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On the full coupling between thermo-plasticity and thermo-damage in thermodynamic modeling of dissipative materials

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ABSTRACT

The phenomenological model of dissipative material in the small strain range is developed in the framework of thermodynamics of irreversible processes with internal state variables and local state method. The possibilities of the model are illustrated in the example of thermo-elastic–plastic damage material. Particular emphasis is put on including in the description of the full coupling between plasticity and damage in nonisothermal conditions. The consequences of thermal-plastic-damage coupling in consistency conditions and loading/unloading conditions are studied in detail.

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1. Introduction

The increasing demands for high performance materials require the adequate constitutive modeling, as well as the appropriate predictions of the overall failure mechanisms under complex thermo-mechanical loads. When engineering materials classified as elastic–plastic–damage (for example polycrystalline metals) are subjected to external loading, the material degradation connected with slip rearrangements of crystallographic planes through dislocation motion, observed at the macro-scale as plastic behavior (Chaboche, 2008), is accompanied by the development of other microscopic defects, like micro-cracks and micro-voids (Lemaitre, 1992; Abu Al-Rub and Voyiadjis, 2003). The nucleation, growth and interaction of these micro-defects under external loads result in a deterioration process on the macro-scale and, as a consequence, change of the constitutive properties of the material.

If the elastic–plastic–damage material is loaded so that not only inelastic strains develop, but also the temperature is changed, then thermo-elasticity, thermo-plasticity and thermo-damage are encountered. The experimental results (Bednarek and Kamocka, 2006) proved that not only the temperature itself but also the heating rate makes a significant impact on parameters that determine carrying capacity at elevated temperatures, and that heating rate should be accounted for in the strength analysis of structures exposed to high temperatures. The need for the additional term, proportional to the temperature rate in the evolution equation for the back-stress was already considered by Prager (1958), introduced

also by Chaboche (1997b) in the unified viscoplastic constitutive equations using the Armstrong–Frederic format. In Chaboche (2008) the discussion is made for the necessity of temperature rate terms in the context of hardening rules.

Ganczarski and Skrzypek (2009) take into account the temperature dependence of all material functions that characterize plasticity and damage components, which results in extended thermo-plastic–damage equations, with the additional temperature rate terms in all evolution equations of thermodynamic conjugate forces. More general case of the non-associated plasticity and non-associated damage, when not only temperature-softening but also damage-softening is taken into account is due to Egner (2009).

In the present analysis a general phenomenological model, based on the irreversible thermodynamics, is formulated and used to describe the dissipative elastic–plastic–damage material in the small strain range. A special attention is paid to the proper description of coupling between heating rate and two dissipative phenomena: plasticity and damage, taking place in the material subjected to non-isothermal conditions. Both thermal softening and damage softening are accounted for and the consequences of coupling in consistency conditions and loading/unloading conditions are studied in detail.

2. General thermodynamical model of dissipative materials

2.1. Basic assumptions

We consider a closed thermodynamic system that is susceptible of several possibly coupled dissipative phenomena (like plasticity, damage, phase changes, frictional slips on closed crack lips etc.,)

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that are formalized on the macroscopic level by the use of a proper set of state variables. The motions of the system obey the fundamental laws of continuum mechanics (conservation of mass, conservation of linear momentum, conservation of angular momentum) and two laws of thermodynamics written here in the local form:

- Conservation of energy

$$\rho \dot{u} - \dot{\varepsilon}_{ij} \sigma_{ij} - r + q_{i,i} = 0 \quad (1)$$

- Clausius–Duhem inequality

$$\rho \dot{s} - \frac{r}{\theta} + \frac{q_{i,i}}{\theta} - q_i \frac{\theta_{,i}}{\theta^2} \geq 0 \quad (2)$$

where ρ is the mass density per unit volume; $\underline{\sigma}$ is the stress tensor; u is the internal energy per unit mass; $\underline{\varepsilon}$ is the strain tensor; r is the distributed heat source per unit volume; \underline{q} is the outward heat flux; s is the internal entropy production per unit mass and θ is the absolute temperature.

Depending on the scale, different approaches may be used in order to describe an overall structural response of a dissipative structure on the macro-scale. In general, micro-mechanical models relate the macro-properties and the macro-response of a structure to its microstructure. In such approach the rearrangements of micro-structure are discrete and stochastic phenomena induced by a number of weakly or strongly interacting micro-changes that influence the overall structural response. The micro-mechanical models have the advantage of being able to sustain heterogeneous structural details on the micro-scale and meso-scale, and to allow a micro-mechanical formulation of the evolution equations based on the accurate micro-changes growth processes involved (cf. Voyiadjis et al., 2007; Boudifa et al., 2009; Aboudi, in press).

Continuum mechanics approach, applied in the present work, provides the constitutive and damage evolution equations in the framework of thermodynamics of irreversible processes. The material heterogeneity (on the micro- and meso-scale) is smeared out over the representative volume element (RVE) of the piece-wise discontinuous material. The true state of material within RVE, represented by the topology, size, orientation and number of micro-changes, is mapped to a material point of the quasi-continuum. The true distribution of micro-changes within the RVE, and the correlation between them are measured by the change of the effective constitutive properties. The micro-structural rearrangements are defined by the set of state variables of the scalar, vectorial or tensorial nature (cf. Murakami and Ohno, 1980; Chaboche, 1997a; Skrzypek et al., 2008; Ganczarski et al., 2010).

In the case of infinitesimal deformation the total strain $\underline{\varepsilon}$ can be expressed as the sum of the elastic (reversible) strain $\underline{\varepsilon}^E$, inelastic (irreversible) strain $\underline{\varepsilon}^I$, and thermal strain $\underline{\varepsilon}^\theta$:

$$\varepsilon_{ij} = \varepsilon_{ij}^E + \varepsilon_{ij}^I + \varepsilon_{ij}^\theta \quad (3)$$

In the process of deformation, various microstructural rearrangements of material structure may be induced, for example the changes in density and configuration of dislocations, the development of microscopic cavities, changes from primary to secondary phase etc. All these rearrangements may contribute to both reversible and irreversible strains (cf. Abu Al-Rub and Voyiadjis, 2003), therefore:

$$\varepsilon_{ij}^E = \varepsilon_{ij}^e + \sum_{k=1}^n \varepsilon_{ij}^{Ek}, \quad \varepsilon_{ij}^I = \sum_{k=1}^n \varepsilon_{ij}^{Ik}, \quad \varepsilon_{ij}^k = \varepsilon_{ij}^{Ek} + \varepsilon_{ij}^{Ik}, \quad k = 1, 2, \dots, n \quad (4)$$

where ε_{ij}^e is a “pure” elastic strain, and $\varepsilon_{ij}^{Ek}, \varepsilon_{ij}^{Ik}$ are respectively the reversible and irreversible components of the total strain ε_{ij}^k induced

by k -th dissipative phenomenon (see Fig. 1), e.g. plastic flow ($k = p$), damage ($k = d$), phase change ($k = ph$) etc. For example, in the case of thermo-elastic–plastic–damage material the total strain tensor is expressed as

$$\varepsilon_{ij} = \underbrace{\varepsilon_{ij}^e + \varepsilon_{ij}^{Ed}}_{\varepsilon_{ij}^E} + \underbrace{\varepsilon_{ij}^p + \varepsilon_{ij}^{Id}}_{\varepsilon_{ij}^I} + \varepsilon_{ij}^\theta \quad (5)$$

while its damage induced component, ε_{ij}^d , consists of both reversible (Ed) and irreversible (Id) damage strain terms

$$\varepsilon_{ij}^d = \varepsilon_{ij}^{Ed} + \varepsilon_{ij}^{Id} \quad (6)$$

2.2. State variables

The irreversible rearrangements of the internal structure can be represented by a group of state variables describing the current state of material microstructure:

$$\{A^k\}, \quad k = p, d, ph, \dots \quad (7)$$

where A^k may be scalars, vectors or even rank tensors. For damage description, in the case where the damaged material remains isotropic, the current state of damage is often represented by the scalar variable A^d denoting the volume fraction of cracks and voids in the total volume.

Damage acquired orthotropy requires a second order tensor, for example the classical (Murakami and Ohno, 1980) tensor:

$$\underline{\underline{A}}^d = \underline{\underline{D}} = \sum_{i=1}^3 D_i n_i \otimes n_i \quad (8)$$

where $D_i = dA_i^d/dA_i$ denotes the ratio of cracks and voids area to the total area on the principal plane of normal unit vector \underline{n} . In the most general case of anisotropy the description of damage needs to be embodied in an eight-order tensor (cf. Cauvin and Testa, 1999), while the principle of strain equivalence allows using fourth-order tensors.

For the phase transformation analysis the scalar variable

$$A^{ph} = \xi = \frac{dV^s}{dV^0} \quad (9)$$

is commonly adopted (cf. Egner and Skoczec, 2010), which denotes the volume fraction V^s of the martensite in the total volume V^0 of the martensite-austenite representative volume element, if martensitic transformation $\gamma \rightarrow \alpha'$ is considered. However, a scalar variable is not capable of describing the acquired anisotropy due to partially directional nature of the martensitic inclusions in the austenitic matrix. Therefore, instead of scalar variable (9) a second-order tensor can be defined in analogy to (8):

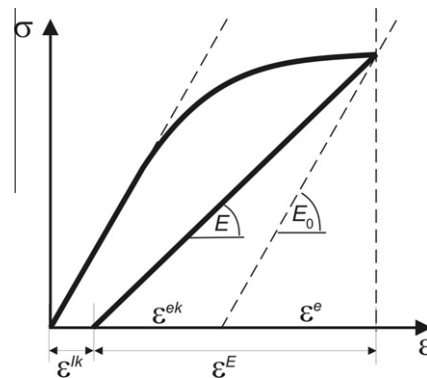


Fig. 1. Components of the strain tensor induced by k -th dissipative phenomenon.

$$\underline{\underline{\Lambda}}^{ph} = \underline{\underline{\xi}} = \sum_{i=1}^3 \xi_i n_i \otimes n_i, \quad (10)$$

where $\xi_i = dA_i^s/dA_i$ describes the ratio of the secondary phase area to the total area on the principal plane of normal unit vector \underline{n} (cf. Egnér, 2010).

Another group of state variables consists of internal (hidden) variables corresponding to the modifications of loading surfaces

$$\{h^k\} = \{r^k, \alpha_{ij}^k, l_{ijpq}^k, g_{ijpqmn}^k\}, \quad k = p, d, ph, \dots \quad (11)$$

where r^k corresponds to isotropic expansion of the loading surface, α_{ij}^k affects translatoric displacements of the loading surface, l_{ijpq}^k is a hardening tensor of the fourth order which includes varying lengths of axes and rotation of the loading surface, and g_{ijpqmn}^k describes changes of the curvature of the loading surface (distortion) related to k -th dissipative phenomenon (cf. Kowalsky et al., 1999; see Fig. 2).

The complete set of state variables $\{V_{st}\}$ reflecting the current state of the thermodynamic system consists of observable variables: elastic (or total) strain tensor ϵ_{ij}^E and absolute temperature θ , and two groups of microstructural $\{A^k\}$ and hardening $\{h^k\}$ state variables:

$$\{V_{st}\} = \{\epsilon_{ij}^E, \theta, A^k, h^k\}, \quad k = p, d, ph, \dots \quad (12)$$

When thermo-elastic-plastic-damage two phase material is considered, the exemplary set of state variables is listed in Table 1.

By the use of state variables (12) the Helmholtz free energy of the material can be written as a sum of elastic (e), plastic (p), damage (d), phase change (ph) etc. terms (cf. Lemaitre and Chaboche, 1990; Abu Al-Rub and Voyiadjis, 2003):

$$\rho\psi = \rho\psi(V_{st}) = \sum_{j=1}^n \rho\psi^j, \quad j = e, p, d, ph, \dots \quad (13)$$

By eliminating all the reversible processes from the Clausius–Duhem inequality (2) the following state equations which express the thermodynamic forces conjugated to the observable state variables are obtained:

$$\sigma_{ij} = \frac{\partial(\rho\psi)}{\partial\epsilon_{ij}^E} \quad (14)$$

$$s = -\frac{\partial\psi}{\partial\theta} \quad (15)$$

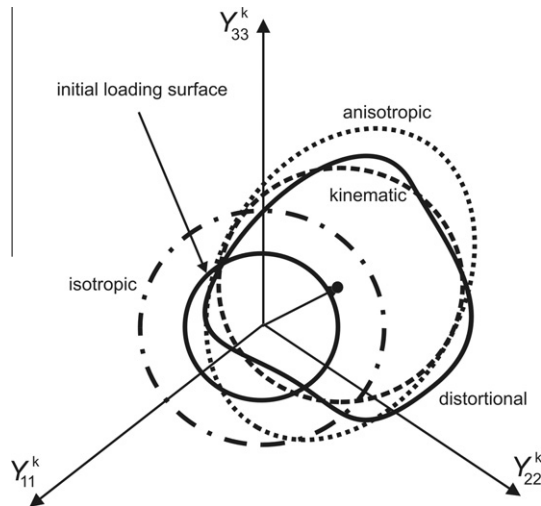


Fig. 2. Modifications of the loading surface related to the k -th dissipative phenomenon in the space of thermodynamic conjugate force Y^k .

In addition, the pairs of forces (Y^k, H^k) conjugated to other microstructural and hidden state variables (A^k, h^k) are postulated in a similar form to (14) and (15) (cf. Chaboche, 1997a):

$$-Y^k = \frac{\partial(\rho\psi)}{\partial A^k}, \quad k = 1, 2, \dots \quad (16)$$

$$H^k = \frac{\partial(\rho\psi)}{\partial h^k}, \quad k = 1, 2, \dots \quad (17)$$

In the above equations Y^k stand for thermodynamic forces conjugated to microstructural state variables A^k , whereas H^k are hardening forces conjugated to hidden state variables h^k (see Table 1).

2.3. Dissipation potentials and evolution rules

To derive the kinetic equations it is assumed that at any given temperature and state of microstructure, the rate at which any specific microstructural rearrangement occurs is fully determined by the thermodynamic force associated with this rearrangement (cf. Rice, 1971). Additionally, the existence of several dissipation potentials F^k is assumed, corresponding to each k -th microstructural rearrangement (due to plastic flow F^p , damage growth F^d , phase change F^{ph} etc.) and defined independently but partly coupled (weak dissipation coupling), (Chaboche, 1997a).

Dissipation functions F^k in general can be expressed in the following nonassociated form:

$$F^k = f^k + g_{iso}^k(R^k) + g_{kin}^k(\underline{\underline{X}}^k) + g_{rot}^k(\underline{\underline{L}}^k) + g_{dist}^k(\underline{\underline{G}}^k), \quad k = p, d, ph, \dots \quad (18)$$

where $g_{iso}^k, g_{kin}^k, g_{rot}^k$ and g_{dist}^k are functions corresponding to respectively isotropic, kinematic, rotational and distortional recovery effects of partial progressive return to the original microstructure (Kuo and Lin, 2007; Mirzakhani et al., 2009). Only the first two terms, related to isotropic and kinematic recovery, are used in the majority of existing models. Usually the recovery functions are defined as quadratic functions of thermodynamic forces conjugated to hardening variables, $R^k, \underline{\underline{X}}^k, \underline{\underline{L}}^k, \underline{\underline{G}}^k$. In (18) f^k stands for loading sur-

face related to k -th dissipative phenomenon. Loading functions f^k , described by relevant thermodynamic forces which are tensors of different order, can be listed in a polynomial hierarchy with increasing complexity and hardening properties, see (Kowalsky et al., 1999).

The kinetic equations for state variables are obtained by the use of the generalized normality rule (Chaboche, 1997a; Egnér, 2010):

$$\dot{\epsilon}_{ij}^E = \sum_{k=1}^n \lambda^k \frac{\partial F^k}{\partial \sigma_{ij}} = \lambda^p \underbrace{\frac{\partial F^p}{\partial \sigma_{ij}}}_{\dot{\epsilon}_{ij}^p} + \lambda^d \underbrace{\frac{\partial F^d}{\partial \sigma_{ij}}}_{\dot{\epsilon}_{ij}^d} + \lambda^{ph} \underbrace{\frac{\partial F^{ph}}{\partial \sigma_{ij}}}_{\dot{\epsilon}_{ij}^{ph}} + \dots \quad (19)$$

$$\dot{A}^k = \sum_{i=1}^n \lambda^i \frac{\partial F^i}{\partial Y^k} = \lambda^p \underbrace{\frac{\partial F^p}{\partial Y^k}}_{\dot{A}^{pk}} + \lambda^d \underbrace{\frac{\partial F^d}{\partial Y^k}}_{\dot{A}^{dk}} + \lambda^{ph} \underbrace{\frac{\partial F^{ph}}{\partial Y^k}}_{\dot{A}^{phk}} + \dots \quad (20)$$

$$-\dot{h}^k = \sum_{i=1}^n \lambda^i \frac{\partial F^i}{\partial H^k} = \lambda^p \underbrace{\frac{\partial F^p}{\partial H^k}}_{\dot{h}^{pk}} + \lambda^d \underbrace{\frac{\partial F^d}{\partial H^k}}_{\dot{h}^{dk}} + \lambda^{ph} \underbrace{\frac{\partial F^{ph}}{\partial H^k}}_{\dot{h}^{phk}} + \dots \quad (21)$$

where λ^i are non-negative consistency multipliers and n is a number of dissipative phenomena, like plastic flow, damage growth, phase change etc., taking place in the material. For rate-independent problems the consistency multipliers may be calculated from the consistency conditions

$$\dot{f}^k = 0, \quad k = 1, 2, \dots, n. \quad (22)$$

Table 1
State variables and corresponding thermodynamic conjugated forces for the elastic–plastic–damage two phase material.

	Variables	Corresponding thermodynamic conjugated forces
Observable	ε_{ij}^E θ	σ_{ij} s
Micro-structural	$A_{ij}^d = D_{ij}$ (damage)	$-Y_{ij}^d$
hardening	$A_{ij}^{ph} = \xi_{ij}$ (phase transformation)	$-Y_{ij}^{ph}$
	r^p	R^p
	σ_{ij}^p	X_{ij}^p
	p_{ijkl}^p	L_{ijkl}^p
	\mathcal{G}_{ijklmn}^p	G_{ijklmn}^p
	r^d	R^d
	σ_{ij}^d	X_{ij}^d
	p_{ijkl}^d	L_{ijkl}^d
	\mathcal{G}_{ijklmn}^d	G_{ijklmn}^d
	r^{ph}	R^{ph}
	σ_{ij}^{ph}	X_{ij}^{ph}
	p_{ijkl}^{ph}	L_{ijkl}^{ph}
$\mathcal{G}_{ijklmn}^{ph}$	G_{ijklmn}^{ph}	

The parameters $\dot{\lambda}^i$ are assumed to obey the classical Kuhn–Tucker loading/unloading conditions:

$$f^k \leq 0 \text{ and } \dot{f}^k \begin{cases} < 0 \text{ and } \dot{\lambda}^k = 0 \Rightarrow \text{passive loading} \\ = 0 \text{ and } \dot{\lambda}^k = 0 \Rightarrow \text{neutral loading} \\ = 0 \text{ and } \dot{\lambda}^k > 0 \Rightarrow \text{active loading} \end{cases} \quad (23)$$

When the classical approach based on the normality rule is used, the rate of a given state variable is derived from one dissipation function, related to dissipative phenomenon represented by this variable. On the other hand, if another approach, based on the postulate of maximum dissipation is applied, coupling between dissipation phenomena is possible to represent in evolution equations, however only associated theories are then described, since the kinetic laws result from side conditions of a minimization Lagrange problem, which are imposed on the loading functions and not dissipation functions. Note that Eqs. (19)–(21) describe both coupling between dissipation phenomena (so that all dissipation functions may appear in each kinetic law) and non-associated rules. Consequently, the inelastic strain rate consists not only of the plastic strain rate, but also of strain rates related to other dissipative phenomena. As well, the rates of microstructural state variables, $\dot{\lambda}^k$ include terms resulting from coupling of k -th dissipation phenomenon with other dissipation phenomena. At the same time the description of nonassociated theories is possible. Therefore, the evolution rules (19)–(21) may be considered as the generalization of classical normality rules and approaches based on the postulate of maximum dissipation.

The comparison between kinetic Eqs. (19)–(21) and approaches presented in Abu Al-Rub and Voyiadjis (2003) (postulate of maximum dissipation) and in Chaboche (1997a) (generalized normality rule) for elastic–plastic–damage material is presented in Table 2.

2.4. Thermo-mechanical coupling

To determine the temperature distribution within the body the heat equation is used, derived from the first law of thermodynamics (1) by substituting into it the internal energy density

$$u = \psi + \theta s \quad (24)$$

together with Fourier's law

$$q_i = -k_{ij}\theta_{,j} \quad (25)$$

where \underline{k} is the thermal conductivity tensor.

The law of energy conservation (1) takes then the following general form (Ottoen and Ristinmaa, 2005):

$$\begin{aligned} \rho c_e^0 \dot{\theta} = & (k_{ij}\theta_{,j})_{,i} + r + \rho \theta \frac{\partial^2 \psi}{\partial \varepsilon_{ij} \partial \theta} (\dot{\varepsilon}_{ij} - \dot{\varepsilon}_{ij}^l) + \rho \frac{\partial \psi}{\partial \varepsilon_{ij}} \dot{\varepsilon}_{ij}^l \\ & + \left(\sigma_{ij} - \rho \frac{\partial \psi}{\partial \varepsilon_{ij}} \right) \dot{\varepsilon}_{ij} - \rho \left(\frac{\partial \psi}{\partial \lambda^k} - \theta \frac{\partial^2 \psi}{\partial \theta \partial \lambda^k} \right) \dot{\lambda}^k \\ & - \rho \left(\frac{\partial \psi}{\partial h^k} - \theta \frac{\partial^2 \psi}{\partial \theta \partial h^k} \right) \dot{h}^k. \end{aligned} \quad (26)$$

In the above equation c_e^0 stands for the specific heat capacity at constant strain. According to Eq. (26) the determination of the temperature distribution within the body is coupled not only to the total strain rate $\dot{\underline{\varepsilon}}$ but also to the inelastic rates, $\dot{\underline{\varepsilon}}^l$, and fluxes $\dot{\lambda}^k$ and \dot{h}^k . Therefore the problem can be solved only if the equations for $\dot{\underline{\varepsilon}}$, $\dot{\underline{\varepsilon}}^l$, $\dot{\lambda}^k$ and \dot{h}^k are solved simultaneously.

3. Example of application: modeling of coupling between thermo-plasticity and thermo-damage

3.1. Temperature effects on material characteristics

The most characteristic feature of the temperature influence on the mechanical properties of conventional engineering materials is that the yield stress changes with temperature. The effect of yield stress drop with temperature is evident for example from the experimental results of Phillips and Tang (1972) shown in Fig. 3a. Generally speaking, degradation of mechanical properties is observed (referred to as thermal softening), accompanied by increasing values of the thermal properties: thermal expansion coefficient α^0 , thermal conductivity k^0 , and specific heat c^0 , cf f.ex. Ottoen and Ristinmaa (2005). Neglecting temperature dependence of the material properties may result in highly erroneous predictions of the material behavior. Fig. 3b presents predicted yield surfaces for several temperatures for $[\pm 45]_s$ laminate of SiC/Ti (Herakovich and Aboudi, 1999). If temperature-dependent properties (TDP) are used, the yield surface contracts with increasing temperature. However, if temperature-independent properties (TIP) are considered, the yield surface expands with increasing temperature and at 300°C it is much larger than at 21°C, which is an erroneous and nonconservative prediction.

Also heating rate makes a significant impact on the material characteristics and phenomena occurring in some engineering materials (Bednarek and Kamocka, 2006). For different heating rates the microstructure of S235JRG2 steel shows significant differences among grain shapes, and reveals empty spaces of various location and size in the vicinity of fracture (see Fig. 4a and b).

Table 2

Comparison between evolution rules proposed in the present paper and in Abu Al-Rub and Voyiadjis (2003) and in Chaboche (1997a).

Present article	Abu Al-Rub and Voyiadjis (2003)	Chaboche (1997a)
(a) $\dot{\epsilon}_{ij}^d = \dot{\lambda}^p \frac{\partial F^p}{\partial \sigma_{ij}} + \dot{\lambda}^d \frac{\partial F^d}{\partial \sigma_{ij}} = \dot{\epsilon}_{ij}^p + \dot{\epsilon}_{ij}^d$ <u>nonassociated and coupled with damage</u> flow rule for inelastic strain	$\dot{\epsilon}_{ij}^d = \dot{\lambda}^p \frac{\partial F^p}{\partial \sigma_{ij}} + \dot{\lambda}^d \frac{\partial F^d}{\partial \sigma_{ij}} = \dot{\epsilon}_{ij}^p + \dot{\epsilon}_{ij}^d$ coupled with damage but associated with plastic and damage loading surfaces; equivalent to (a) if recovery terms in (30) are independent of σ_{ij}	$\dot{\epsilon}_{ij}^d = \dot{\lambda}^p \frac{\partial F^p}{\partial \sigma_{ij}}$ nonassociated, but uncoupled from damage , $\dot{\epsilon}_{ij}^d$ neglected
(b) $\dot{D}_{ij} = \dot{\lambda}^p \frac{\partial F^p}{\partial Y_{ij}^d} + \dot{\lambda}^d \frac{\partial F^d}{\partial Y_{ij}^d} = \dot{D}_{ij}^p + \dot{D}_{ij}^d$ <u>nonassociated and coupled with plasticity</u> flow rule for damage variable	$\dot{D}_{ij} = \dot{\lambda}^p \frac{\partial F^p}{\partial Y_{ij}^d} + \dot{\lambda}^d \frac{\partial F^d}{\partial Y_{ij}^d} = \dot{D}_{ij}^p + \dot{D}_{ij}^d$ coupled with plasticity but associated with plastic and damage loading surfaces; equivalent to (b) if recovery terms in (30) are independent of Y_{ij}^d	$\dot{D}_{ij} = \dot{\lambda}^d \frac{\partial F^d}{\partial Y_{ij}^d}$ nonassociated, but uncoupled from plasticity , \dot{D}_{ij}^p neglected
(c) $\dot{\alpha}_{ij}^p = -\dot{\lambda}^p \frac{\partial F^p}{\partial X_{ij}^p} - \dot{\lambda}^d \frac{\partial F^d}{\partial X_{ij}^p}$ nonassociated and coupled with damage flow rule for plastic kinematic hardening variable	$\dot{\alpha}_{ij}^p = -\dot{\lambda}^p \frac{\partial F^p}{\partial X_{ij}^p}$ nonassociated, but uncoupled from damage ; equivalent to (c) if damage dissipation potential F^d does not depend on plastic hardening variable X_{ij}^p	$\dot{\alpha}_{ij}^p = -\dot{\lambda}^p \frac{\partial F^p}{\partial X_{ij}^p} - \dot{\lambda}^s \frac{\partial F^s}{\partial X_{ij}^p}$ nonassociated , F^s is additional, static microstructural evolution potential, but uncoupled from damage
(d) $\dot{r}^p = -\dot{\lambda}^p \frac{\partial F^p}{\partial R^p} - \dot{\lambda}^d \frac{\partial F^d}{\partial R^p}$ nonassociated and coupled with damage flow rule for plastic isotropic hardening variable	$\dot{r}^p = \dot{p} = -\dot{\lambda}^p \frac{\partial F^p}{\partial R^p}$ nonassociated, but uncoupled from damage ; in a general case inconsistent with (a) (see Eq. (54))	$\dot{r}^p = -\dot{\lambda}^p \frac{\partial F^p}{\partial R^p} - \dot{\lambda}^s \frac{\partial F^s}{\partial R^p}$ nonassociated , F^s is additional, static microstructural evolution potential, but uncoupled from damage
(e) $\dot{\alpha}_{ij}^d = -\dot{\lambda}^p \frac{\partial F^p}{\partial X_{ij}^d} - \dot{\lambda}^d \frac{\partial F^d}{\partial X_{ij}^d}$ nonassociated and coupled with plasticity flow rule for damage kinematic hardening variable	$\dot{\alpha}_{ij}^d = -\dot{\lambda}^d \frac{\partial F^d}{\partial X_{ij}^d}$ nonassociated, but uncoupled from plasticity ; equivalent to (e) if plastic dissipation potential F^p does not depend on damage hardening variable X_{ij}^d	$\dot{\alpha}_{ij}^d = -\dot{\lambda}^d \frac{\partial F^d}{\partial X_{ij}^d}$ nonassociated, but uncoupled from plasticity ; equivalent to (e) if plastic dissipation potential F^p does not depend on damage hardening variable X_{ij}^d
(f) $\dot{r}^d = -\dot{\lambda}^p \frac{\partial F^p}{\partial R^d} - \dot{\lambda}^d \frac{\partial F^d}{\partial R^d}$ nonassociated and coupled with plasticity flow rule for damage isotropic hardening variable	$\dot{r}^d = \dot{r} = -\dot{\lambda}^d \frac{\partial F^d}{\partial R^d}$ nonassociated, but uncoupled from plasticity ; in a general case inconsistent with (a)	$\dot{r}^d = -\dot{\lambda}^d \frac{\partial F^d}{\partial R^d}$ nonassociated, but uncoupled from plasticity

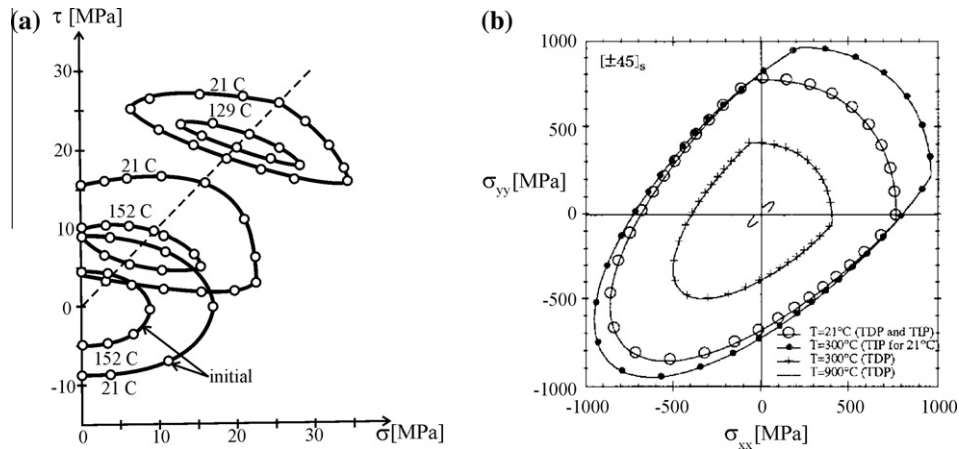


Fig. 3. (a) Effect of temperature on the yield surfaces of pure aluminum subjected to combined tension and torsion (after Phillips and Tang, 1972); (b) Predicted yield surfaces for SiC/Ti (after Herakovich and Aboudi, 1999).

Increasing the heating rate results in decreasing the slip along grain boundaries and leads to the creation of local empty spaces, which decrease the cross-section area and give reasons for more brittle cracking than in the case of a long-time low heating rate. The low heating rate creates a significant grain deformation within the pearlite-and-ferrite areas, accompanied by the ductile damage, while the high heating rate causes small grain deformation accompanied by the brittle damage.

3.2. Damage effect on mechanical and thermal modules

The influence of damage on material characteristics (referred to as damage softening) is much less recognized in the existing literature than the influence of temperature, and most often only the elastic stiffness is considered as affected by damage.

However, it seems justified to accept that not only elastic stiffness but also other mechanical characteristics are affected by damage. The influence of damage on plastic behaviour of metals is well

visible in cyclic loading. Fig. 5a and b present cycle fatigue stress-strain behaviour for AISI 316L stainless steel (Lemaitre, 1992) and aluminium alloy Al-2024 (Abdul-Latif and Chadli, 2007). Detailed analysis of the subsequent strain-stress loops confirms an elastoplastic behaviour of both materials and strong influence of damage. During the initial cycles the materials exhibit plastic hardening leading to the stabilized cycle, then asymmetric drop of both the stress amplitude and the modulus of elasticity reveals following damage growth. This process is accompanied by a gradual decrease of the hysteresis area and a change of shape of subsequent hysteresis loops, associated with a formation of the characteristic inflection point on their lower branches. The mathematical model to describe such cycle fatigue behavior, based on continuous damage deactivation in which microcracks close gradually, was proposed by Ganczarski and Cegielski (2010).

The considerations on the effect of damage on the thermal properties of materials were performed by Skrzypek and Ganczarski (1998) and Ganczarski (1999).

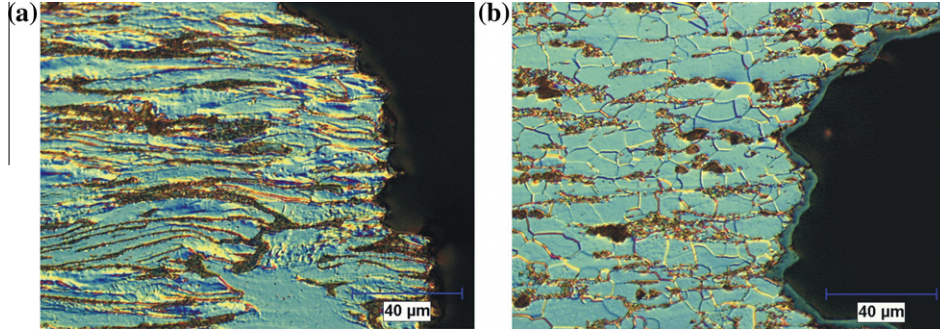


Fig. 4. S235JRG2 steel microstructure after mechanical tests: (a) 5°C/min heating rate; (b) 50°C/min heating rate (after Bednarek and Kamocka, 2006).

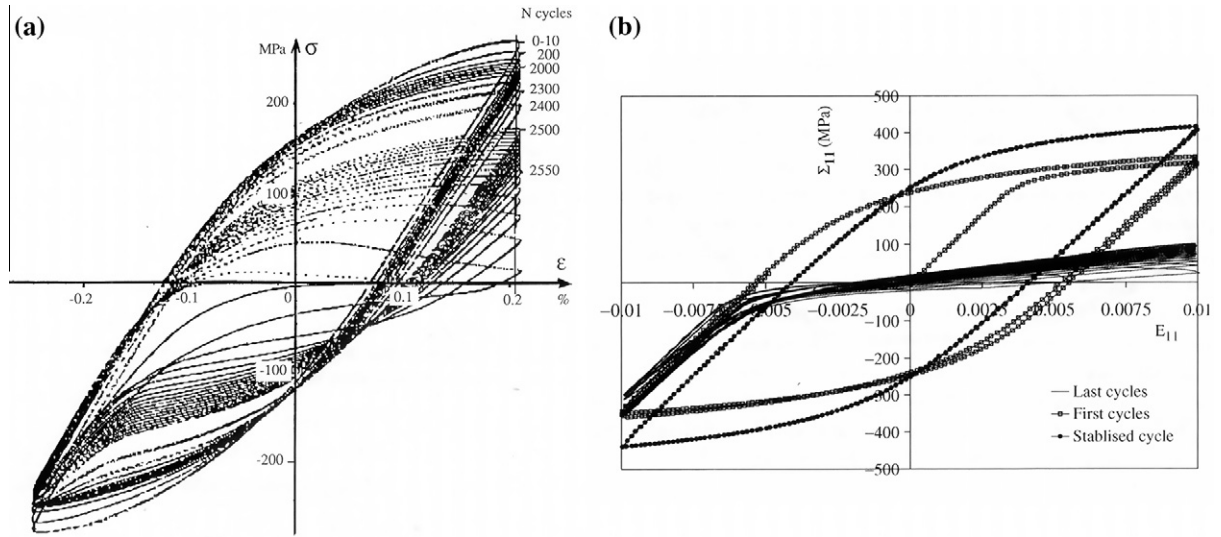


Fig. 5. Cycle fatigue stress–strain behaviour for (a) AISI 316L stainless steel (experimental results by Dufailly in: Lemaitre, 1992); (b) aluminium alloy Al-2024 (test by Abdul-Latif and Chadli, 2007).

It is of interest to obtain knowledge about the variation of all material parameters when micro-damage evolves. So far, the influence of damage on most of material characteristics is usually not accounted for in the models due to the existing gap between the formulated constitutive equations and the possibilities to identify the material parameters. However, fast development of computational possibilities allows to simulate numerically even very complex problems. In addition, with the increased attention paid to many innovative materials of complex microstructure, and a deeper understanding of the meaning of material characteristics, together with the development of advanced experimental techniques which allow for the determination of structural features such as size and volume fractions of microstructural inhomogeneities in a variety of materials, the identification becomes much more well-founded.

3.3. Extended equations of thermo-elastic–plastic–damage materials

In view of above experimental observations it seems justified to extend the common formulations for coupled thermo-elastic–plastic–damage behavior, accounting not only for damage effect on the elastic modules but also on plastic and thermal characteristics. Additionally, the effect of both temperature and damage rates has to be included. The complete set of state variables (12) for the thermo-elastic–plastic–damage material:

$$\{V_{st}\} = \left\{ \underline{\underline{\epsilon}}_{ij}^E; \theta; D_{ij}; r^p; \alpha_{ij}^p, r^d, \alpha_{ij}^d \right\} \quad (27)$$

consists in the present example of elastic strain $\underline{\underline{\epsilon}}_{ij}^E$ and absolute temperature θ ; microstructural damage variable D_{ij} (see Eq. (8)), internal variables: kinematic and isotropic plastic hardening variables α_{ij}^p and r^p , and kinematic and isotropic damage hardening variables α_{ij}^d and r^d . For simplicity, the anisotropic and distortional hardening of plastic and damage loading surfaces is not here considered.

3.3.1. State equations

The state equations result from the assumed form of the state potential, which is here the Helmholtz free energy (13), decomposed into thermo-elastic ($\rho\psi^{te}$), thermo-plastic ($\rho\psi^{tp}$) and thermo-damage ($\rho\psi^{td}$) terms, after (Abu Al-Rub and Voyiadjis, 2003) and Chaboche (1989):

$$\rho\psi(V_{st}) = \rho\psi^{te}(\underline{\underline{\epsilon}}^E; \theta; \underline{\underline{D}}) + \rho\psi^{tp}(\theta; \underline{\underline{D}}; r^p, \underline{\underline{\alpha}}^p) + \rho\psi^{td}(\theta; \underline{\underline{D}}; r^d, \underline{\underline{\alpha}}^d) \quad (28)$$

The following functions for $\rho\psi^{tk}$ are here assumed:

$$\begin{aligned} \rho\psi^{te} &= \rho h(\theta) + \frac{1}{2} \underline{\underline{\epsilon}}_{ij}^E E_{ijkl}(\theta, \underline{\underline{D}}) \underline{\underline{\epsilon}}_{kl}^E - \beta_{ij}(\theta, \underline{\underline{D}}) \underline{\underline{\epsilon}}_{ij}^E (\theta - \theta_0), \\ \beta_{ij}(\theta, \underline{\underline{D}}) &= E_{ijkl}(\theta, \underline{\underline{D}}) \alpha_{kl}^p(\theta, \underline{\underline{D}}) \end{aligned} \quad (29)$$

$$\rho\psi^{tp} = \frac{1}{3} C^p(\theta, \underline{\underline{D}}) \alpha_{ij}^p \alpha_{ij}^p + R_\infty^p(\theta, \underline{\underline{D}}) \left[r^p + \frac{e^{-b^p(\theta) r^p}}{b^p(\theta, \underline{\underline{D}})} \right] \quad (30)$$

$$\rho\psi^{td} = \frac{1}{2} C^d(\theta, \underline{\underline{D}}) \alpha_{ij}^d \alpha_{ij}^d + R_\infty^d(\theta, \underline{\underline{D}}) \left[r^d + \frac{e^{-b^d(\theta) r^d}}{b^d(\theta, \underline{\underline{D}})} \right] \quad (31)$$

In Eqs. (29)–(31) $h(\theta)$ is the function of temperature, $\underline{\alpha}^{\theta}(\theta, \underline{\mathbf{D}})$ is the thermal expansion tensor; $\underline{\mathbf{E}}(\theta, \underline{\mathbf{D}})$ is the elastic stiffness tensor; $C^p(\theta, \underline{\mathbf{D}})$, $C^d(\theta, \underline{\mathbf{D}})$, $R_{\infty}^p(\theta, \underline{\mathbf{D}})$, $R_{\infty}^d(\theta, \underline{\mathbf{D}})$, $b^p(\theta, \underline{\mathbf{D}})$, $b^d(\theta, \underline{\mathbf{D}})$, are material parameters, which in general may be temperature and damage dependent. Symbol θ_0 stands for the reference temperature at which no thermal strains exists.

The expression for elastic term of the Helmholtz free energy (29) was written in the simplified form which does not account for the unilateral damage effect (cf. Krajcinovic, 1996; Bielski et al., 2006). In the present example, which is focused on coupling between thermo-plasticity and thermo-damage, this effect is neglected for simplicity.

State equations can be written as follows (see Eqs. (14)–(17)):

$$\sigma_{ij} = \frac{\partial(\rho\psi)}{\partial \varepsilon_{ij}^E} = E_{ijkl} \varepsilon_{kl}^E - \beta_{ij}(\theta - \theta_0) \quad (32)$$

$$X_{ij}^p = \frac{\partial(\rho\psi)}{\partial \alpha_{ij}^p} = \frac{2}{3} C^p \alpha_{ij}^p \quad (33)$$

$$R^p = \frac{\partial(\rho\psi)}{\partial r^p} = R_{\infty}^p (1 - e^{-b^p r^p}) \quad (34)$$

$$-Y_{ij}^d = \frac{\partial(\rho\psi)}{\partial D_{ij}} = -(Y_{ij}^{ed} + Y_{ij}^{pd} + Y_{ij}^{dd}) \quad (35)$$

$$X_{ij}^d = \frac{\partial(\rho\psi)}{\partial \alpha_{ij}^d} = C^d \alpha_{ij}^d \quad (36)$$

$$R^d = \frac{\partial(\rho\psi)}{\partial r^d} = R_{\infty}^d (1 - e^{-b^d r^d}) \quad (37)$$

Note that the damage driving force Y_{ij}^d in the presence of coupling between thermo-elasticity, thermo-plasticity and thermo-damage in the state potential (28), consists of three terms:

$$Y_{ij}^{ed} = -\frac{\partial[\rho\psi^{te}(\theta, \underline{\mathbf{D}})]}{\partial D_{ij}} = Y_{ij}^{ed}(\underline{\alpha}; \theta, \underline{\mathbf{D}}), \quad (38)$$

$$Y_{ij}^{pd} = -\frac{\partial[\rho\psi^{tp}(\theta, \underline{\mathbf{D}})]}{\partial D_{ij}} = Y_{ij}^{pd}(\underline{\mathbf{X}}^p, R^p; \theta, \underline{\mathbf{D}}), \quad (39)$$

$$Y_{ij}^{dd} = -\frac{\partial[\rho\psi^{td}(\theta, \underline{\mathbf{D}})]}{\partial D_{ij}} = Y_{ij}^{dd}(\underline{\mathbf{X}}^d, R^d; \theta, \underline{\mathbf{D}}), \quad (40)$$

which stand here for the elastic, plastic and damage strain energy release rates, respectively, as the extension of commonly used definitions of the elastic strain energy release rate only (cf. Lemaitre, 1992).

3.3.2. Evolution equations

Potentials of dissipation, plastic (F^p) and damage (F^d) are here assumed not equal to plastic yield surface (non-associated thermo-plasticity) and damage surface (non-associated thermo-damage), respectively. This allows obtaining non-linear plastic and damage hardening rules, which give more realistic description of the material response:

$$F^p = f^p + \frac{3}{4} \frac{\gamma^p(\theta, \underline{\mathbf{D}})}{C^p(\theta, \underline{\mathbf{D}})} \tilde{X}_{ij}^p \tilde{X}_{ij}^p + \frac{1}{2} \eta^p(\theta, \underline{\mathbf{D}}) (\tilde{R}^p)^2 \quad (41)$$

$$F^d = f^d + \frac{1}{2} \frac{\gamma^d(\theta, \underline{\mathbf{D}})}{C^d(\theta, \underline{\mathbf{D}})} X_{ij}^d X_{ij}^d + \frac{1}{2} \eta^d(\theta, \underline{\mathbf{D}}) (R^d)^2 \quad (42)$$

where $\gamma^p(\theta, \underline{\mathbf{D}})$, $\gamma^d(\theta, \underline{\mathbf{D}})$, $\eta^p(\theta, \underline{\mathbf{D}})$ and $\eta^d(\theta, \underline{\mathbf{D}})$ are material parameters. $f^p = 0$ is the von Mises-type plastic yield surface, and $f^d = 0$ is the damage surface:

$$f^p = \sqrt{\frac{3}{2}} (\tilde{s}_{ij} - \tilde{X}_{ij}^p) (\tilde{s}_{ij} - \tilde{X}_{ij}^p) - (R_0^p + \tilde{R}^p) = 0 \quad (43)$$

$$f^d = \sqrt{(Y_{ij}^d - X_{ij}^d) (Y_{ij}^d - X_{ij}^d)} - (R_0^d + R^d) = 0 \quad (44)$$

Since plasticity can only affect the undamaged material skeleton, it seems justified to define the plastic potential in terms of the effective variables:

$$\begin{aligned} \tilde{X}_{ij}^p &= \left(M_{ijkl} - \frac{1}{3} M_{rkrk} \delta_{ij} \right) X_{kl}^p, \\ \tilde{s}_{ij} &= \left(M_{ijkl} - \frac{1}{3} M_{rkrk} \delta_{ij} \right) \sigma_{kl}, \end{aligned} \quad (45)$$

$$\tilde{R}^p = \frac{R^p}{1 - D_{eq}}, \quad D_{eq} = \sqrt{D_{ij} D_{ij}}$$

In the special case when $\eta^p(\theta, \underline{\mathbf{D}}) = 0$ and $\eta^d(\theta, \underline{\mathbf{D}}) = 0$ the dissipation functions (41) and (42) are equivalent to functions proposed in Abu Al-Rub and Voyiadjis (2003).

A concept of the fourth-rank damage effect tensor $\underline{\mathbf{M}}(\underline{\mathbf{D}})$ is introduced in (45) that transforms the thermodynamic forces σ_{ij} and X_{ij}^p in the physical space of RVE to the effective forces \tilde{s}_{ij} and \tilde{X}_{ij}^p in the fictitious pseudo-undamaged space of quasi-continuum, basing on the adopted equivalence hypothesis between physical and fictitious spaces. Many different expressions for $\underline{\mathbf{M}}(\underline{\mathbf{D}})$ exist in literature. A comprehensive review of the most common formulations and equivalence hypotheses can be found for example in Skrzypek and Ganczarski (1999).

The rates of state variables are obtained by the use of the generalized normality rule, Eqs. (19)–(21):

$$\dot{\varepsilon}_{ij}^d = \dot{\lambda}^p \frac{\partial F^p}{\partial \sigma_{ij}} + \dot{\lambda}^d \frac{\partial F^d}{\partial \sigma_{ij}} = \dot{\varepsilon}_{ij}^p + \dot{\varepsilon}_{ij}^d \quad (46)$$

$$\dot{D}_{ij} = \dot{\lambda}^p \frac{\partial F^p}{\partial Y_{ij}^d} + \dot{\lambda}^d \frac{\partial F^d}{\partial Y_{ij}^d} = \dot{D}_{ij}^p + \dot{D}_{ij}^d \quad (47)$$

$$\dot{\alpha}_{ij}^p = -\dot{\lambda}^p \frac{\partial F^p}{\partial X_{ij}^p} - \dot{\lambda}^d \frac{\partial F^d}{\partial X_{ij}^p} = \dot{\alpha}_{ij}^{pp} + \dot{\alpha}_{ij}^{dp} \quad (48)$$

$$\dot{r}^p = -\dot{\lambda}^p \frac{\partial F^p}{\partial R^p} - \dot{\lambda}^d \frac{\partial F^d}{\partial R^p} = \dot{r}^{pp} + \dot{r}^{dp} \quad (49)$$

$$\dot{\alpha}_{ij}^d = -\dot{\lambda}^p \frac{\partial F^p}{\partial X_{ij}^d} - \dot{\lambda}^d \frac{\partial F^d}{\partial X_{ij}^d} = \dot{\alpha}_{ij}^{pd} + \dot{\alpha}_{ij}^{dd} \quad (50)$$

$$\dot{r}^d = -\dot{\lambda}^p \frac{\partial F^p}{\partial R^d} - \dot{\lambda}^d \frac{\partial F^d}{\partial R^d} = \dot{r}^{pd} + \dot{r}^{dd} \quad (51)$$

where $\dot{\lambda}^p$ and $\dot{\lambda}^d$ are non-negative consistency multipliers.

Note that isotropic plastic hardening variable r^{pp} is in general not equal to classical accumulated plastic strain defined as:

$$p = \int_0^t \sqrt{\frac{2}{3} \dot{\varepsilon}_{ij}^p \dot{\varepsilon}_{ij}^p} dt \quad (52)$$

In the case of non-associated plasticity and/or coupling with damage in Eq. (41) the definition of plastic hardening variable r^{pp} as being equal to accumulated plastic strain (52) leads to inconsistency between Eqs. (46) and (49). According to Eqs. (46) and (52) we can write

$$\dot{\lambda}^p = \frac{\sqrt{\frac{2}{3} \dot{\varepsilon}_{ij}^p \dot{\varepsilon}_{ij}^p}}{\sqrt{\frac{2}{3} \frac{\partial F^p}{\partial \sigma_{ij}} \frac{\partial F^p}{\partial \sigma_{ij}}}} = \frac{\dot{p}}{\sqrt{\frac{2}{3} \frac{\partial F^p}{\partial \sigma_{ij}} \frac{\partial F^p}{\partial \sigma_{ij}}}} = \frac{\dot{p}}{\sqrt{M_{ijkl} M_{ijkl}}} \quad (53)$$

Substitution of Eq. (53) into Eq. (49) leads to the following relation between \dot{r}^{pp} and \dot{p} :

$$\dot{r}^{pp} = \dot{p} \frac{1 - D_{eq} - \eta^p R^p}{(1 - D_{eq})^2 \sqrt{M_{ijkl} M_{ijkl}}} \quad (54)$$

It can be seen from (54) that generalized accumulated plastic strain r^{pp} reduces to classical parameter p only in the special case of associated plasticity ($\eta^p = 0$) without damage ($D_{eq} = 0, M_{ijkl} = I_{ijkl}$). Also

r^{dd} is in general not equal to accumulated damage strain $q = \int_0^t \sqrt{\dot{\epsilon}_{ij}^d \dot{\epsilon}_{ij}^d} dt$ for similar reasons.

The evolution equations for thermodynamic conjugate forces are derived taking the time rate of state Eqs. (32)–(37), see Table 3.

By taking into account the temperature and damage dependence of material characteristics the additional terms appear in rate equations of thermodynamic forces (see Table 3), which may play a significant role when solving high temperature and/or damage rate problems, such as fire conditions or thermal shock problems.

3.3.3. Consistency conditions

Since the thermodynamic conjugate forces are functions of temperature and damage, the consistency relations (22) for development of plasticity and damage take the following forms:

$$\dot{f}^p = \frac{\partial f^p}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f^p}{\partial X_{ij}^p} \dot{X}_{ij}^p + \frac{\partial f^p}{\partial R^p} \dot{R}^p + \frac{\partial f^p}{\partial D_{ij}} \dot{D}_{ij} + \frac{\partial f^p}{\partial \theta} \dot{\theta} = 0 \quad (55)$$

$$\dot{f}^d = \frac{\partial f^d}{\partial Y_{ij}^d} \dot{Y}_{ij}^d + \frac{\partial f^d}{\partial X_{ij}^d} \dot{X}_{ij}^d + \frac{\partial f^d}{\partial R^d} \dot{R}^d + \frac{\partial f^d}{\partial D_{ij}} \dot{D}_{ij} + \frac{\partial f^d}{\partial \theta} \dot{\theta} = 0 \quad (56)$$

However, as a consequence of coupling between thermo-plasticity and thermo-damage the damage driving force Y^d is a function of all other thermodynamic forces. Therefore, the consistency condition for damage (56) has an extended form:

$$\begin{aligned} \dot{f}^d = & \frac{\partial f^d}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f^d}{\partial X_{ij}^d} \dot{X}_{ij}^d + \frac{\partial f^d}{\partial R^d} \dot{R}^d + \frac{\partial f^d}{\partial X_{ij}^p} \dot{X}_{ij}^p + \frac{\partial f^d}{\partial R^p} \dot{R}^p + \frac{\partial f^d}{\partial D_{ij}} \dot{D}_{ij} \\ & + \frac{\partial f^d}{\partial \theta} \dot{\theta} = 0 \end{aligned} \quad (57)$$

Using the chain rule and equations from Table 3 the consistency conditions may be transformed to the following form:

$$\dot{f}^p = \frac{\partial f^p}{\partial \sigma_{ij}} E_{ijkl} \dot{\epsilon}_{kl} - \dot{\lambda}^p h_{11} - \dot{\lambda}^d h_{12} + S^p \dot{\theta} = 0 \quad (58)$$

$$\dot{f}^d = \frac{\partial f^d}{\partial \sigma_{ij}} E_{ijkl} \dot{\epsilon}_{kl} - \dot{\lambda}^p h_{21} - \dot{\lambda}^d h_{22} + S^d \dot{\theta} = 0 \quad (59)$$

The quantities h_{11} , h_{12} , h_{21} and h_{22} are the generalized hardening moduli (cf. Ottosen and Ristinmaa, 2005) which are shown in details in Egner (2011)). The following cases are (mathematically) possible to describe: plastic-hardening thermo-plasticity ($h_{11} > 0$), plastic-ideal thermo-plasticity ($h_{11} = 0$), plastic-softening thermo-plasticity ($h_{11} < 0$), damage-hardening thermo-plasticity ($h_{12} > 0$), damage-ideal thermo-plasticity ($h_{12} = 0$), damage-softening thermo-plasticity ($h_{12} < 0$), plastic-hardening thermo-damage ($h_{21} > 0$), plastic-ideal thermo-damage ($h_{21} = 0$), plastic-softening thermo-damage ($h_{21} < 0$), damage-hardening thermo-damage ($h_{22} > 0$), damage-ideal thermo-damage ($h_{22} = 0$), damage-softening thermo-damage ($h_{22} < 0$).

Table 3

Evolution equations for thermodynamic conjugate forces with additional terms resulted from coupling with damage and temperature.

	No coupling	Coupling with damage	Coupling with temperature
$\dot{\sigma}_{ij} =$	$E_{ijkl} \dot{\epsilon}_{kl}^E$	$+ \left[\frac{\partial E_{ijkl}}{\partial D_{kl}} \epsilon_{pq}^E - \frac{\partial \beta_{ij}}{\partial D_{kl}} (\theta - \theta_0) \right] \dot{D}_{kl}$	$+ \left[\frac{\partial E_{ijkl}}{\partial \theta} \epsilon_{kl}^E - \frac{\partial \beta_{ij}}{\partial \theta} (\theta - \theta_0) + \beta_{ij} \right] \dot{\theta}$
$\dot{X}_{ij}^p =$	$\frac{2}{3} C^p \dot{\epsilon}_{ij}^p$	$+ \frac{X_{ij}^p}{C^p} \frac{\partial C^p}{\partial D_{kl}} \dot{D}_{kl}$	$+ \frac{X_{ij}^p}{C^p} \frac{\partial C^p}{\partial \theta} \dot{\theta}$
$\dot{R}^p =$	$R_{\infty}^p b^p e^{-b^p r^p} \dot{r}^p$	$+ \left[\frac{R_{\infty}^p}{R_{\infty}^p} \frac{\partial R_{\infty}^p}{\partial D_{ij}} + \frac{(R_{\infty}^p - R^p)}{b^p} \frac{\partial b^p}{\partial D_{ij}} \ln \left(\frac{R_{\infty}^p}{R_{\infty}^p - R^p} \right) \right] \dot{D}_{ij}$	$+ \left[\frac{R_{\infty}^p}{R_{\infty}^p} \frac{\partial R_{\infty}^p}{\partial \theta} + \frac{(R_{\infty}^p - R^p)}{b^p} \frac{\partial b^p}{\partial \theta} \ln \left(\frac{R_{\infty}^p}{R_{\infty}^p - R^p} \right) \right] \dot{\theta}$
$\dot{X}_{ij}^d =$	$C^d \dot{\epsilon}_{ij}^d$	$+ \frac{X_{ij}^d}{C^d} \frac{\partial C^d}{\partial D_{kl}} \dot{D}_{kl}$	$+ \frac{X_{ij}^d}{C^d} \frac{\partial C^d}{\partial \theta} \dot{\theta}$
$\dot{R}^d =$	$R_{\infty}^d b^d e^{-b^d r^d} \dot{r}^d$	$+ \left[\frac{R_{\infty}^d}{R_{\infty}^d} \frac{\partial R_{\infty}^d}{\partial D_{ij}} + \frac{(R_{\infty}^d - R^d)}{b^d} \frac{\partial b^d}{\partial D_{ij}} \ln \left(\frac{R_{\infty}^d}{R_{\infty}^d - R^d} \right) \right] \dot{D}_{ij}$	$+ \left[\frac{R_{\infty}^d}{R_{\infty}^d} \frac{\partial R_{\infty}^d}{\partial \theta} + \frac{(R_{\infty}^d - R^d)}{b^d} \frac{\partial b^d}{\partial \theta} \ln \left(\frac{R_{\infty}^d}{R_{\infty}^d - R^d} \right) \right] \dot{\theta}$

The temperature sensitivity parameters S^p and S^d (see Egner, 2011) express how the yield and damage surfaces change with temperature:

$$S^p \begin{cases} > 0 \Rightarrow \text{yield surface contracts} \\ = 0 \Rightarrow \text{yield surface remains constant} \\ < 0 \Rightarrow \text{yield surface expands,} \end{cases} \quad (60)$$

$$S^d \begin{cases} > 0 \Rightarrow \text{damage surface contracts} \\ = 0 \Rightarrow \text{damage surface remains constant} \\ < 0 \Rightarrow \text{damage surface expands.} \end{cases} \quad (61)$$

With reference to the majority of experiments the physical meaning seems to have the case when the yield surface contracts with increasing temperature and damage.

3.3.4. Loading/unloading criteria

To obtain the general loading/unloading criteria let us first observe that for $f^p < 0$ and $f^d < 0$ a thermo-elastic response occurs. Thermo-plasticity requires $f^p = 0$ and $\dot{\lambda}^p \geq 0$, while thermo-damage demands $f^d = 0$ and $\dot{\lambda}^d \geq 0$. Taking into account (58) and (59) we have:

$$\dot{\lambda}^p = \frac{1}{w} \left[\left(h_{22} \frac{\partial f^p}{\partial \sigma_{ij}} - h_{12} \frac{\partial f^d}{\partial \sigma_{ij}} \right) E_{ijkl} \dot{\epsilon}_{kl} + (h_{22} S^p - h_{12} S^d) \dot{\theta} \right] = \frac{1}{w} \left(A_{ij}^p E_{ijkl} \dot{\epsilon}_{kl} + \bar{S}^p \dot{\theta} \right), \quad (62)$$

$$\dot{\lambda}^d = \frac{1}{w} \left[\left(-h_{21} \frac{\partial f^p}{\partial \sigma_{ij}} + h_{11} \frac{\partial f^d}{\partial \sigma_{ij}} \right) E_{ijkl} \dot{\epsilon}_{kl} + (h_{11} S^d - h_{21} S^p) \dot{\theta} \right] = \frac{1}{w} \left(A_{ij}^d E_{ijkl} \dot{\epsilon}_{kl} + \bar{S}^d \dot{\theta} \right), \quad (63)$$

where:

$$w = h_{11} h_{22} - h_{21} h_{12} > 0, \quad (64)$$

Therefore, for coupled thermo-plasticity and thermo-damage we arrive at the following loading/unloading criteria (23):

$$f^p = 0 \text{ and } \begin{cases} A_{ij}^p E_{ijkl} \dot{\epsilon}_{kl} + \bar{S}^p \dot{\theta} > 0 \Rightarrow \text{thermo-plastic loading} \\ A_{ij}^p E_{ijkl} \dot{\epsilon}_{kl} + \bar{S}^p \dot{\theta} = 0 \Rightarrow \text{neutral loading} \\ A_{ij}^p E_{ijkl} \dot{\epsilon}_{kl} + \bar{S}^p \dot{\theta} < 0 \Rightarrow \text{elastic unloading} \end{cases} \quad (65)$$

$$f^d = 0 \text{ and } \begin{cases} A_{ij}^d E_{ijkl} \dot{\epsilon}_{kl} + \bar{S}^d \dot{\theta} > 0 \Rightarrow \text{thermo-damage growth} \\ A_{ij}^d E_{ijkl} \dot{\epsilon}_{kl} + \bar{S}^d \dot{\theta} = 0 \Rightarrow \text{thermo-damage initiation} \\ A_{ij}^d E_{ijkl} \dot{\epsilon}_{kl} + \bar{S}^d \dot{\theta} < 0 \Rightarrow \text{elastic unloading.} \end{cases} \quad (66)$$

It should be noticed here that conditions (65), (66), in the presence of the full thermo-plasticity and thermo-damage coupling, are significantly different from corresponding conditions for uncoupled isothermal case (see Fig. 6). Namely, if we consider purely thermo-elastic behaviour, where $\dot{\lambda}^p = 0$ and $\dot{\lambda}^d = 0$, we obtain the thermo-elastic stress rate which results for a given total strain rate $\dot{\epsilon}_{kl}$

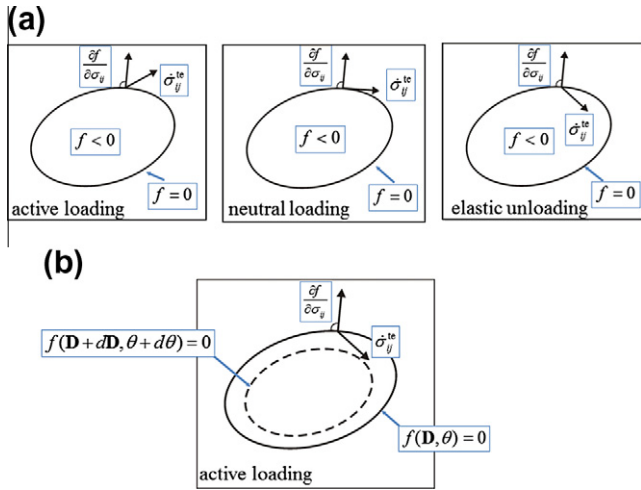


Fig. 6. Illustration of loading/unloading conditions for (a) uncoupled isothermal process, (b) coupled thermo/plastic/damage process: loading surface contracting with increasing temperature and damage.

and a given temperature rate $\dot{\theta}$ provided that the material responds thermo-elastically (Ottosen and Ristinmaa, 2005), see Table 3:

$$\dot{\sigma}_{ij}^{te} = E_{ijkl} \dot{\epsilon}_{kl} + \left[\frac{\partial E_{ijkl}}{\partial \theta} \epsilon_{kl}^E - \frac{\partial \beta_{ij}}{\partial \theta} (\theta - \theta_0) + \beta_{ij} \right] \dot{\theta} \quad (67)$$

Now if the considered loading surface contracts with increasing temperature and/or damage, then even if $(\partial f^p / \partial \sigma_{ij}) \dot{\sigma}_{ij}^{te} < 0$ the expression (62) may still have positive value and active plastic loading then occurs (see Fig. 6b), and even if $(\partial f^d / \partial \sigma_{ij}) \dot{\sigma}_{ij}^{te} < 0$ the expression (63) may still have positive value and active damage loading then occurs.

3.3.5. Heat balance equation

In the case of thermo-elastic-plastic-damage material, for which the number of state variables (12) is reduced to (27), the general coupled heat Eq. (26) takes the following form:

$$\begin{aligned} \rho c_e^d \dot{\theta} = & (k_{ij} \theta_{,j})_{,i} + r + \rho \theta \frac{\partial^2 \psi}{\partial \epsilon_{ij} \partial \theta} (\dot{\epsilon}_{ij} - \dot{\epsilon}_{ij}^l) + \rho \frac{\partial \psi}{\partial \epsilon_{ij}} \dot{\epsilon}_{ij}^l \\ & + \left(\sigma_{ij} - \rho \frac{\partial \psi}{\partial \epsilon_{ij}} \right) \dot{\epsilon}_{ij} - \rho \left(\frac{\partial \psi}{\partial D_{ij}} - \theta \frac{\partial^2 \psi}{\partial \theta \partial D_{ij}} \right) \dot{D}_{ij} \\ & - \rho \left(\frac{\partial \psi}{\partial r^p} - \theta \frac{\partial^2 \psi}{\partial \theta \partial r^p} \right) \dot{r}^p - \rho \left(\frac{\partial \psi}{\partial \alpha_{ij}^p} - \theta \frac{\partial^2 \psi}{\partial \theta \partial \alpha_{ij}^p} \right) \dot{\alpha}_{ij}^p \\ & - \rho \left(\frac{\partial \psi}{\partial r^d} - \theta \frac{\partial^2 \psi}{\partial \theta \partial r^d} \right) \dot{r}^d - \rho \left(\frac{\partial \psi}{\partial \alpha_{ij}^d} - \theta \frac{\partial^2 \psi}{\partial \theta \partial \alpha_{ij}^d} \right) \dot{\alpha}_{ij}^d \end{aligned} \quad (68)$$

which is nonlinear and fully coupled to mechanical problem.

4. Conclusions

In the presented article a general thermodynamic framework for the formulation of a coupled constitutive model for dissipative materials in the small strain range was presented:

- Two additional hardening variables were included into the general set of state variables to account for the effects of rotation and distortion of the specific loading surface.
- Also, the recovery effects related to rotational and distortional hardening were consistently indicated in the general formulation of dissipation potentials.
- A new consistent normality rule for deriving the kinetic equations was proposed, as a generalization of the classical normal-

ity rule (Chaboche, 1997a) and the maximum dissipation principle approach (Abu Al-Rub and Voyiadjis, 2003).

- It was also shown that the classical accumulated plastic strain and accumulated damage strain are not the properly defined hardening state variables in a general case of non-associated coupled elastic-plastic-damage formulations.

A systematic construction of a special case of thermo-elastic-plastic-damage constitutive model derived from the general formulation, and destined for solving high temperature and damage rate problems was shown in detail.

- A special attention was paid to complete and consistent incorporation of temperature and damage softening into the kinetic equations, which results in additional, temperature and damage rate dependent terms, most often neglected in the existing models. However, for high temperature and/or damage rates these terms may play a significant role.
- The ability of the proposed model to describe different cases of plastic/damage hardening/softening combinations was presented by introducing the generalized hardening moduli.
- It was indicated, that in the case of plastic and/or damage softening the classical loading/unloading conditions have to be extended with additional terms accounting for thermal-plastic-damage coupling, otherwise the recognition of active/neutral/passive processes may be false.

The application of the presented considerations to a general dissipative material requires better experimental recognition of the influence of different dissipative phenomena (like damage, phase transformations etc) on the material characteristics. If a simple case of thermo-plasticity is considered, the experimental identification of material parameters in a wide range of temperatures may be found for example in Velay et al. (2006), while in a constitutive model presented there the temperature terms are neglected. For such cases the analysis presented in the paper also forms a good basis for the extension to general nonisothermal applications.

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