we get a triangle equivalence $R(T)_{\tau^{-1}[1]}$ between the corresponding cluster categories. This triangle equivalence provides a realization of "truncate reflection functors" in [FZ4,MRZ], if we identify the cluster categories in each side with the set of almost positive roots of corresponding Kac–Moody Lie algebras. We will explain simply in the following, for details, we refer to [Z1,MRZ].

Let $(\Gamma,\mathbf{d})$ be a valued graph without cycles, Ω an orientation. For any vertex $k\in\Gamma$, we can define a new orientation $s_k\Omega$ of $(\Gamma,\mathbf{d})$ by reversing the direction of arrows along all edges containing k. A vertex $k\in\Gamma$ is said to be a sink (or a source) with respect to Ω if there are no arrows starting (or ending) at vertex k.

Let K be a field and $(\Gamma,\mathbf{d},\Omega)$ a valued quiver. Let $\mathbf{M}=(F_i,{}_iM_j)_{i,j\in\Gamma}$ be a reduced K-species of type Ω; that is, for all $i,j\in\Gamma$, ${}_iM_j$ is an F_i–F_j-bimodule, where F_i and F_j are finite extensions of K and $\dim({}_iM_j)_{F_j}=d_{ij}$ and $\dim_K F_i=\varepsilon_i$. An K-representation $V=(V_i,{}_j\varphi_i)$ of $\mathbf{M}$ consists of F_i-vector space V_i, $i\in\Gamma$, and of an F_j-linear map ${}_j\varphi_i:V_i\otimes{}_iM_j\to V_j$ for each arrow $i\to j$. Such representation is called finite-dimensional if $\sum_{i\in\Gamma}\dim_K V_i<\infty$. The category of finite-dimensional representations of $\mathbf{M}$ over K is denoted by $\mathrm{rep}(\mathbf{M},\Gamma,\Omega)$.

Now we fix an K-species $\mathbf{M}$ of a given valued quiver $(\Gamma,\mathbf{d},\Omega)$. Given a sink, or a source k of the quiver $(\Gamma,\mathbf{d},\Omega)$, we are going to recall the reflection functor $S_k^{\pm}$:

$$S_k^{+}:\mathrm{rep}(\mathbf{M},\Gamma,\Omega)\longrightarrow\mathrm{rep}(\mathbf{M},\Gamma,s_k\Omega),\quad \text{if } k \text{ is a sink,}$$

or

$$S_k^{-}:\mathrm{rep}(\mathbf{M},\Gamma,\Omega)\longrightarrow\mathrm{rep}(\mathbf{M},\Gamma,s_k\Omega),\quad \text{if } k \text{ is a source.}$$

We assume k is a sink. For any representation $V=(V_i,\phi_\alpha)$ of $(\mathbf{M},\Gamma,\Omega)$, the image of it under S_k^+ is by definition, $S_k^+V=(W_i,{}_j\psi_i)$, a representation of $(\mathbf{M},\Gamma,s_k\Omega)$, where $W_i=V_i$, if $i\neq k$; and W_k is the kernel in the diagram:

$$0\longrightarrow W_k\xrightarrow{({}_j\chi_k)_j}\bigoplus_{j\in\Gamma}V_j\otimes{}_jM_k\xrightarrow{({}_k\phi_j)_j}V_k\qquad(*)$$

${}_j\psi_i={}_j\phi_i$ and ${}_j\psi_k={}_j\bar\chi_k:W_k\otimes{}_kM_j\to X_j$, where ${}_j\bar\chi_k$ corresponds to ${}_j\chi_k$ under the isomorphism $\mathrm{Hom}_{F_j}(W_k\otimes{}_kM_j,V_j)\approx\mathrm{Hom}_{F_i}(W_k,V_j\otimes{}_jM_i)$.

If $\boldsymbol{\alpha}=(\alpha_i):V\to V'$ is a morphism in $\mathrm{rep}(\mathbf{M},\Gamma,\Omega)$, then $S_k^+\boldsymbol{\alpha}=\beta=(\beta_i)$, where $\beta_i=\alpha_i$ for $i\neq k$ and $\beta_k:W_k\to W_k'$ as the restriction of $\bigoplus_{j\in\Gamma}(\alpha_j\otimes 1)$ given in the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & W_k \arrow[r, "({}_j\chi_k)_j"] \arrow[d, "\beta_k"] & \bigoplus_{j\in\Gamma}V_j\otimes{}_jM_k \arrow[r, "({}_k\phi_j)_j"] \arrow[d, "\oplus_j(\alpha_j\otimes 1)"] & V_k \arrow[d, "\alpha_k"] \\
0 \arrow[r] & W_k' \arrow[r, "({}_j\chi_k')_j"] & \bigoplus_{j\in\Gamma}V_j'\otimes{}_jM_k \arrow[r, "({}_k\phi_j')_j"] & V_k'
\end{tikzcd}$$

If k is source, the definition of $S_k^- V$ is dual to that of $S_k^+ V$, we omit it and refer to [DR].

In the rest of the section, we denote by $\mathcal{H}$ the category $\mathrm{rep}(\mathbf{M}, \Gamma, \Omega)$ and by $\mathcal{H}'$ the category $\mathrm{rep}(\mathbf{M}, \Gamma, s_k\Omega)$, where k is a sink (or source) of $(\Gamma, \mathbf{d}, \Omega)$. The root categories $D^b(\mathcal{H})/[2]$, $D^b(\mathcal{H}')/[2]$ are denoted by $\mathcal{R}(\Omega)$ and $\mathcal{R}(s_k\Omega)$, respectively. The cluster categories $D^b(\mathcal{H})/F$, $D^b(\mathcal{H}')/F$ are denoted by $\mathcal{C}(\Omega)$ and $\mathcal{C}(s_k\Omega)$, respectively.

Let P_i (or P_i') be the projective indecomposable representation $\mathcal{H}$ (respectively $\mathcal{H}'$) corresponding to the vertex $i \in \Gamma_0$, and E_k (or E_k') the simple representation of $\mathcal{H}$ (respectively $\mathcal{H}'$) corresponding to the vertex k. We denote by H (or H') the tensor algebra of $(\mathbf{M}, \Gamma, \Omega)$ ($(\mathbf{M}, \Gamma, s_k\Omega)$, respectively). Note that if k is a sink, then $P_k = E_k$.

Let $T = \bigoplus_{i \in \Gamma - k} P_i \oplus \tau^{-1} P_k$. Suppose k is a sink, then T is a tilting module in $\mathcal{H} = \mathrm{rep}(\mathbf{M}, \Gamma, \Omega)$ which is called BGP-tilting module (or APR-tilting). $S_k^+ = \mathrm{Hom}(T, -)$ as functors.

The following lemma is proved in [Z1], for completeness, we give a proof here.

Lemma 4.2. *Let k be a sink (or a source) of a valued quiver $(\Gamma, \mathbf{d}, \Omega)$. Then the BGP-reflection functor induces a triangle equivalence $R_{\tau^{-1}[1]}(S_k^+)$ (respectively, $R_{\tau^{-1}[1]}(S_k^-)$) from $\mathcal{C}(\Omega)$ to $\mathcal{C}(s_k\Omega)$. Moreover, $R_{\tau^{-1}[1]}(S_k^+)(\widetilde{E_k}) = \widetilde{P_k'}[1]$, $R_{\tau^{-1}[1]}(S_k^+)(\widetilde{P}_k[1]) = \widetilde{E_k'}$, and for $j \neq k$, $R_{\tau^{-1}[1]}(S_k^+)(\widetilde{P_j}[1]) = \widetilde{P_j'}[1]$, for indecomposable non-projective H-module X, $R_{\tau^{-1}[1]}(S_k^+)(\tilde{X}) = \widetilde{S_k^+(X)}$.*

Proof. From Corollary 2.5, we have the triangle equivalent functor $R_{\tau^{-1}[1]}(S_k^+)$ from the cluster category $\mathcal{C}(\Omega)$ to $\mathcal{C}(s_k\Omega)$. Now we prove that $R_{\tau^{-1}[1]}(S_k^+)(\widetilde{E_k}) = \widetilde{P_k'}[1]$. From [APR], we have AR-sequence $(*): 0 \to E_k \to X \to \tau^{-1}E_k \to 0$ in H-mod with X and $\tau^{-1}E_k$ being modules without E_k as direct summands. Since S_k^+ is left exact functor, we have the exact sequence $0 \to S_k^+(X) \to S_k^+(\tau^{-1}E_k)$ in H'-mod, in which the cokernel of the injective map is E_k'. Regarded as the stalk complex of degree 0, $E_k^\bullet$ is isomorphic to the complex: $\cdots \to 0 \to X \to \tau^{-1}E_k \to 0 \to \cdots$ in $D^b(\mathcal{H})$. By applying S_k^+ to the complex above, we have that $S_k^+(E_k^\bullet) = \cdots \to 0 \to S_k^+(X) \to S_k^+(\tau^{-1}E_k) \to 0 \to \cdots$. It follows that the complex $\cdots \to 0 \to S_k^+(X) \to S_k^+(\tau^{-1}E_k) \to 0 \to \cdots$ is quasi-isomorphic to the stalk complex $E_k'^\bullet[-1]$ of degree -1. It follows $R(S_k^+)(\widetilde{E_k}) = \widetilde{E_k'}[-1]$. Since $\tau\widetilde{P_k'} = \widetilde{E_k'}[-1]$, $R_{\tau^{-1}[1]}(S_k^+)(\widetilde{E_k}) = \tau\widetilde{P_k'} = \widetilde{F^{-1}(P_k')}[1] = \widetilde{P_k'}[1]$. The proof for others is easy: in derived category $D^b(\mathcal{H})$, we have that $S_k^+(P_i) = P_i'$ for any $i \neq k$, $S_k^+(E_k[1]) = E_k'$. It follows that $R_{\tau^{-1}[1]}(S_k^+)(\widetilde{P}_i) = \widetilde{P_i'}$ for any $i \neq k$ and $R_{\tau^{-1}[1]}(S_k^+)(\widetilde{E_k[1]}) = \widetilde{E_k'}$. Let $X \in \mathrm{ind}\, H$ be non-projective representation, $R_{\tau^{-1}[1]}(S_k^+)(\tilde{X}) = \widetilde{R(S_k^+)(X)} = \widetilde{S_k^+(X)}$. The proof is finished. $\square$

Let $(\Gamma, \mathbf{d}, \Omega)$ be a valued quiver. We denote by Φ the set of roots of the corresponding Kac–Moody Lie algebra. Let $\Phi_{\geqslant -1}$ denote the set of almost positive roots, i.e., the positive roots together with the negatives of the simple roots. When Γ is of Dynkin type, the cluster variables of type Γ are in 1-1 correspondence with the elements of $\Phi_{\geqslant -1}$ (compare [FZ3,FZ4]). We define the map γ_Ω from $\mathrm{ind}\,\mathcal{C}(\Omega)$ and $\Phi_{\geqslant -1}$ as follows (compare [BMRRT]): Let $X \in \mathrm{ind}(\mathrm{mod}\, H \vee H[1])$.

$$\gamma_\Omega(\tilde{X}) = \begin{cases} \mathbf{dim}\, X & \text{if } X \in \mathrm{ind}\, H, \\ -\mathbf{dim}\, E_i & \text{if } X = P_i[1], \end{cases}$$

where $\mathbf{dim}\, X$ denotes the dimension vector of representation X. When Γ is of Dynkin type, $\Phi_{\geqslant -1}$ is a bijection which sends basic tilting objects to clusters in $\Phi_{\geqslant -1}$ (compare [Z1]).

Let s_i be the Coxeter generator of Weyl group of Φ corresponding to $i \in \Gamma$. We recall from [FZ3,FZ4] that the "truncated reflection" σ_i of $\Phi_{\geqslant -1}$ is defined as follows:

$$\sigma_i(\alpha) = \begin{cases} \alpha & \alpha = -\alpha_j,\ j \neq i, \\ s_i(\alpha) & \text{otherwise.} \end{cases}$$

By using Lemma 4.2, one gets the following commutative diagram which explains the $R_{\tau^{-1}[1]}(S_k^+)$ is the realization of "truncate reflections" in [FZ3,FZ4].

Proposition 4.3. *Let k be a sink (or a source) of a valued quiver $(\Gamma, \mathbf{d}, \Omega)$. Then we have the commutative diagram:*

$$\begin{tikzcd} \operatorname{ind}\mathcal{C}(\Omega) \arrow[r, "R_{\tau^{-1}[1]}(S_k^+)", "(R_{\tau^{-1}[1]}(S_k^-),\ \textit{respectively})"'] \arrow[d, "\gamma_\Omega"'] & \operatorname{ind}\mathcal{C}(s_k\Omega) \arrow[d, "\gamma_{s_k\Omega}"] \\ \Phi_{\geqslant -1} \arrow[r, "\sigma_k"] & \Phi_{\geqslant -1} \end{tikzcd}$$

Remark 4.4. If $(\Gamma, \mathbf{d}, \Omega)$ is a simply-laced quiver of Dynkin type, say Q, for a sink or source k, there are functors Σ_k^+ and Σ_k^-, respectively, defined in [MRZ] which give a realization of σ_k via "decorated" quiver representation. We remark that the functors Σ_k^+ and Σ_k^- defined in [MRZ] are not equivalent. In this case, the functors $R_{\tau^{-1}[1]}(S_k^+)$ also satisfy some community with functors Σ_k^+ in the following diagram: in the diagram, rep $\tilde{Q}$ denotes the category of decorated representations of Q and $\mathbf{sdim}(M)$ its signed dimension vector (refer [MRZ]). For Ψ_Q, we refer [BMRRT, Section 4].

$$\begin{tikzcd} \operatorname{ind}\mathcal{C}(Q) \arrow[r, "R_{\tau^{-1}[1]}(S_k^+)", "(R_{\tau^{-1}[1]}(S_k^-),\ \textit{respectively})"'] \arrow[d, "\Psi_Q"'] & \operatorname{ind}\mathcal{C}(s_k Q) \arrow[d, "\Psi_{s_k Q} V"] \\ \operatorname{indrep} \tilde{Q} \arrow[r, "\Sigma_k^+", "(\Sigma_k^-,\ \textit{respectively})"'] \arrow[d, "\mathbf{sdim}"'] & \operatorname{indrep} \widetilde{s_k Q} \arrow[d, "\mathbf{sdim}"] \\ \Phi_{\geqslant -1} \arrow[r, "\sigma_k"] & \Phi_{\geqslant -1} \end{tikzcd}$$

We now apply Corollary 2.5 to the root categories $\mathcal{R}(\Omega)$, where $(\mathbf{M}, \Gamma, \Omega)$ is a valued quiver with species $\mathbf{M}$. The Grothendieck group $K_0(\mathcal{R}(\Omega))$ is $\mathbf{Z}^n$, where n is the number of vertices of Γ. For $M \in K_0(\mathcal{R}(\Omega))$, we denote by $\mathbf{dim}\, M$ the canonical image of M in $K_0(\mathcal{R}(\Omega))$. It is easy to see that $\mathbf{dim}\, M[1] = -\mathbf{dim}\, M$. It follows from Kac's theorem [Ka,DX] that we have a map $\mathbf{dim} : \operatorname{ind}\mathcal{R}(\Omega) \to \Phi$, which is surjective. In case Γ is of Dynkin type, the map $\mathbf{dim}$ is bijective. From Corollaries 2.4, 2.5, we get a similar commutative diagram to Proposition 4.3 (compare [XZZ]).

Proposition 4.5. *Let k be a sink (or a source) of a valued quiver $(\Gamma, \mathbf{d}, \Omega)$ (of any type). Then the BGP-reflection functor induces a triangle equivalence $R_{[2]}(S_k^+)$ ($R_{[2]}(S_k^-)$, respectively) from $\mathcal{R}(\Omega)$ to $\mathcal{R}(s_k\Omega)$, with $R_{[2]}(S_k^+)(\widetilde{E_k}) = \widetilde{E'_k}[1]$. Moreover, we have the commutative diagram*:

$$\begin{tikzcd} \operatorname{ind}\mathcal{R}(\Omega) \arrow[r, "R_{[2]}(S_k^+)", "(R_{[2]}(S_k^-),\ \textit{respectively})"'] \arrow[d, "\mathbf{dim}"'] & \operatorname{ind}\mathcal{R}(s_k\Omega) \arrow[d, "\mathbf{dim}"] \\ \Phi \arrow[r, "s_k"] & \Phi \end{tikzcd}$$

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References

[APR] M. Auslander, M.I. Platzeck, I. Reiten, Coxeter functors without diagrams, Trans. Amer. Math. Soc. 250 (1979) 1–46.
[ARS] M. Auslander, I. Reiten, S. Smalø, Representation Theory of Artin Algebras, Stud. Adv. Math., vol. 36, Cambridge Univ. Press, 1995.
[BFZ] A. Berenstein, S. Fomin, A. Zelevinsky, Cluster algebras III: Upper bounds and double Bruhat cells, Duke Math. J. 126 (1) (2005) 1–52.
[BG] K. Bongartz, P. Gabriel, Covering spaces in representation theory, Invent. Math. 65 (1982) 331–378.
[BGP] I.N. Bernstein, I.M. Gelfand, V.A. Ponomarev, Coxeter functors and Gabriel theorem, Uspekhi Mat. Nauk 28 (2 (170)) (1973) 19–33; Russian Math. Surveys 28 (1973) 17–32.
[BMR] A. Buan, R. Marsh, I. Reiten, Cluster-tilted algebras, preprint, arXiv: math.RT/0402054, 2004; Trans. Amer. Math. Soc., in press.
[BMRRT] A. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov, Tilting theory and cluster combinatorics, preprint, arXiv: math.RT/0402075, 2004, Adv. Math., in press.
[CCS] P. Caldero, F. Chapoton, R. Schiffler, Quivers with relations arising from clusters (A_n case), Trans. Amer. Math. Soc. 358 (2006) 1347–1364.
[CFZ] F. Chapoton, S. Fomin, A. Zelevinsky, Polytopal realizations of generalized associahedra, Canad. Math. Bull. 45 (4) (2002) 537–566.
[CK] P. Caldero, B. Keller, From triangulated categories to cluster algebras II, preprint, arXiv: math.RT/0510251.
[DX] B.M. Deng, J. Xiao, A new approach to Kac’s theorem on representations of valued quivers, Math. Z. 245 (2003) 183–199.
[DR] V. Dlab, C.M. Ringel, Indecomposable representations of graphs and algebras, Mem. Amer. Math. Soc. 173 (1976).
[FGR] R.M. Fossum, P.A. Griffith, I. Reiten, Trivial Extensions of Abelian Categories, Lecture Notes in Math., vol. 456, Springer-Verlag, Berlin, 1975.
[FZ1] S. Fomin, A. Zelevinsky, Cluster algebras I: Foundations, J. Amer. Math. Soc. 15 (2) (2002) 497–529.
[FZ2] S. Fomin, A. Zelevinsky, The Laurent phenomenon, Adv. in Appl. Math. 28 (2) (2002) 119–144.
[FZ3] S. Fomin, A. Zelevinsky, Cluster algebras II: Finite type classification, Invent. Math. 154 (1) (2003) 63–121.
[FZ4] S. Fomin, A. Zelevinsky, Y-systems and generalized associahedra, Ann. of Math. 158 (3) (2003) 977–1018.
[GSV] M. Gekhtman, M. Shapiro, A. Vainshtein, Cluster algebras and Poisson geometry, Mosc. Math. J. 3 (3) (2003) 899–934.

[H1] D. Happel, Triangulated Categories in the Representation Theory of Quivers, London Math. Soc. Lecture Note Ser., vol. 119, Cambridge Univ. Press, Cambridge, 1988.

[H2] D. Happel, A characterization of hereditary categories with tilting object, Invent. Math. 144 (2001) 381–398.

[H3] D. Happel, Tilting sets on cylinders, Triangulated categories in the representation theory of quivers, Proc. London Math. Soc. 51 (3) (1985) 21–55.

[HRS] D. Happel, I. Reiten, S. Smalø, Tilting in abelian categories and quasitilted algebras, Mem. Amer. Math. Soc. 575 (1996).

[HU] D. Happel, L. Unger, On the set of tilting objects in hereditary categories, in: Representations of Algebras and Related Topics, in: Fields Inst. Commun., vol. 45, Amer. Math. Soc., Providence, RI, 2005, pp. 141–159.

[Ka] V. Kac, Infinite root systems, representations of quivers and invariant theory, Invent. Math. 56 (1980) 57–92.

[Ke1] B. Keller, Deriving DG categories, Ann. Sci. École Norm. Sup. 27 (1994) 63–102.

[Ke2] B. Keller, Triangulated orbit categories, Doc. Math. 10 (2005) 551–581.

[KZ] S. König, A. Zimmermann, Derived Equivalences for Group Rings, Lecture Notes in Math., vol. 1685, Springer-Verlag, Berlin, 1998.

[MRZ] R. Marsh, M. Reineke, A. Zelevinsky, Generalized associahedra via quiver representations, Trans. Amer. Math. Soc. 355 (10) (2003) 4171–4186.

[Ric] J. Rickard, Morita theory for derived categories, J. London Math. Soc. 39 (1989) 436–456.

[Rin] C.M. Ringel, Tame Algebras and Integral Quadratic Forms, Lecture Notes in Math., vol. 1099, Springer-Verlag, Berlin, 1984.

[XZ1] J. Xiao, B. Zhu, Relations for the Grothendieck groups of triangulated Categories, J. Algebra 257 (2002) 37–50.

[XZ2] J. Xiao, B. Zhu, Locally finite triangulated categories, J. Algebra 290 (2005) 473–490.

[XZZ] J. Xiao, G.L. Zhang, B. Zhu, BGP-reflection functors in root categories, Sci. China Ser. A 48 (8) (2005) 1033–1045.

[Z1] B. Zhu, BGP-reflection functors and Cluster combinatorics, preprint. Also see arXiv: math.RT/0511380, 2004.

[Z2] B. Zhu, Applications of BGP-reflection functors: Isomorphisms of cluster algebras, preprint, arXiv: math.RT/0511384.