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Inverse M-matrices, II

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ABSTRACT

This is an update of the 1981 survey by the first author. In the meantime, a considerable amount has been learned about the very special structure of the important class of inverse M-matrices. Developments since the earlier survey are emphasized, but we have tried to be somewhat complete; and, some results have not previously been published. Some proofs are given where appropriate and references are given for others. After some elementary preliminaries, results are grouped by certain natural categories.

0. Introduction

By an M-matrix, we mean an n-by-n matrix A with nonpositive off-diagonal entries that has an entry-wise nonnegative inverse. This is equivalent to A being of the form $\alpha I - B$, in which B is entry-wise nonnegative and $\alpha > \rho(B)$, the spectral radius of B (or A having nonpositive off-diagonal entries and being positive stable, i.e., each eigenvalue has positive real part). A nonnegative matrix that occurs as the inverse of an M-matrix is called an *inverse M-matrix*. We denote the n-by-n entry-wise nonnegative matrices by $\mathcal{N}$, the n-by-n positive (nonnegative) diagonal matrices by $\mathcal{D}$ ($\overline{\mathcal{D}}$), the n-by-n matrices with

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nonpositive off-diagonal entries by $\mathcal{Z}$, the M-matrices by $\mathcal{M}$, and the inverse M-matrices by $\mathcal{IM}$. Much is known about each of these important classes [1,22,30,34]. It was shown in [22] that $A \in \mathcal{M}$ if and only if $A \in \mathcal{Z}$ and A has positive principal minors. From this it follows that if $A \in \mathcal{IM}$, then $\det A > 0$ and A has positive diagonal entries. Many equivalent conditions for a Z-matrix to be an M-matrix may be found in the above references.

For any m-by-n matrix A, $\alpha \subseteq M = \{1, \ldots, m\}$ and $\beta \subseteq N = \{1, \ldots, n\}$, we denote the submatrix lying in rows α and columns β by $A[\alpha, \beta]$. If $m = n$, the principal submatrix $A[\alpha, \alpha]$ is abbreviated $A[\alpha]$. Similarly, $A(\alpha, \beta)$ denotes the submatrix obtained from A by deleting the rows indexed by α and the columns indexed by β and if $m = n$, the submatrix $A(\alpha, \alpha)$ is abbreviated $A(\alpha)$. We denote the cardinality of α, the complement of α, and the relative complement of α in β by $|\alpha|$, α^c, and $\beta - \alpha$, respectively. For $i \in N$, we abbreviate $\alpha - \{i\}$ by $\alpha - i$, $\alpha \cup \{i\}$ by $\alpha + i$, $A(\{i\}, \{j\})$ by $A(i, j)$, and $A[\{i\}, \{j\}]$ by $A[i, j]$ or a_{ij}. Inequalities between matrices are entry-wise throughout.

1. Preliminary facts

A number of facts follow from the definition of $\mathcal{IM}$-matrices and are presented without proof.

Theorem 1.1. *If $A \in \mathcal{N}$ is invertible, then $A \in \mathcal{IM}$ if and only if $A^{-1} \in \mathcal{Z}$.*

Corollary 1.1.1. *If $A, B \in \mathcal{IM}$, then $AB \in \mathcal{IM}$ if and only if $(AB)^{-1} \in \mathcal{Z}$.*

Example 1.1.2. Consider the $\mathcal{IM}$ matrices

$$\mathbf{A} = \begin{bmatrix} 21 & 14 & 13 \\ 13 & 24 & 20 \\ 14 & 13 & 24 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 24 & 20 & 13 \\ 13 & 24 & 14 \\ 14 & 13 & 24 \end{bmatrix}.$$

AB is not $\mathcal{IM}$ since the $(1, 3)$ entry of $(AB)^{-1}$ is positive.

Remark. Multiplicative closure can be shown to hold for $n = 2$ (since A, B, and AB have positive determinant).

Corollary 1.1.3. *If $A, B \in \mathcal{IM}$, then $A + B \in \mathcal{IM}$ if and only if $A + B$ is invertible and $(A + B)^{-1} \in \mathcal{Z}$.*

Additive closure does not even hold for $\mathcal{IM}$ matrices of order 2.

Example 1.1.4. Consider the $\mathcal{IM}$ matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}.$$

If $B = A^T$ (the transpose of A), then $A + B$ is not even invertible.

However, if $\det(A + B) > 0$, then $A + B$ is $\mathcal{IM}$ in the 2-by-2 case.

From theorem 1.1 and the cofactor form of the inverse we have

Theorem 1.2. *If $A \in \mathcal{N}$, then $A \in \mathcal{IM}$ if and only if $\det A > 0$ and either $\det A(i, j) = 0$ or sgn $\det A(i, j) = (-1)^{i+j+1}$ for $1 \leq i, \ j \leq n, i \neq j$.*

Theorem 1.2.1. *If P is a permutation matrix, then $A \in \mathcal{IM}$ if and only if $P^T AP \in \mathcal{IM}$.*

Theorem 1.2.2. *$A \in \mathcal{IM}$ if and only if $A^T \in \mathcal{IM}$.*

Theorem 1.2.3. *If $D, E \in \mathcal{D}$, then $A \in \mathcal{IM}$ if and only if $DAE \in \mathcal{IM}$.*

An n-by-n complex matrix $A = (a_{ij})$ is said to be *diagonally dominant of its rows* if

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|, \quad i = 1, \ldots, n,$$

and *diagonally dominant of its columns* if A^T is diagonally dominant of its rows. A is said to be *diagonally dominant of its row entries* if

$$|a_{ii}| > |a_{ij}|, \quad j \neq i,\ i = 1, \ldots, n,$$

and *diagonally dominant of its column entries* if A^T is diagonally dominant of its row entries.

Theorem 1.3. *If $A \in \mathcal{IM}$, then*

(i) *there is a $D \in \mathcal{D}$ such that DA is diagonally dominant of its column entries,*
(ii) *there is an $E \in \mathcal{D}$ such that AE is diagonally dominant of its row entries, and*
(iii) *there are $G, H \in \mathcal{D}$ such that GAH is diagonally dominant of both its row and column entries.*

Proof. Let $A \in \mathcal{IM}$ so that $B = A^{-1} \in \mathcal{M}$. It follows from the Perron–Frobenius theorem [29] that there is a $D \in \mathcal{D}$ such that $R = BD$ is diagonally dominant of its rows. Let $S = R^{-1} = D^{-1}A = (s_{ij})$ and suppose $i, j \in N$ with $i \neq j$. Then it follows from the cofactor expansion of R^{-1} and from $R \in \mathcal{Z}$ that

$$|s_{ii}| - |s_{ji}| = \frac{\det R[N-i] + \det R[N-i, N-j]}{\det R} = \frac{\det T}{\det R}$$

in which T is obtained from $R[N-i]$ by adding the column $\pm R[N-i, i]$ to the jth column. Since R is diagonally dominant of its rows and has positive diagonal, $\det T > 0$ and so $|s_{ii}| - |s_{ji}| > 0$. Thus, $S = D^{-1}A$ is diagonally dominant of its column entries and (i) holds.

The proof of (ii) is similar and (iii) follows from (i) and (ii). □

Of course, because of theorem 1.2.3, any $\mathcal{IM}$ matrix may be diagonally scaled to one with 1's on the diagonal, a *normalized* $\mathcal{IM}$ matrix. In fact, the scaled $\mathcal{IM}$ matrix may be taken to have 1's on the diagonal and entries < 1 off [46].

It is known that $\mathcal{M}$ matrices have diagonal Lyapunov solutions; that is, for each $B \in \mathcal{M}$, there is a $D \in \mathcal{D}$ such that $DB + B^T D$ is positive definite. (This follows from the fact that an M-matrix may be scaled to have both row and column diagonal dominance [38].) This allows us to prove the analogous fact for $\mathcal{IM}$ matrices.

Theorem 1.4. *For each $A \in \mathcal{IM}$, there is a $D \in \mathcal{D}$ such that $DA + A^T D$ is positive definite.*

Proof. If $A \in \mathcal{IM}$, then $A^{-1} \in \mathcal{M}$ and thus there is $D \in \mathcal{D}$ such that $DA^{-1} + (A^{-1})^T D$ is positive definite. Hence, $A^T(DA^{-1} + (A^{-1})^T D)A = DA + A^T D$ is also. □

Our next fact is immediate from the corresponding property for M-matrices.

Theorem 1.5. *Each $A \in \mathcal{IM}$ is positive stable.*

Theorem 1.6. *If $A \in \mathcal{IM}$ and E_{ii} is the n-by-n matrix with a 1 in the (i, i) position and 0's elsewhere, then $A + tE_{ii} \in \mathcal{IM}$ for any $t \geq 0$.*

Proof. Let $A \in \mathcal{IM}$ and let $A^{-1} = (\alpha_{ij})$. Without loss of generality, we may assume that $i = 1$. Let $B = A + te_1e_1^T$ in which e_1 denotes the first standard basis vector. Then, from [29, p. 19–20], we have

$$\begin{aligned} B^{-1} &= A^{-1} - \frac{1}{1 + e_1^T A^{-1} e_1} A^{-1} e_1 e_1^T A^{-1} \\ &= A^{-1} - \frac{1}{1+\alpha_{11}} \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \vdots \\ \alpha_{n1} \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \end{bmatrix} \\ &= A^{-1} + \begin{bmatrix} - & + & + & . & . & + \\ + & - & - & . & . & - \\ + & - & - & . & . & - \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ + & - & - & . & . & - \end{bmatrix} \\ &= A^{-1} + C. \end{aligned}$$

For $j \neq 1$, $|\alpha_{1j}| \geq \frac{\alpha_{11}}{1+\alpha_{11}}|\alpha_{1j}| = c_{1j}$ and $|a_{j1}| \geq \frac{\alpha_{11}}{1+\alpha_{11}}|\alpha_{j1}| = c_{j1}$. Thus, $B^{-1} \in \mathcal{Z}$ which implies $B \in \mathcal{IM}$ and completes the proof. □

Closure under addition of a nonnegative diagonal matrix follows immediately.

Theorem 1.7. *If $A \in \mathcal{IM}$ and $D \in \overline{\mathcal{D}}$, then $A + D \in \mathcal{IM}$.*

For two $\mathcal{IM}$ matrices that result from inversion of comparable $\mathcal{IM}$ matrices, there is a natural inequality that results from multiplication on the left (right) by the nonnegative matrix $A^{-1}(B^{-1})$.

Theorem 1.8. *If $A \geq B$ are $\mathcal{M}$ matrices, then for the $\mathcal{IM}$ matrices B^{-1} and A^{-1}, we have*

$$B^{-1} \geq A^{-1}.$$

When A and B are $\mathcal{IM}$ matrices, the corresponding statement is not generally valid.

2. Partitioned $\mathcal{IM}$ matrices

If A is square and $A[\alpha]$ is invertible, the *Schur complement* of $A[\alpha]$ in A, denoted $A/A[\alpha]$, is defined by

$$A/A[\alpha] = A[\alpha^c] - A[\alpha^c, \alpha]A[\alpha]^{-1}A[\alpha, \alpha^c].$$

Let $\alpha = \{1, \dots, k\}$. (Due to theorem 1.2.1, there is no difference between $\alpha = \{1, \dots, k\}$ and a general α.) It was shown in [8] that if $A/A[\alpha] = B = (b_{ij})$, then, for $k + 1 \leq i, j \leq n$,

$$b_{ij} = \frac{\det A[\alpha + i, \alpha + j]}{\det A[\alpha]} = \frac{s_{ij}}{\det A[\alpha]},$$

in which $S = (s_{ij})$ is Sylvester's matrix of "bordered" minors [24], i.e.,

$$s_{ij} = \det \begin{bmatrix} A[\alpha] & A[\alpha, j] \\ A[i, \alpha] & a_{ij} \end{bmatrix}.$$

Thus, $S = (\det A[\alpha])(A/A[\alpha])$.

We shall make use of the Schur complement form of the inverse [29] given in the following form. Let the square matrix A be partitioned as

$$\mathbf{A} = \begin{bmatrix} A[\alpha] & A[\alpha, \alpha^c] \\ A[\alpha^c, \alpha] & A[\alpha^c] \end{bmatrix} \tag{2.1}$$

in which A, $A[\alpha]$, and $A[\alpha^c]$ are all invertible. Then

$$\begin{aligned} \mathbf{A}^{-1} &= \begin{bmatrix} (A/A[\alpha^c])^{-1} & -A[\alpha]^{-1}A[\alpha, \alpha^c](A/A[\alpha])^{-1} \\ -(A/A[\alpha])^{-1}A[\alpha^c, \alpha]A[\alpha]^{-1} & (A/A[\alpha])^{-1} \end{bmatrix} \\ &= \begin{bmatrix} (A/A[\alpha^c])^{-1} & -(A/A[\alpha^c])^{-1}A[\alpha, \alpha^c]A[\alpha^c]^{-1} \\ -A[\alpha^c]^{-1}A[\alpha^c, \alpha](A/A[\alpha^c])^{-1} & (A/A[\alpha])^{-1} \end{bmatrix}. \end{aligned} \tag{2.2}$$

We now make use of the fact that M-matrices are closed under extraction of principal submatrices and under extraction of Schur complements (Schur complementation) [47].

Theorem 2.3. *Let $A \geq 0$ be partitioned as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$. Then, $A \in \mathcal{IM}$ if and only if*

(i) $A/A_{11} \in \mathcal{I}M$;
(ii) $A/A_{22} \in \mathcal{I}M$;
(iii) $(A_{11})^{-1}A_{12}(A/A_{11})^{-1} \geq 0$;
(iv) $(A/A_{11})^{-1}A_{21}(A_{11})^{-1} \geq 0$;
(v) $(A_{22})^{-1}A_{21}(A/A_{22})^{-1} \geq 0$;
(vi) $(A/A_{22})^{-1}A_{12}(A_{22})^{-1} \geq 0$.

Proof. For necessity, suppose $A \in \mathcal{IM}$ and consider the Schur complement form of its inverse. Since $A^{-1} \in \mathcal{M}$ and M-matrices are closed under extraction of principal submatrices, $(A/A_{11})^{-1}$ and $(A/A_{22})^{-1}$ are in $\mathcal{M}$ and (i) and (ii) follow. Statements (iii)–(vi) follow since $A^{-1} \in \mathcal{Z}$.

For sufficiency, observe that (i) and (ii) and either (iii) and (iv) or (v) and (vi) ensure that $A^{-1} \in \mathcal{Z}$. This completes the proof. □

Corollary 2.3.1. *$\mathcal{IM}$ matrices are closed under extraction of Schur complements.*

Corollary 2.3.2. *$\mathcal{IM}$ matrices are closed under extraction of principal submatrices.*

These follow from theorem 2.3 and the Schur complement form of $(A^{-1})^{-1}$, respectively. In turn, corollary 2.3.2 implies that

Corollary 2.3.3. *$\mathcal{IM}$ matrices have positive principal minors.*

Notice also that theorem 2.3 allows us to zero out any row or column of an $\mathcal{IM}$ matrix off the diagonal and remain $\mathcal{IM}$, i.e., if $A = \begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathcal{IM}$ and $B = \begin{bmatrix} a_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, then $B^{-1} =$

$\begin{bmatrix} (a_{11})^{-1} & -(a_{11})^{-1}A_{12}(A_{22})^{-1} \\ 0 & (A_{22})^{-1} \end{bmatrix} \in \mathcal{Z}$ since $a_{11}, A_{12}(A_{22})^{-1} \geq 0$. This fact can also be shown by applying theorem 1.2.3 and theorem 1.7, i.e., multiply the first column of A by some $t, 0 < t < 1$, then add $a_{11} - ta_{11} > 0$ to the $(1, 1)$ entry to obtain the $\mathcal{IM}$ matrix $A(t) = \begin{bmatrix} a_{11} & A_{12} \\ tA_{21} & A_{22} \end{bmatrix}$. By continuity, $\begin{bmatrix} a_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} (a_{11})^{-1} & -(a_{11})^{-1}A_{12}(A_{22})^{-1} \\ 0 & (A_{22})^{-1} \end{bmatrix} \in \mathcal{IM}$.

We also have

Theorem 2.4. *Let $A \geq 0$ be partitioned as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$. Then, if $A \in \mathcal{IM}$,*

(i) $A_{11} \in \mathcal{IM}$;
(ii) $A/A_{11} \in \mathcal{IM}$;
(iii) $A_{22} \in \mathcal{IM}$;
(iv) $A/A_{22} \in \mathcal{IM}$;
(v) $(A_{11})^{-1}A_{12} \geq 0$;
(vi) $A_{21}(A_{11})^{-1} \geq 0$;
(vii) $(A_{22})^{-1}A_{21} \geq 0$;
(viii) $A_{12}(A_{22})^{-1} \geq 0$.
(ix) $A_{12}(A/A_{11})^{-1} \geq 0$;
(x) $(A/A_{11})^{-1}A_{21} \geq 0$;
(xi) $A_{21}(A/A_{22})^{-1} \geq 0$;
(xii) $(A/A_{22})^{-1}A_{12} \geq 0$.

Proof. Observe that (i)–(iv) follow from the preceding remarks. Then, (v)–(xii) follow from (iii) to (vi) of theorem 2.3 upon multiplying by the appropriate choice of $A/A_{11}, A/A_{22}, A_{11}$, or A_{22} which completes the proof. □

For $\mathcal{IM}$ matrices of order 2 or 3, it is obvious from the remarks preceding theorem 2.4 that we can zero out any reducing block (given a block matrix $\mathbf{A} = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$. the block C, respectively, D, is a reducing block provided A, B, E are square) or zero out either triangular part and remain $\mathcal{IM}$. However, these properties do not hold in general.

Example 2.5. Consider the $\mathcal{IM}$ matrix

$$\mathbf{A} = \begin{bmatrix} 20 & 8 & 11 & 5 \\ 19 & 20 & 19 & 12 \\ 17 & 8 & 20 & 5 \\ 14 & 9 & 14 & 20 \end{bmatrix}.$$

Neither $\mathbf{B} = \begin{bmatrix} 20 & 8 & 0 & 0 \\ 19 & 20 & 0 & 0 \\ 17 & 8 & 20 & 5 \\ 14 & 9 & 14 & 20 \end{bmatrix}$ nor $\mathbf{C} = \begin{bmatrix} 20 & 0 & 0 & 0 \\ 19 & 20 & 0 & 0 \\ 17 & 8 & 20 & 0 \\ 14 & 9 & 14 & 20 \end{bmatrix}$ is in $\mathcal{IM}$ since the $(4, 1)$ entry of the inverse of each is positive.

We will utilize Schur's formula [29], which states that

$$\det A = (detA[\alpha])(\det A/A[\alpha])$$

provided $A[\alpha]$ is nonsingular.

We will also need a special case of Sylvester's identity for determinants. Let A be an n-by-n matrix, $\alpha \subseteq N$, and suppose $|\alpha| = k$. Define the $(n-k)$-by-$(n-k)$ matrix $B = (b_{ij})$ by setting $b_{ij} = det\, A[\alpha + i, \alpha + j]$, for every $i, j \in \alpha^c$. Then Sylvester's identity for determinants (see [29]) states that for each $\delta, \gamma \subseteq \alpha^c$, with $|\delta| = |\gamma| = m$,

$$det\, B[\delta, \gamma] = (det\, A[\alpha])^{m-1} det\, A[\alpha \cup \delta, \alpha \cup \gamma]. \tag{2.6}$$

We will utilize the following special case of this identity: let A be an n-by-n matrix partitioned as follows :

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12}^T & a_{13} \\ a_{21} & A_{22} & a_{23} \\ a_{31} & a_{32}^T & a_{33} \end{bmatrix}, \tag{2.7}$$

in which A_{22} is $(n-2)$-by-$(n-2)$ and a_{11}, a_{33} are scalars. Define the matrices

$$\mathbf{B} = \begin{bmatrix} a_{11} & a_{12}^T \\ a_{21} & A_{22} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} a_{12}^T & a_{13} \\ A_{22} & a_{23} \end{bmatrix}, \mathbf{D} = \begin{bmatrix} a_{21} & A_{22} \\ a_{31} & a_{32}^T \end{bmatrix}, \mathbf{E} = \begin{bmatrix} A_{22} & a_{23} \\ a_{32}^T & a_{33} \end{bmatrix}.$$

If we let $b = det\, B$, $c = det\, C$, $d = det\, D$, and $e = det\, E$, then, by equation (2.6), it follows that $det \begin{bmatrix} b & c \\ d & e \end{bmatrix} = det\, A_{22}\ det\, A$. Hence, provided $det\, A_{22} \neq 0$, we have

$$det\, A = \frac{det\, B\ det\, E - det\, C\ det\, D}{det\, A_{22}}. \tag{2.8}$$

With certain nonnegativity/positivity assumptions $\mathcal{IM}$ matrices can be characterized in terms of Schur complements [46].

Theorem 2.9. *Let $A \geq 0$. Then $A \in \mathcal{IM}$ if and only if A has positive diagonal entries, all Schur complements are nonnegative, and all Schur complements of order 1 are positive.*

In fact, these conditions can be somewhat relaxed.

Theorem 2.9.1. *Let $A \geq 0$. Then $A \in \mathcal{IM}$ if and only if*

(i) *A has at least one positive diagonal entry,*
(ii) *all Schur complements of order 2 are nonnegative, and*
(iii) *all Schur complements of order 1 are positive.*

Proof. For necessity, assume that $A \in \mathcal{IM}$. Then A has positive diagonal entries and, since $\mathcal{I}M$ matrices are closed under extraction of Schur complements, each Schur complement is nonnegative. So we just need to show those of order 1 are positive. But this follows from Schur's formula since A has positive principal minors.

For sufficiency, suppose (i), (ii), and (iii) hold, say $a_{ii} > 0$. Observe that (by considering all Schur complements $A[\{i, j\}]/a_{ii}$ in which $j \neq i$), (iii) implies that A has positive diagonal entries.

Claim. All principal minors of A are positive.

Proof of Claim. If A is 1-by-1, then the claim certainly holds. So, inductively, assume the claim holds for all matrices of order $< n$ satisfying (i), (ii), and (iii) and let $A = \begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$. Thus, all principal submatrices of order $< n$ have positive determinant and it suffices to prove that $\det A > 0$. By Schur's formula, $\det A = (\det A_{22})(a_{11} - A_{12}(A_{22})^{-1}A_{21})$. The inductive hypothesis implies $\det A_{22} > 0$ and thus the positivity of $\det A$ follows from (iii), completing the proof of the claim.

Now let $A^{-1} = B = (b_{ij})$ and consider b_{ij}, $i \neq j$, and assume, without loss of generality, that $i < j$. Define the sequences

$$\alpha = \langle 1, \ldots, i-1, i+1, \ldots, j-1, j+1, \ldots, n \rangle,$$

$$\alpha_1 = \langle 1, \ldots, i-1, i+1, \ldots, j-1, j+1, \ldots, n, i \rangle,$$

and

$$\alpha_2 = \langle 1, \ldots, i-1, i+1, \ldots, j-1, j+1, \ldots, n, j > .$$

Then,

$$\begin{aligned} b_{ij} &= (-1)^{i+j} \frac{\det A(j,i)}{\det A} \\ &= (-1)^{i+j}(-1)^{n-i-1}(-1)^{n-j} \frac{\det A[\alpha_1, \alpha_2]}{\det A} \\ &= -\left(a_{ij} - A[i,\alpha](A[\alpha])^{-1}A[\alpha, j]\right) \frac{\det A[\alpha]}{\det A} \\ &\leq 0. \end{aligned}$$

The latter inequality holds since $A/A[\alpha]$ is a Schur complement of order 2 and hence is nonnegative. Thus, $A^{-1} \in \mathcal{Z}$ which implies $A \in \mathcal{IM}$ and completes the proof. □

From the latter part of the proof of theorem 2.9.1 we obtain another characterization of $\mathcal{IM}$ matrices.

Theorem 2.9.2. *Let $A \geq 0$. Then $A \in \mathcal{IM}$ if and only if*

(i) $\det A > 0$ *and*
(ii) *for each principal submatrix B of order $n-2$,* $\det B > 0$ *and* $A/B \geq 0$.

As noted in corollary 2.3.1 and corollary 2.3.2, $\mathcal{IM}$ matrices are closed under extraction of Schur complements and under extraction of principal submatrices. Conversely, if $A \geq 0$ with principal submatrix B and both B and A/B are $\mathcal{IM}$, then A is not necessarily $\mathcal{IM}$. A counterexample was provided in [32] as well as added restrictions on A and B so as to ensure that A is $\mathcal{IM}$.

3. Submatrices

An *almost principal minor (APM)* of a square matrix $A = (a_{ij})$ is the determinant of an (almost principal) submatrix $A[\alpha, \beta]$ in which $|\alpha| = |\beta|$ and β differs from α in exactly one index; i.e., $|\alpha \cap \beta| = |\alpha| - 1$. Of course, a k-by-k almost principal submatrix of A is a maximal non-principal submatrix of a $(k+1)$-by-$(k+1)$ principal submatrix of A, since $A[\alpha, \beta]$ sits in $A[\alpha \cup \beta]$. Since the maximal proper principal minors and the maximal *APM*'s of a square, invertible matrix are the numerators of the inverse entries via the co-factor representation of the inverse, the *APM*'s of an $\mathcal{IM}$ matrix have a special sign structure, because of theorem 1.2 and theorem 2.4/theorem 1.2.1. We record these facts and further results here. Recall that if A is n-by-n $\mathcal{IM}$ matrix and α is a proper subset of N, then $A[\alpha]$ is $\mathcal{IM}$.

If A is an n-by-n $\mathcal{IM}$ matrix, then the *APM* $\det A(i, j)$ is 0 or has sign $(-1)^{i+j+1}$, according to theorem 1.2. If $\det A[\alpha, \beta]$ is an *APM* in A, then it is also an *APM* in $A[\alpha \cup \beta]$. We then have

Theorem 3.1. *Let A be an n-by-n $\mathcal{IM}$ matrix and $\alpha, \beta, \gamma \subseteq N$ in which $\alpha = \gamma - i$, $\beta = \gamma - j$ $(i \neq j)$ so that $\gamma = \alpha \cup \beta$. Then,* $\det A[\alpha, \beta]$ *is a APM in A and either equals 0 or has sign $(-1)^{r+s+1}$ in which r (resp. s) is the number of indices in α (resp. β) less than or equal to i (resp. j).*

(We note that an analogous statement to theorem 3.1 can be made concerning the *APM* $A[\alpha + i, \alpha + j]$ and that analogous statements can be made about M-matrices upon replacing $r + s + 1$ with $r + s$.)

For an individual minor of an $\mathcal{IM}$ matrix that is neither principal nor an *APM*, there is no constraint upon the sign.

Example 3.2. Consider the $\mathcal{IM}$ matrices

$$\mathbf{A} = \begin{bmatrix} 200 & 50 & 25 & 12 \\ 50 & 200 & 50 & 25 \\ 25 & 50 & 200 & 50 \\ 12 & 25 & 50 & 200 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 200 & 50 & 25 & 13 \\ 50 & 200 & 50 & 25 \\ 25 & 50 & 200 & 50 \\ 13 & 25 & 50 & 200 \end{bmatrix}.$$

$\det A[\{1, 2\}, \{3, 4\}]$ is positive while $\det B[\{1, 2\}, \{3, 4\}]$ is negative.

APM's in $\mathcal{IM}$ matrices can be 0, but the pattern of 0's is far from arbitrary. An essentially complete description of the possibilities is given in [47], which we summarize here. There are also corresponding inequalities involving minors that we mention. First, we note that "small" vanishing *APM*'s imply that "larger" ones also vanish.

Theorem 3.3. *If $\phi \neq \alpha \subseteq \beta \subseteq N - \{i, j\}$ and A is an n-by-n $\mathcal{IM}$ matrix such that the APM* $\det A[\alpha + i, \alpha + j] = 0$, *then the APM* $\det A[\beta + i, \beta + j] = 0$.

There are also relationships among vanishing *APM*'s of the same"size".

Theorem 3.4. *If $n \geq 3$ and A is an n-by-n $\mathcal{IM}$ matrix, let i, j, k be distinct indices in N. If $\alpha \subseteq N - \{i, j, k\}$, then, if* $\det A[\alpha + i, \alpha + j] = 0$, *either*

(i) $\det A[\alpha + i, \alpha + k] = 0$ *or*
(ii) $\det A[\alpha + k, \alpha + j] = 0$.

Related inequalities include the following.

Theorem 3.5. *If A is $\mathcal{IM}$ and $\phi \neq \alpha \subseteq N$, then*

(i) $\left(A^{-1}[\alpha]\right)^{-1} \leq A[\alpha]$; *and*
(ii) $A[\alpha]^{-1} \leq A^{-1}[\alpha]$.

For particular minors of different sizes in a *normalized* $\mathcal{IM}$ matrix, there are inequalities generalizing theorem 3.5.

Theorem 3.6. *If A is an n-by-n, normalized $\mathcal{IM}$ matrix and $\phi \neq \alpha \subseteq \beta \subseteq N - \{i, j\}$, then*

(i) $\det A[\beta] \leq \det A[\alpha]$*; and*
(ii) $|\det A[\beta + i, \beta + j]| \leq |\det A[\alpha + i, \alpha + j]|$.

Thus, generally, "smaller" *PM*'s or *APM*'s are bigger (have a larger determinant) in an $\mathcal{IM}$ matrix. Heretofore there does not seem to have been any work on nontrivial inequalities involving nonprincipal minors of $\mathcal{IM}$ matrices like those for *APM*'s in theorem 3.6(ii) above.

Another way in which vanishing minors are related in an $\mathcal{IM}$ matrix is the following.

Theorem 3.7. *Suppose that A is an n-by-n $\mathcal{IM}$ matrix and that $\gamma = N - i$. If $A[\gamma]^{-1}[\alpha, \beta] = 0$, then either $A^{-1}[\alpha, \beta + i] = 0$ or $A^{-1}[\alpha + i, \beta] = 0$. Thus, if $A[\gamma]^{-1}$ has a p-by-q, off-diagonal* 0 *block, then A^{-1} must have one of size $(p + 1)$-by-q or p-by-$(q + 1)$.*

4. The path product property

Let $A = (a_{ij})$ be an n-by-n nonnegative matrix with positive diagonal entries. We call A a *path product* (*PP*) matrix if, for any triple of indices $i, j, k \in N$,

$$\frac{a_{ij}a_{jk}}{a_{jj}} \leq a_{ik} \tag{4.1}$$

a *strict path product* (*SPP*) matrix if there is strict inequality whenever $i \neq j$ and $k = i$ [46]. In [46] it was noted that any $\mathcal{IM}$ matrix is *SPP* and that for $n \leq 3$ (but not greater) the two classes are the same. See also [77]. If $a_{ii} = 1, i = 1, \ldots, n$, we call a *PP* (resp. *SPP*) matrix A *normalized*.

For a *PP* matrix A and any path $i_1 \to i_2 \to \cdots \to i_{k-1} \to i_k$ in the complete graph K_n on n vertices, we have

$$\frac{a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k}}{a_{i_2 i_2} a_{i_3 i_3} \cdots a_{i_{k-1} i_{k-1}}} \leq a_{i_1 i_k} \tag{4.2}$$

and, if in addition, A is *SPP*, then the inequality is strict. We call inequality (4.2) the *path product inequalities* and, if $i_1 = i_k$ in inequality (4.2), the *cycle product inequalities*. We call a product $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k}$ an (i_1, i_k) *path product* (of length $k - 1$) and, if $i_k = i_1$, an (i_1, i_k) *cycle product* (of length $k - 1$).

In inequality (4.2), if $i_k = i_1$, we see that the product of entries around any cycle is no more than the corresponding diagonal product, i.e.,

$$a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_1} \leq a_{i_1 i_1} a_{i_2 i_2} \cdots a_{i_{k-1} i_{k-1}}. \tag{4.3}$$

It follows that in a normalized *PP* matrix no cycle product is more than 1 and that in a strictly normalized *PP* matrix all cycles of length two or more have product less than 1. From this we have [46].

Theorem 4.4. *If A is a normalized PP (resp. normalized SPP) matrix, then there is a normalized PP (resp. normalized SPP) matrix $\hat{A}$ diagonally similar to A in which all (resp. off-diagonal) entries are $\leq$ (resp. $<$) 1.*

PP matrices are closed under: extraction of principal submatrices, permutation similarity, Hadamard (entry-wise) multiplication, left (right) multiplication by a positive diagonal matrix (and hence positive diagonal congruence), and positive diagonal similarity but not under Schur complementation, addition, or ordinary multiplication [46]. Moreover, a *PP* matrix remains *PP* upon the addition of a nonnegative diagonal matrix.

There is a strong connection between $\mathcal{IM}$ matrices and *SPP* matrices as the next several results from [46] illustrate.

Theorem 4.5. *Every $\mathcal{IM}$ matrix is SPP.*

Theorem 4.6. *If A is an n-by-n SPP matrix, $n \leq 3$, then A is $\mathcal{IM}$.*

For $n \geq 4$, sufficiency no longer holds.

Example 4.7. The *SPP*-matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0.10 & 0.40 & 0.30 \\ 0.40 & 1 & 0.40 & 0.65 \\ 0.10 & 0.20 & 1 & 0.60 \\ 0.15 & 0.30 & 0.60 & 1 \end{bmatrix}$$

is not $\mathcal{IM}$, as the $(2, 3)$ entry of A^{-1} is positive.

Given an n-by-n matrix A, $G(A)$, the (directed) *graph of A*, is the graph with vertices N and satisfying: (i, j) is an edge of $G(A)$ if and only if $a_{ij} \neq 0$. *PP* matrices can be used to deduce the following important facts about $\mathcal{IM}$ matrices [46], some of which we will use later.

Theorem 4.8

(i) *If an $\mathcal{IM}$ matrix has a 0 entry, then it is reducible.*
(ii) *An $\mathcal{IM}$ matrix has a transitive graph.*
(iii) *An $\mathcal{IM}$ matrix can be scaled by positive diagonal matrices D, E so that DAE has all diagonal entries equal to* 1 *and all off-diagonal entries less than* 1.
(iv) *Every $\mathcal{I}M$ matrix satisfies the strict cycle product inequalities*

$$a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{k-1}i_1} < a_{i_1i_1}a_{i_2i_2}\cdots a_{i_{k-1}i_{k-1}}. \tag{4.9}$$

In fact, it follows from theorem 4.8(i) that every 0 entry of an $\mathcal{IM}$ matrix lies in an off diagonal reducing block. The known fact (see [34] and references) that the 0-pattern of an $\mathcal{IM}$ matrix is power-invariant also follows from this discussion. Each is simply part of a reducing 0-block.

A *P*-matrix is a real n-by-n matrix whose principal minors are all positive. In [46] it was shown that *SPP* matrices are not necessarily *P*-matrices in contrast to *M*- and $\mathcal{IM}$ matrices.

A path product $a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{k-1}i_k}$ is called an (i_1, i_k) *path product* and we say it is *even* (*odd*) if the path $i_1 \to i_2 \to \cdots \to i_{k-1} \to i_k$ in $G(A)$ has even (odd) length (equivalently, if k is odd(even)). Note that if $A \geq 0$, then the (i, j) entry of A^k, $k = 1, 2, \ldots$, equals the sum of the (i, j) path products of length k.

Theorem 4.10 *Let A be an n-by-n triangular normalized SPP matrix. Then, $A \in \mathcal{I}M$ if and only if the sum of the even length (i, j) path products is at most the sum of the odd length (i, j) path products for all i and j with $i \neq j$.*

Proof. Let $A = I + T$ be an n-by-n triangular normalized *SPP* matrix. Then,

$$\begin{aligned} A^{-1} &= (I+T)^{-1} \\ &= I - T + T^2 - \cdots \pm T^{n-1} \\ &= I - \sum_{k\ odd} T^k + \sum_{k\ even} T^k. \end{aligned}$$

Thus, we see that, for all i and j with $i \neq j$, the (i, j) entry of A^{-1} is the sum of the even (i, j) path products minus the sum of the odd (i, j) path products. Hence, $A^{-1} \in \mathcal{M}$ if and only if the sum of the even (i, j) path products is at most the sum of the odd (i, j) path products for all i and j with $i \neq j$. □

We identify the case in which no path product equalities occur, i.e., all inequalities inequalities (4.1) are strict, and call such *SPP* matrices *totally strict path product* (*TSPP*) [51]. Observe that *TSPP* matrices are necessarily positive, but may not be $\mathcal{IM}$ and that $\mathcal{IM}$ matrices may not be *TSPP*.

Example 4.11 The *TSPP* matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0.50 & 0.35 & 0.40 \\ 0.50 & 1 & 0.50 & 0.26 \\ 0.35 & 0.50 & 1 & 0.50 \\ 0.40 & 0.26 & 0.50 & 1 \end{bmatrix}$$

is not $\mathcal{IM}$ since the (2, 4) entry of the inverse is positive while the $\mathcal{IM}$ matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 0.50 & 0.25 \\ 0.50 & 1 & 0.50 \\ 0.25 & 0.50 & 1 \end{bmatrix}$$

is not *TSPP*.

Consider the following condition on the collection of path product inequalities: for all distinct indices $i, j, k \in N$ and, for all $m \in N - \{i, j, k\}$,

$$a_{ik} = a_{ij}a_{jk} \quad \text{implies that either} \quad a_{im} = a_{ij}a_{jm} \quad \text{or} \quad a_{mk} = a_{mj}a_{jk}. \tag{4.12}$$

If implication (4.12) is satisfied by an *SPP* matrix, we say that A is *purely strict path product* (*PSPP*) [51]. We will see (Section 8) that, in *PSPP* matrices, path product equalities force certain cofactors to vanish.

We note that *PSPP* and *SPP* coincide (vacuously) when $n \leq 3$ and that generally the *TSPP* matrices are contained in the *PSPP* matrices (vacuously). Also observe that, if A is *TSPP* (*PSPP*), then so is any normalization of A. Lastly, we note that an $\mathcal{IM}$ matrix is necessarily *PSPP* (this was proved for positive normalized $\mathcal{IM}$ matrices in [51] and the same proof applies in the general case), but the converse does not hold (by example 4.11, for instance).

Theorem 4.13 *Any normalized $\mathcal{IM}$ matrix is PSPP.*

Proof. Let $A = (a_{ij})$ be a normalized $\mathcal{IM}$ matrix. By theorem 4.5, A is *SPP*. If $n \leq 3$, then A is *PSPP* vacuously. So we may assume that $n \geq 4$. Also, assume that $a_{ik} = a_{ij}a_{jk}$ for the distinct indices i, j, k of N and let $m \in N - \{i, j, k\}$ and consider the principal submatrix

$$A[\{m, j, k, i\}] = \begin{bmatrix} 1 & a_{mj} & a_{mk} & a_{mi} \\ a_{jm} & 1 & a_{jk} & a_{ji} \\ a_{km} & a_{kj} & 1 & a_{ki} \\ a_{im} & a_{ij} & a_{ik} & 1 \end{bmatrix}.$$

This submatrix is $\mathcal{IM}$ by inheritance. So (via equation 2.8, for instance) the k, i cofactor $c_{ki} = (a_{mk} - a_{mj}a_{jk})(a_{im} - a_{ik}a_{km}) \leq 0$. Hence, by inequalities (4.1), either $a_{mk} = a_{mj}a_{jk}$ or $a_{im} = a_{ij}a_{jm}$. Thus, A is *PSPP*. □

5. Triangular factorization

PP matrices do not necessarily admit an *LU*-factorization in which the factors are *PP*.

Example 5.1 The *PP* matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

has *LU* factorization

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

but $U = (u_{ij})$ is not *PP* since $u_{23}u_{34} \not\leq u_{24}$.

However, *M*-matrices and $\mathcal{IM}$ matrices have *LU*- (*UL*-) factorizations within their respective classes. To see this, first suppose that $A \in \mathcal{M}$. If A is 1-by-1, A trivially factors as LU in which $L, U > 0$. So assume $n > 1$. Since *M*-matrices are closed under extraction of principal submatrices, we may assume (inductively) that $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ in which $A_{11} = L_{11}U_{11}$ with L_{11} (U_{11}) being an $(n-1)$-by-$(n-1)$ lower- (upper-) triangular *M*-matrix. Therefore, since A has positive principal minors $A = LU = \begin{bmatrix} L_{11} & 0 \\ L_{21} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & u_{nn} \end{bmatrix} = \begin{bmatrix} L_{11}U_{11} & L_{11}U_{12} \\ L_{21}U_{11} & L_{21}U_{12} + u_{nn} \end{bmatrix}$ in which $u_{nn} = A/A_{11} > 0$ (by Schur's formula). Thus, $L_{11}U_{12}, L_{21}U_{11} \leq 0$. Since $L_{11}^{-1}, U_{11}^{-1} \geq 0$, it follows that $U_{12}, L_{21} \leq 0$. Thus, $L, U \in Z$. And since

$$L^{-1} = \begin{bmatrix} L_{11}^{-1} & 0 \\ -L_{21}L_{11}^{-1} & 1 \end{bmatrix} \quad \text{and} \quad U^{-1} = \begin{bmatrix} U_{11}^{-1} & -\frac{1}{u_{nn}}U_{11}^{-1}U_{12} \\ 0 & \frac{1}{u_{nn}} \end{bmatrix}$$

are both nonnegative, $L, U \in \mathcal{M}$. Similarly, it can be shown that A has a *UL*-factorization within the class of *M*-matrices. Observe that if $A = LU$ (UL) in which $L, U \in \mathcal{M}$, then $A^{-1} = U^{-1}L^{-1}$ ($L^{-1}U^{-1}$) with $L^{-1}, U^{-1} \in \mathcal{IM}$. Thus, $\mathcal{IM}$ matrices also have *LU*- and *UL*-factorizations within their class. However, it is not the case that if L, U are, respectively, lower- and upper-triangular $\mathcal{IM}$ matrices, then LU and/or UL is $\mathcal{IM}$.

Example 5.2 Consider the $\mathcal{IM}$ matrices

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

Neither LU nor UL is $\mathcal{IM}$ since the inverse of the former is positive in the $(2, 1)$ entry and the inverse of the latter is positive in the $(3, 2)$ entry.

6. Sums, products, and closure

Suppose that $A, B, \ldots$, and C are n-by-n $\mathcal{IM}$ matrices, throughout this section. It is natural to ask about closure under a variety of operations. Neither the sum $A+B$ (see corollary 1.1.2) nor the product AB (see example 1.1.4) need generally be in $\mathcal{IM}$. In addition, the conventional powers $A^t, t \in \mathbb{R}$ and > 1 are nonnegative and need not be in $\mathcal{IM}$ for $n > 3$. Recall that A^t, when t is not an integer, is defined naturally for an M-matrix, via power series, and thus for an IM-matrix, as in [34], or, equivalently, via principal powers, as in [30].

Example 6.1 Consider the $\mathcal{IM}$ matrix

$$\mathbf{A} = \begin{bmatrix} 90 & 59 & 44 & 71 \\ 56 & 96 & 48 & 88 \\ 64 & 60 & 88 & 84 \\ 42 & 43 & 36 & 95 \end{bmatrix}.$$

Since the inverse of A^3 has a positive 1, 4 entry, A^3 is not an $\mathcal{IM}$ matrix.

For m-by-n matrices $A = (a_{ij})$ and $B = (b_{ij})$, the *Hadamard (entry-wise) product* $A \circ B$ is defined by $A \circ B = (a_{ij}b_{ij})$. The Hadamard product, $A \circ B$ need not be $\mathcal{IM}$ for $n > 3$ (see the example from [78] in Section 8).

However, in each of these cases, there is an aesthetic condition for the result to be $\mathcal{IM}$, even when the number of summands or factors is more than two. Note that, since $A, B, \ldots,$ and $C \geq 0$ (entry-wise), necessarily $A+B+\cdots+C$, $AB\cdots C$, and $A \circ B \circ \cdots \circ C \geq 0$. The second of these is necessarily invertible, while the first and third need not be. But, invertibility plus nonnegativity mean that the result is $\mathcal{IM}$ if and only if the inverse has nonpositive off-diagonal entries. This gives

Theorem 6.2 *If $A, B, \ldots, C$ are n-by-n $\mathcal{IM}$ matrices, then $AB\cdots C$ (resp. $A+B+\cdots+C$, $A \circ B \circ \cdots \circ C$, if they are invertible) is $\mathcal{IM}$ if and only if the off-diagonal entries of its inverse are nonpositive.*

Also,

Theorem 6.3 *If $t > 1$ is an integer, then A^t is $\mathcal{IM}$ if and only if the off-diagonal entries of the inverse of A^t are nonpositive.*

The above discussion leaves powers A^t, $0 \leq t < 1$, for consideration and Hadamard powers $A^{(t)}$, $t > 1$ and $0 < t < 1$. The Hadamard powers $A^{(t)}$, $t > 1$, are discussed in Section 8 and interestingly, are always $\mathcal{IM}$! For $n > 2$ and $0 < t < 1$, Hadamard powers $A^{(t)}$ need not be $\mathcal{IM}$.

Example 6.4 Consider the $\mathcal{IM}$ matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0.001 & 0.064 & 0.027 \\ 0.064 & 1 & 0.064 & 0.274625 \\ 0.001 & 0.008 & 1 & 0.216 \\ 0.003375 & 0.027 & 0.216 & 1 \end{bmatrix}.$$

The Hadamard cube root of A, given by

$$\mathbf{A}^{(1/3)} = \begin{bmatrix} 1 & 0.1 & 0.4 & 0.3 \\ 0.4 & 1 & 0.4 & 0.65 \\ 0.1 & 0.2 & 1 & 0.6 \\ 0.15 & 0.3 & 0.6 & 1 \end{bmatrix},$$

is not $\mathcal{IM}$ since the 2, 3 entry of its inverse is positive.

The remaining issue of "conventional" powers (roots) A^t, $0 < t < 1$ has an interesting resolution. Based upon a power series argument given in [34], the M-matrix $B = A^{-1}$ always has a kth root $B^{1/k}$, $k = 1, 2, \ldots$, such that $(B^{1/k})^k = B$, that is an M-matrix. The same argument or continuity gives a natural $B^t = (A^{-1})^t$, $0 \le t < 1$, that is an M-matrix. As the laws of exponents are valid, this means

Theorem 6.5 *If A is $\mathcal{IM}$, then for each t, $0 \le t < 1$, there is a natural A^t such that A^t is $\mathcal{IM}$. If $t = p/q$, with p and q positive integers and $0 \le p \le q$, then*

$$(A^t)^q = A^p.$$

7. Spectral structure

Let $A \in \mathcal{IM}$ and let $\sigma(A)$ denote the spectrum of A. It follows from Perron–Frobenius theory that $|\lambda| \le \rho(A)$ for all $\lambda \in \sigma(A)$ – the spectrum of A – with equality only for $\lambda = \rho(A)$ and that the M-matrix $B = A^{-1} = \alpha I - P$ in which $P \ge 0$ and $\alpha > \rho(P)$. Thus, $q(B) = \frac{1}{\rho(A)} = \alpha - \rho(P)$ is the eigenvalue of B with minimum modulus, $\sigma(B)$ is contained in the disc $\{z \in \mathbb{C} : |z - \alpha| \le \rho(P)\}$, and $Re(\lambda) \ge q(B)$ for all $\lambda \in \sigma(B)$ with equality only for $\lambda = q(B)$. Moreover, $\sigma(B)$ is contained in the open wedge

$$W_n \equiv \left\{ z = re^{i\theta} : r > 0, |\theta| < \frac{\pi}{2} - \frac{\pi}{n} \right\}$$

in the right half-plane if $n > 2$, and in $(0, \infty)$ if $n = 2$ [30]. So $\sigma(A)$ is contained in the wedge W_n if $n > 2$ and in $(0, \infty)$ if $n = 2$. Under the transformation $f(z) = \frac{1}{z}$, circles are mapped to circles and lines to lines (see [60] for details). This in turn implies that $\sigma(A)$ is contained in the disc $\{z \in C : |z - \beta| \le R\}$ in which $\beta = \frac{\alpha}{\alpha^2 - (\rho(P))^2}$ and $R = \frac{\rho(P)}{\alpha^2 - (\rho(P))^2}$.

It was noted in [33,23] that if A and B are M-matrices, then the Hadamard product $A \circ B^{-1}$ is also an M-matrix. A real n-by-n matrix A is *diagonally symmetrizable* if there exists a diagonal matrix D with positive diagonal entries such that DA is symmetric. In [33] it was shown that if the M-matrix A is diagonally symmetrizable, then $q(A \circ A^{-1}) = 1$.

For an M-matrix A, there has been a great deal of interest in bounds for $q(A \circ A^{-1})$. For instance, $q(A \circ A^{-1}) \le 1$ was proved in [17] and, moreover, it was asked whether $q(A \circ A^{-1}) \ge \frac{1}{n}$. This latter question was answered affirmatively in [18] and the authors conjectured that $q(A \circ A^{-1}) \ge \frac{2}{n}$. Lower bounds were also studies in [57]. Also, for M-matrices A and B, a lower bound for $q(A \circ B^{-1})$ was determined. The conjecture in [18] was later established independently in [74,79,7].

8. Hadamard products and powers

Many facts about the Hadamard (entry-wise) product may be found in [27,34].

Since n-by-n $\mathcal{IM}$ matrices A and B are entry-wise nonnegative, it is natural to ask whether $A \circ B$ is again $\mathcal{IM}$. For $n \le 3$, this is so, as $\mathcal{IM}$ is equivalent to *SPP* for $n = 3$ (Section 3) and the Hadamard product of *SPP* matrices is *SPP*; and for $n = 1, 2$, the claim is immediate. It has long been known that such a Hadamard product is not always $\mathcal{IM}$. Examples for $n = 6$ (and hence for larger n) may be found in [30,36] and more recently, it was noted in [78] that the two 4-by-4 symmetric matrices

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 2 & 1 & 3 \\ 2 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 4 \end{bmatrix}$$

are $\mathcal{IM}$, while $A \circ B$ is not. This entirely resolves the question of dimensions in which there is Hadamard product closure.

Theorem 8.1 *The n-by-n $\mathcal{IM}$ matrices are closed under Hadamard product if and only if $n \leq 3$.*

This leaves the question of which pairs of $\mathcal{IM}$ matrices have an $\mathcal{IM}$ Hadamard product. Counterexamples seem not so common, in part because the Hadamard product is *SPP*, but also because the ideas in [50] indicate that a bounded multiple of I (at worst) need be added to make the Hadamard product $\mathcal{IM}$. Nonetheless, better descriptions of such pairs would be of interest. In [9,50] the *dual* of the $\mathcal{IM}$ matrices was defined to be

$$\mathcal{IM}^{(D)} = \{A \in M_n(R) : A \circ B \in \mathcal{IM} \text{ for all } B \in \mathcal{IM}\}.$$

It is not hard to see that $\mathcal{IM}^{(D)} \subseteq \overline{\mathcal{IM}}$, but an effective characterization of $\mathcal{IM}^{(D)}$ would be of interest.

For $A = (a_{ij}) \in \mathcal{IM}$, another (more special) natural question is whether $A^{(2)} \equiv A \circ A \in \mathcal{IM}$ also [70]. This was also conjectured elsewhere. More generally, is $A^{(k)} = A \circ A \circ \cdots \circ A \in \mathcal{IM}$ for all positive integers k, and, if so, is $A^{(t)} = (a_{ij}^t) \in \mathcal{I}M$ for all $t \geq 1$ where for each real number t, the tth Hadamard power of A is defined by $A^{(t)} \equiv (a_{ij}^t)$. A constructive proof for $A^{(2)}$ when $n = 4$ was given by the authors. Then, the following was shown in [4].

Theorem 8.2 *All positive integer Hadamard powers of an $\mathcal{IM}$ matrix are $\mathcal{IM}$.*

In [78], as well as elsewhere, it was conjectured that, for an $\mathcal{IM}$ matrix A, the tth Hadamard power of A is $\mathcal{IM}$ for all real $t \geq 1$. This was recently proven in [5].

Theorem 8.3 *If A is an $\mathcal{IM}$ matrix and $t \geq 1$, $A^{(t)}$ is $\mathcal{IM}$.*

The question of which $\mathcal{IM}$ matrices satisfy $A^{(t)}$ is $\mathcal{IM}$ for all $t > 0$ is still open.

The strong results about Hadamard powers remaining $\mathcal{IM}$, raises a question about which (nonnegative) matrices become $\mathcal{IM}$ via Hadamard powering. We call an n-by-n $A \geq 0$ *eventually inverse M* ($\mathcal{EIM}$) if there exists a $T > 0$ such that $A^{(T)}$ is $\mathcal{IM}$. Of course, then $A^{(t)}$ is $\mathcal{IM}$ for all $t \geq T$. Since, for any $t > 0$, $A^{(t)}$ is *SPP* if and only if A is *SPP*, the property *SPP* is necessary for $\mathcal{EIM}$, but it is not quite sufficient.

Example 8.4 Consider the normalized *SPP* matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0.5 & 0.7 & 0.4 \\ 0.5 & 1 & 0.5 & 0.25 \\ 0.7 & 0.5 & 1 & 0.5 \\ 0.4 & 0.25 & 0.5 & 1 \end{bmatrix}.$$

Since the 2, 4 cofactor of $A^{(t)}$ is $c_{24}^{(t)} = [(0.5)^t - (0.35)^t][(0.4)^t - (0.35)^t]$ which is positive for all $t > 0$, we see that A is not $\mathcal{EIM}$.

The following result was proved for positive matrices in [51]. Here, we extend it to the general (nonnegative) case.

Theorem 8.5 *Let A be an n-by-n SPP matrix. Then there exists $T > 0$ such that* $\det A^{(t)} > 0$ *for all $t > T$.*

Proof. Without loss of generality, let A be an n-by-n normalized *SPP* matrix and let $max_{i \neq j} a_{ij} = M < 1$. Denote the set of permutations of N by S_n and the identity permutation by *id*. Then,

$$\begin{aligned} det\,A^{(t)} &= \textstyle\sum_{\tau\in S_n} sgn(\tau)a^t_{1,\tau(1)}a^t_{2,\tau(2)}\cdots a^t_{n,\tau(n)} \\ &= 1+\textstyle\sum_{\substack{\tau\in S_n\\ \tau\neq id}} sgn(\tau)a^t_{1,\tau(1)}a^t_{2,\tau(2)}\cdots a^t_{n,\tau(n)} \\ &\geq 1-\textstyle\sum_{\substack{\tau\in S_n\\ \tau\neq id}} a^t_{1,\tau(1)}a^t_{2,\tau(2)}\cdots a^t_{n,\tau(n)} \\ &\geq 1-(n!-1)M^{nt}. \end{aligned}$$

It is clear that there exists $T > 0$ such that for all $t > T$, $1-(n!-1)M^{nt} > 0$, completing the proof. □

By applying theorem 8.5, we obtain [51].

Theorem 8.6 *If A is an TSPP matrix, A is* $\mathcal{EIM}$.

Recall that *TSPP* matrices are necessarily positive. While the condition *TSPP* is sufficient for $\mathcal{EIM}$, positive *EIM* matrices are not necessarily *TSPP* (see example 4.11). Rather, the correct necessary and sufficient condition, *PSPP* (see Section 4), was given in [51] for positive matrices. The general (nonnegative) case follows by the same argument and we have

Theorem 8.7 *Let A be a nonnegative n-by-n matrix. A is* $\mathcal{EIM}$ *if and only if A is PSPP.*

It follows from theorem 4.13 and theorem 8.7 that

Theorem 8.8 *If A is an* $\mathcal{IM}$ *matrix, then there exists* $T > 0$ *such that* $A^{(t)} \in \mathcal{IM}$ *for all* $t > T$.

Putting these ideas together we have the following complete result.

Theorem 8.9 *For an n-by-n nonnegative matrix A, either*

(i) *there is no* $t > 0$ *such that* $A^{(t)}$ *is* $\mathcal{IM}$ *or*
(ii) *there is a critical value* $T > 0$ *such that* $A^{(t)}$ *is* $\mathcal{IM}$ *for all* $t > T$ *and* $A^{(t)}$ *is not* $\mathcal{IM}$ *for all* $0 \leq t < T$.

The situation for $\mathcal{IM}$ matrices (which includes the symmetric ones) should be contrasted with *doubly nonnegative* (*DN*) matrices, i.e., those matrices that are symmetric positive semi-definite and entry-wise nonnegative. Note that a symmetric $\mathcal{IM}$ matrix is *DN*. For the *n*-by-*n DN* matrices as a class, there is a *critical exponent T* such that $A \in DN$ implies $A^{(t)} \in DN$ for all $t \geq T$ and T is a minimum over all *DN* matrices [30]. That critical exponent is $n-2$ [28,30]. All positive integer Hadamard powers of *DN* matrices are *DN* (because the positive semi-definite matrices are closed under Hadamard product), but it is possible for non-integer powers to leave the class, until the power increases to $n-2$. This, curiously, cannot happen for (symmetric) $\mathcal{IM}$ matrices, as the “critical exponent" for the entire $\mathcal{IM}$ class is simply 1.

As mentioned, a positive matrix (certainly) need not be *SPP* and *SPP* matrices need not be $\mathcal{IM}$. However, it is worth noting that addition of a multiple of the identity can “fix" both of these failures. If $A > 0$ is *n*-by-*n*, it is shown in [50] that there is an $\alpha \geq 0$ such that $\alpha I + A$ is *SPP*; in addition, either A is already *SPP* or a value $\beta > 0$ may be calculated such that $\alpha I + A$ is *SPP* for all $\alpha > \beta$. Moreover, if $A \geq 0$ is *n*-by-*n* and *SPP*, then there is a minimal $\beta \geq 0$ such that $\alpha I + A$ is $\mathcal{IM}$ for all $\alpha > \beta$. In fact, if A is normalized *SPP*, which may always be arranged (Section 4), then $\beta \leq n-2$. This means that if we consider $A \circ B$ with both A and B $\mathcal{IM}$, then either $A \circ B$ will be $\mathcal{IM}$ (if, for example, one of the matrices is already of the form: a large multiple of I plus an $\mathcal{IM}$) or may be made $\mathcal{IM}$ by the addition of a positive diagonal matrix (that is not too big).

9. Perturbation of $\mathcal{IM}$-matrices

How may a given $\mathcal{IM}$ matrix be altered so as to remain $\mathcal{IM}$? And how may a non-$\mathcal{IM}$ matrix be changed so as to become $\mathcal{IM}$. As mentioned theorem 1.8, addition of a nonnegative diagonal matrix to an $\mathcal{I}M$ matrix results in an $\mathcal{I}M$ matrix. We add that some nonnegative matrices that are not $\mathcal{I}M$ may be made $\mathcal{I}M$ via a nonnegative diagonal addition.

If A is the inverse of an M-matrix that has no zero minors, then each entry (column or row) of A may be changed, at least a little, so as to remain $\mathcal{I}M$. By linearity of the determinant, the set of possibilities for a particular entry (column or row) is an interval (convex set), which suggests the question of determination of this interval (convex set).

We begin by discussing positive, rank 1 perturbation of a given $\mathcal{I}M$ matrix. There is a nice result, found in [37].

Theorem 9.1 *Let A be an $\mathcal{I}M$ matrix, let p and q^T be arbitrary nonnegative vectors, and for $t \geq 0$, define*

$$x = Ap,$$

$$y^T = q^T A,$$

and

$$s = 1 + tq^T Ap.$$

We then have

(i) $(A + txy^T)^{-1} = A^{-1} - \frac{1}{s}pq^T$ *is an M-matrix;*
(ii) $(A + txy^T)^{-1}x = \frac{1}{s}p \geq 0$ *and* $y^T(A + txy^T)^{-1} = \frac{1}{s}q^T \geq 0$*;*
(iii) $y^T(A + txy^T)^{-1}x = \frac{1}{s}q^T Ap < \frac{1}{t}$.

The perturbation result may be used to show very simply that so called *strictly ultrametric* (*SU*) matrices [13] are $\mathcal{IM}$ (see Section 12).

If we consider a particular column of an $\mathcal{I}M$ matrix, then the set of replacements of that column that result in an $\mathcal{I}M$ matrix is a convex set. This convex set may be viewed as the intersection of $n^2 - n + 2$ half-spaces via theorem 1.2. Without loss of generality, we may assume the column is the last, so that, partitioned by columns,

$$A(x) = [a_1 a_2 \cdots a_{n-1} x],$$

with $a_1, a_2, \ldots, a_{n-1} \geq 0$. Then the half-spaces are given by the linear constraints

$$x \geq 0,$$

$$(-1)^{i+j+1} \det A(x)(i,j) \geq 0, \quad 1 \leq i \leq n,\ 1 \leq j < n,\ i \neq j,$$

and

$$\det A(x) > 0.$$

A similar analysis may be given for a single off-diagonal entry. It may be taken to be the 1, n entry, so that

$$\mathbf{A}(x) = \begin{bmatrix} a_{11} & x \\ A_{21} & A_{22} \end{bmatrix}$$

with x a scalar. Now the interval for x is determined by the inequalities

$$(-1)^{i+j+1} \det A(x)(i,j) \geq 0, \quad 1 < i \leq n,\ 1 \leq j < n,\ i \neq j,$$

and

$$\det A(x) > 0.$$

In [41] conditions are given on

$$A = [a_1 a_2 \cdots a_{n-1}]$$

such that there exist an $x \geq 0$ so that

$$A(x) = [a_1 a_2 \cdots a_{n-1} x]$$

is $\mathcal{IM}$. If A is re-partitioned as

$$\mathbf{A} = \begin{bmatrix} A_{11} \\ a_{21} \end{bmatrix}$$

in which A_{11} is square, the conditions are the following subset of those in theorem 2.4:

(i) A_{11} is $\mathcal{IM}$;
(ii) $a_{21} \geq 0$;
(iii) $a_{21} A_{11}^{-1} \geq 0$.

What about diagonal perturbation? By theorem 1.8, if $D \in \overline{\mathcal{D}}$, then $A + D$ is *IM* whenever A is. So what if A is not $\mathcal{IM}$? If A is irreducible, we need to assume that $A > 0$. Is anything else necessary to make $A + D$ $\mathcal{IM}$ for some $D \in \overline{\mathcal{D}}$? Interestingly, the answer is no. An $A > 0$ may always be made *PP* and *SPP* by a sufficiently large diagonal addition. In [50] it is shown that

Theorem 9.2 *If A is an n-by-n normalized SPP matrix, then* $A + sI$ *is* $\mathcal{IM}$ *for* $s \geq n - 3$.

Furthermore, the $n - 3$ is best possible. Thus, for any $A > 0$, there is diagonal matrix D such that $A + E$ is $\mathcal{IM}$ for all diagonal $E \geq D$.

Another perturbation question, posed by J. Garloff [25], is whether an "interval" defined by two *IM* matrices is contained in the $\mathcal{IM}$ matrices. If $A, B \in R^{n \times n}$, the *interval from A to B*, denoted by *I(A,B)* is the set of matrices $C = (c_{ij}) \in R^{n \times n}$ satisfying $\min\{a_{ij}, b_{ij}\} \leq c_{ij} \leq \max\{a_{ij}, b_{ij}\}$ for all i and j, while the set of *vertices (vertex matrices) derived from A and B*, denoted *V(A,B)*, is the set of matrices $C = (c_{ij}) \in R^{n \times n}$ such that $c_{ij} = a_{ij}$ or b_{ij} for all i and j. If $C = (c_{ij})$ in which $c_{ij} = min\{a_{ij}, b_{ij}\}$ ($c_{ij} = max\{a_{ij}, b_{ij}\}$), $i, j = 1, \ldots, n$, then C is called the *left endpoint matrix* (*right endpoint matrix*). Note that there are at most 2^{n^2} distinct vertex matrices. We were motivated by a question raised in [24]: given two $\mathcal{IM}$ matrices A and B, when is $I(A, B) \subseteq \mathcal{IM}$? This was answered fully in [49] as follows.

We note first that $I(A, B) \subseteq \mathcal{IM}$ does not hold in general.

Example 9.3 Consider the $\mathcal{IM}$ matrices

$$\mathbf{A} = \begin{bmatrix} 1 & .4 & .3 \\ .6 & 1 & .6 \\ .4 & .6 & 1 \end{bmatrix} \leq \mathbf{B} = \begin{bmatrix} 1 & .9 & .6 \\ .6 & 1 & .6 \\ .4 & .6 & 1 \end{bmatrix}.$$

The matrix $\mathbf{C} = \begin{bmatrix} 1 & .6 & .3 \\ .6 & 1 & .6 \\ .4 & .6 & 1 \end{bmatrix}$ satisfies $A \leq C \leq B$ (entrywise), yet C is not an $\mathcal{IM}$ matrix since $(C^{-1})_{13} > 0$.

A *line* in a matrix is a single row or column. In [49] it was proved

Theorem 9.4 *Suppose that* $A = (a_{ij})$ *and* $B = (b_{ij}) \in \mathcal{IM}$, *and that* $a_{ij} = b_{ij}$ *except perhaps for the entries in one line. Then, for* $0 \le t \le 1$, $tA + (1-t)B \in \mathcal{IM}$.

An immediate consequence is

Theorem 9.5 *Suppose that* $A = (a_{ij})$ *and* $B = (b_{ij}) \in \mathcal{IM}$, *and that* $a_{ij} = b_{ij}$ *for* $(i,j) \neq (r,s)$, *i.e., A and B differ in at most the* r, s *entry. Then,* $tA + (1-t)B \in \mathcal{IM}$ *for* $0 \le t \le 1$.

Theorem 9.4 does not necessarily hold if two matrices differ in more than a line.

Example 9.6 Consider the $\mathcal{IM}$ matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ .6 & 1 & .6 \\ .4 & .6 & 1 \end{bmatrix} \le \mathbf{B} = \begin{bmatrix} 1 & .6 & 0 \\ .6 & 1 & 0 \\ .4 & .6 & 1 \end{bmatrix}.$$

For $0 < t < 1$,

$$tA + (1-t)B = \begin{bmatrix} 1 & .6(1-t) & 0 \\ .6 & 1 & .6t \\ .4 & .6 & 1 \end{bmatrix}$$

cannot possibly be $\mathcal{IM}$ since it is irreducible, but contains a zero entry (see theorem 4.8).

The theorem [49] below characterizes those $\mathcal{IM}$ matrices A, B such that $I(A,B) \subseteq \mathcal{IM}$.

Theorem 9.7 *Let* $A, B \in \mathbb{R}^{n \times n}$. *Then* $I(A,B) \subseteq \mathcal{IM}$ *if and only if* $V(A,B) \subseteq \mathcal{IM}$.

10. Determinantal inequalities

Classical determinantal inequalities associated with the names of Hadamard, Fischer, Koteljanskii, and Szasz have long been known for M-matrices. Because of Jacobi's determinantal identity, it is an easy exercise to show that it follows that these also hold for $\mathcal{IM}$ matrices. The most general of these, associated with the name Koteljanskii, is

Theorem 10.1 *If* J, K *are index sets contained in N and* $A \in \mathcal{IM}$ *is n-by-n, then*

$$\det A[J \cup K] \det A[J \cap K] \le \det A[J] \det A[K].$$

Proof. As in the work of Koteljanskii, this may be proven using Sylvester's determinantal inequality, permutation similarity invariance of $\mathcal{IM}$ and the fact that symmetrically placed almost principal minors of $A \in \mathcal{IM}$ are weakly of the same sign. □

The inequalities of Hadamard, Fischer, and Szasz may be deduced from theorem 10.1 and are stated below.

Corollary 10.1.1 (Hadamard) *If* $A \in \mathcal{IM}$, *then* $\det A \le \prod_{i=1}^{n} a_{ii}$.

Corollary 10.1.2 (Fischer) *If* $A \in \mathcal{IM}$ *and* $J \subseteq N$, *then* $\det A \le \det A[J] \det A[J^c]$.

Corollary 10.1.3 (Szasz) *If $A \in \mathcal{I}M$ and Π_k denotes the product of all the $\binom{n}{k}$ principal minors of A of size k-by-k, then*

$$\Pi_1 \geq (\Pi_2)^{\frac{1}{\binom{n-1}{1}}} \geq (\Pi_3)^{\frac{1}{\binom{n-1}{2}}} \geq \cdots \geq \Pi_n$$

in which $\binom{n}{i}$ denotes the ith binomial coefficient, $i = 0, 1, \ldots, n$.

There are inequalities among products of principal minors of any $A \in \mathcal{I}M$ besides 10.1 and its chordal generalizations. Recently, all such inequalities have been characterized [11]. In order to understand this result, consider two collections $\alpha = \{\alpha_1, \ldots, \alpha_p\}$ and $\beta = \{\beta_1, \ldots, \beta_p\}$ of index sets, $\alpha_i, \beta_j \subseteq N,\ i, j \in \{1, \ldots, p\}$. For any index set $J \subseteq N$ and any collection α, define the two functions

$$f_\alpha(J) \equiv \text{ the number of sets } \alpha_i \text{ such that } J \subseteq \alpha_i$$

and

$$F_\alpha(J) \equiv \text{ the number of sets } \alpha_i \text{ such that } \alpha_i \subseteq J.$$

For the two collections α, β, the following two set-theoretic axioms are important:

$$f_\alpha(\{i\}) = f_\beta(\{i\}),\ i = 1, \ldots, n \tag{ST0}$$

and

$$F_\alpha(J) \geq F_\beta(J),\ \text{for all } J \subseteq N. \tag{ST2}$$

A third axiom (ST1) arises only in the characterization of determinantal inequalities for M-matrices. The result for $\mathcal{I}M$ is then

Theorem 10.2 *The following statements about two collections α, β of index sets are equivalent:*

(i) $\frac{\prod_{i=1}^{p} \det A[\alpha_i]}{\prod_{i=1}^{p} \det A[\beta_i]}$ *is bounded over all $A \in \mathcal{I}M$;*
(ii) $\prod_{i=1}^{p} \det A[\alpha_i] \leq \prod_{i=1}^{p} \det A[\beta_i]$ *for all $A \in \mathcal{I}M$; and*
(iii) *the pair of collections α, β satisfy (ST0) and (ST2).*

The proof is given in [11].

The above results leave only the question of whether there are inequalities involving some non-principal minors in a matrix $A \in \mathcal{I}M$. Because $\mathcal{I}M$ matrices are P-matrices, there are some obvious inequalities, e.g.

$$\det A[\alpha + i, \alpha + j] \det A[\alpha + j, \alpha + i] \leq \det A[\alpha + i] \det A[\alpha + j]$$

whenever $i, j \notin \alpha$. There is also a family of nontrivial inequalities involving almost principal minors of $\mathcal{I}M$ matrices that extend those of theorem 3.6. These inequalities exhibit a form of monotonicity already known for principal minors. Recall that if $\alpha \subseteq \beta \subseteq N$, then

$$\det A[\beta] \leq \det A[\alpha] \left(\prod_{k \in \beta - \alpha} a_{kk} \right) \quad \text{for} \quad A \in \mathcal{I}M.$$

This just follows from $\det A[\beta] \leq \det A[\alpha] \det A[\beta - \alpha]$ and Hadamard's inequality (both are special cases of theorem 10.1). If A were normalized, the product of diagonal entries associated with $\beta - \alpha$ would disappear and the above inequality could be paraphrased "bigger minors are smaller". The same holds for our new inequalities which generalize those given in theorem 3.6.

Theorem 10.3 *Let $\alpha \subseteq \beta \subseteq N$ and suppose that $A = (a_{ij}) \in \mathcal{IM}$ is n-by-n. Then, if* $\det A[\alpha+i, \alpha+j] \neq 0$,

$$\frac{|A[\beta+i, \beta+j]|}{|A[\alpha+i, \alpha+j]|} \leq \frac{A[\beta \cup \{i,j\}]}{A[\alpha \cup \{i,j\}]} \leq \det A[\beta-\alpha] \leq \prod_{i \in \beta-\alpha} a_{ii}$$

whenever $i \neq j$, $i, j \notin \beta$. If $\det A[\alpha+i, \alpha+j] = 0$, *then* $\det A[\beta+i, \beta+j] = 0$, *also, for $i \neq j$, $i, j \notin \beta$.*

We note that other determinantal inequalities for $\mathcal{IM}$ matrices were given in [6].

11. Completion theory

A *partial matrix* is an array with some entries specified, and the other, *unspecified*, entries free to be chosen. A *completion* of a partial matrix is the conventional matrix resulting from a particular choice of values for the unspecified entries. Ref. [35] is a good reference on matrix completion problems.

One topic of interest is the completion of *partial PP (SPP) matrices*, i.e., nonnegative matrices such that every specified path satisfies the *PP* (*SPP*) conditions 4.1 [46]. We make the assumption throughout that all diagonal entries are 1's since *PP* matrices are invariant under positive diagonal scaling.

The *SPP* matrix completion problem is fundamental in considering the (difficult) $\mathcal{IM}$-matrix completion problem [44]. Here, a *partial $\mathcal{IM}$-matrix* is a partial nonnegative matrix in which each fully specified principal submatrix is $\mathcal{IM}$ and we wish to determine whether a given partial $\mathcal{IM}$ matrix can be completed to $\mathcal{IM}$. Since every $\mathcal{IM}$ matrix is *SPP*, for such a completion to exist it is necessary for the partial $\mathcal{IM}$ matrix to be partial *SPP* and for any $\mathcal{IM}$ matrix completion to be *SPP*. Thus, the set of *SPP* completions (if they exist) of a partial $\mathcal{IM}$ matrix is a place to start in the search for an $\mathcal{IM}$ matrix completion and it represents a narrowing of the superset of possible completions. From [46] we have

Theorem 11.1 *Every partial PP (SPP) matrix has a PP (SPP) matrix completion.*

Partial $\mathcal{IM}$-matrices are not necessarily partial *PP* as shown by the following example.

Example 11.2 Consider the partial $\mathcal{IM}$ matrix

$$\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{2} & ? & \frac{1}{9} \\ \frac{1}{2} & 1 & \frac{1}{2} & ? \\ ? & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{9} & ? & \frac{1}{2} & 1 \end{bmatrix}.$$

A is not partial *PP* since $a_{12}a_{23}a_{34} = \frac{1}{8} > \frac{1}{9} = a_{14}$. Thus, A cannot be completed to a *PP* matrix and, since every $\mathcal{IM}$ matrix is *PP*, no $\mathcal{IM}$ matrix completion exists.

In fact, even if a partial $\mathcal{IM}$ matrix is partial *SPP*, it may not have an $\mathcal{IM}$ matrix completion. For instance, if in example 4.7, we let the $(1, 4)$ and $(4, 1)$ entries be unspecified, it can be shown [44] that no $\mathcal{IM}$ matrix completion exists.

A chordal graph [26] is *k-chordal* if no two distinct maximal cliques intersect in more than k vertices. In [45] the symmetric $\mathcal{IM}$ (*SIM*) completion problem was studied and it was shown that, for partial *SIM* matrices, 1-chordal graphs guarantee *SIM* completion.

Theorem 11.3 *Let G be a 1-chordal graph on n vertices. Then every n-by-n partial SIM matrix A, the graph of whose specified entries is G, has a SIM completion. Moreover, there is a unique SIM completion A_1 of A whose inverse entries are 0 in every unspecified position of A, and A_1 is the unique determinant maximizing SIM completion of A.*

However, this is not true for chordal graphs in general. Consider the partial *SIM* matrix (which has a 3-chordal graph with two maximal cliques)

$$\mathbf{A} = \begin{bmatrix} 1 & \frac{9}{40} & \frac{1}{5} & \frac{2}{5} & x \\ \frac{9}{40} & 1 & \frac{3}{10} & \frac{1}{2} & \frac{3}{8} \\ \text{noalign}\ \frac{1}{5} & \frac{3}{10} & 1 & \frac{1}{5} & \frac{3}{4} \\ \frac{2}{5} & \frac{1}{2} & \frac{1}{5} & 1 & \frac{1}{4} \\ x & \frac{3}{8} & \frac{3}{4} & \frac{1}{4} & 1 \end{bmatrix}.$$

In [45], using equation 2.8 and the cofactor form of the inverse, it was shown that no value of x yields a *SIM* completion.

Cycle conditions are derived that are necessary for the general $\mathcal{I}M$ completion problem. Further, these conditions are shown to be necessary and sufficient for completability of a partial symmetric $\mathcal{I}M$ matrix, the graph of whose specified entries is a cycle.

Theorem 11.4 *Let*

$$\mathbf{A} = \begin{bmatrix} 1 & a_1 & & & & a_n \\ b_1 & 1 & a_2 & & ? & \\ & a_2 & \cdot & & & \\ & & \cdot & \cdot\ \cdot & & \\ & & & \cdot\ \cdot & \cdot & \\ & ? & & \cdot & \cdot & a_{n-1} \\ a_n & & & & a_{n-1} & 1 \end{bmatrix}$$

be an n-by-n partial SIM matrix, $n \geq 4$. Then A has a SIM completion if and only if the cycle conditions

$$\prod_{j \neq i} a_j \leq a_i, \quad i = 1, \ldots, n$$

are satisfied.

A *block graph* is a graph built from cliques and (simple) cycles as follows: starting with a clique or cycle, sequentially articulate the "next" clique or simple cycle at no more than one vertex of the current graph. A completability criterion for partial *SIM* matrices with block graphs is as follows.

Theorem 11.5 *Let A be an n-by-n partial SIM matrix, the graph of whose specified entries is a block graph. Then A has a SIM completion if and only if all minimal cycles in G satisfy the cycle conditions.*

In addition, other graphs for which partial *SIM* matrices have *SIM* completions are discussed. SIM matrices were also studies in [14].

12. Miscellaneous topics

12.1. Connections between $\mathcal{IM}$, totally nonnegative, and Jacobi matrices

Let $A \geq 0$. A is *totally nonnegative* (*totally positive*) if all minors of all orders of A are nonnegative (positive); further, A is *oscillatory* if A is totally nonnegative and a power of A is totally positive [24]. In [59] a class of totally nonnegative matrices whose inverses are M-matrices are characterized as follows.

Theorem 12.1 *Suppose A is a nonsingular, totally nonnegative matrix. Then A^{-1} is an M-matrix if and only if* $\det A(i,j) = 0$ *for $i + j = 2k$, in which k is a positive integer, and $i \neq j$.*

A special class of oscillatory matrices that are $\mathcal{IM}$ is also investigated.

A is a *Jacobi* matrix if $a_{ij} = 0$ for $|i-j| > 1$ and a Jacobi matrix of order n is a *normal* Jacobi matrix if $a_{i,i+1}, a_{i+1,i} < 0$ for $i = 1, 2, \ldots, n-1$ [55]. In [55] the results in [59] are extended as follows.

Theorem 12.2 *Let A be a matrix. Consider the following three conditions.*

(i) *A^{-1} is totally nonnegative.*
(ii) *A is an M-matrix.*
(iii) *A is a Jacobi matrix.*

Then any two of the three conditions imply the third.

Theorem 12.3 *Let A be a matrix. Consider the following three conditions.*

(i) *A^{-1} is oscillatory.*
(ii) *A is an M-matrix.*
(iii) *A is a normal Jacobi matrix.*

Then any two of the three conditions imply the third.

Theorem 12.4 *Let A be an M-matrix. Then A^{-1} is totally nonnegative if and only if A is a Jacobi matrix, and A^{-1} is oscillatory if and only if A is a normal Jacobi matrix.*

In [73] the class of totally nonnegative matrices that are $\mathcal{IM}$ are fully characterized as follows.

Theorem 12.5 *Let A be a nonsingular totally nonnegative n-by-n matrix. Then the following properties are equivalent.*

(i) *A^{-1} is an M-matrix.*
(ii) $\det A(i,j) = 0$ *if* $|i-j| = 2$.
(iii) *A^{-1} is a tridiagonal matrix.*
(iv) *For any* $k \in \{1, 2, \ldots, n-1\}$, $\det A([i_1, \ldots, i_k], [j_1, \ldots, j_k]) = 0$ *if* $|i_1 - j_l| > 1$ *for some* $l \in \{1, \ldots, k\}$.

We have the following characterization of $\mathcal{IM}$ matrices that are totally positive [73].

Theorem 12.6 *Let A be a nonsingular n-by-n M-matrix. Then the following properties are equivalent.*

(i) *A^{-1} is a totally positive matrix.*
(ii) *A is a tridiagonal matrix.*

12.2. Graph theoretic characterizations of $\mathcal{IM}$ matrices

In [58] graph theoretic characterizations of (0, 1)-matrices that are $\mathcal{IM}$ matrices were obtained. A matrix $A \in \mathbb{R}^{n \times n}$ is *essentially triangular* if for some permutation matrix P, $P^{-1}AP$ is a triangular matrix. Let A be a n-by-n matrix and $G(A) = (V(A) = N, E(A))$ be its associated (adjacency) graph, i.e., $(i,j) \in E(A)$ if and only if $a_{ij} \neq 0$ where $V(A)$ is its set of vertices and $E(A)$ is its set of edges. A path from vertex i to vertex j is called an *(i,j)-path*; a path of length k, k being the number of directed edges in the path, is called an *k-path*; and a path of length k from i to j is called*(i,j | k)-path*. Let H denote the join of the (disjoint) graphs G_1 and G_2 in which G_1 consists of a single vertex and G_2 consists of a 2-path. (H is called a *buckle*.) Then

Theorem 12.7 *An essentially triangular* $(0, 1)$*-matrix* A *is* $\mathcal{IM}$ *if and only if the associated graph is a partial order and, for every* i, j, k*, if an* $(i, j|k)$*-path in* $G(A)$ *is a maximal* (i, j)*-path, then it is a unique* k*-path.*

Theorem 12.8 A $(0, 1)$*-matrix* A *is* $\mathcal{IM}$ *if and only if* $G(A)$ *induces a partial order on its vertices that contains no buckles as induced subgraphs.*

A directed graph G with vertex set $V(G)$ and edge set $E(G)$ is *transitive* if $(i, j), (j, k) \in E(G)$ implies $(i, k) \in E(G)$. For an n-by-n matrix A, we say A is *transitive* if $G(A)$ is transitive. In [53] we have the following result that relates the zero pattern of a transitive invertible matrix and that of its inverse.

Theorem 12.9 *Suppose that* $A = (a_{ij})$ *is an* n*-by-*n $(0, 1)$ *invertible transitive matrix with* $B = (b_{ij}) = A^{-1}$*. If* $i \neq j$ *and* $a_{ij} = 0$*, then* $b_{ij} = 0$*.*

For a matrix B, let $B_{\mathcal{IM}}$ denote the set

$$B_{\mathcal{IM}} = \{\alpha \in \mathbb{R} : \alpha I + B \in \mathcal{IM}\}.$$

$B_{\mathcal{I}M}$ is a (possibly empty) ray on the real line, bounded from below and is nonempty if and only if B is nonnegative and transitive. Then,

Theorem 12.10 *Let* B *be an* n*-by-*n *nonnegative transitive matrix. Then the ray* $B_{\mathcal{IM}}$ *is closed, i.e.,* $B_{\mathcal{IM}} = [\alpha_0, \infty)$*, if and only if there exists* $\beta_0 \in \mathbb{R}$ *such that*

(i) $B + \beta_0 I \notin \mathcal{IM}$,
(ii) $B + \beta_0 I$ *is positive stable and so are its principal submatrices of order* $n - 1$*.*

The authors express the infimum of $B_{\mathcal{IM}}$ as a maximal root of a polynomial which depends on the matrix B, differentiating between the cases in which $B_{\mathcal{IM}}$ is open or closed. Lastly, they apply these results to give a different proof of 12.6.

In [56] 12.8 was generalized as follows. Consider a nonnegative square matrix $A = (a_{ij})$ and its associated adjacency graph $G(A) = (V(A), E(A))$. The *weight* of an edge (i, j) of $G(A)$ is then a_{ij}. The *weight* of a directed path is the product of the weights of its edges. The *weight* of a set of paths is the sum of the weights of its paths. For esssentially triangular $\mathcal{IM}$ matrices, the following result was proved.

Theorem 12.11 *Let* A *be an essentially triangular nonnegative nonsingular matrix, and let* A_1 *be its normalized matrix. Then* $A_1 \in \mathcal{IM}$ *if and only if* $G(A_1)$ *is a partial order and the weight of the collection of even paths between any two vertices in* $G(A)$ *does not exceed the weight of its collection of odd paths.*

Let A be an n-by-n matrix and let γ be a subgraph of $G(A)$ and let $A(\gamma)$ denote the principal minor of A defined by the set of vertices of G which are not in γ.

Let A be a square matrix and let p be a simple path in $G(A)$. If $W(p)$ is the weight of p, then the *generalized weight* of p is defined by $w(p) \equiv W(p)A(p)$.

A graph-theoretic characterization of general $\mathcal{IM}$ matrices was then given.

Theorem 12.12 *A nonnegative square matrix* A *is* $\mathcal{IM}$ *if and only if*

(i) *For every two distinct vertices* i *and* j *in* $G(A)$*, the sum of the even* (i, j)*-paths does not exceed that of the odd* (i, j)*-paths.*
(ii) *All principal minors of* A *are positive.*

A special class of inverse $\mathcal{IM}$ matrices was also characterized.

12.3. Linear interpolation problems

The linear interpolation problem for a class of matrices $\mathcal{C}$ asks for which vectors x, y there exists a matrix $A \in \mathcal{C}$ such that $Ax = y$. In [48] the linear interpolation problem is solved for several important classes of matrices, one of which is $\mathcal{M}$ (and hence it is solved for $\mathcal{IM}$ also). In addition, a transformational characterization is given for M-matrices that refines the known such characterization for P-matrices.

12.4. Newton matrices

For an n-by-n real matrix A with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ let

$$S_k = \sum_{i_1<\cdots<i_k} \lambda_{i_1} \cdots \lambda_{i_k}$$

and $c_k = \frac{1}{\binom{n}{k}} S_k$, with $c_0 = 1$. The matrix A is a *Newton matrix* [40] if $c_k^2 \geq c_{k-1}c_{k+1}$, $k = 1, \ldots, n-1$. If each $c_k > 0$, A is called *p-Newton*. Using the immanantal results of [39], it was observed in [27] that M-matrices are p-Newton. Since p-Newton matrices are closed under inversion [40], it follows that $\mathcal{IM}$ matrices are also p-Newton.

12.5. Perron complements of $\mathcal{IM}$ matrices

In [61,62] the notion of Perron complements was introduced. Specifically, for an n-by-n nonnegative irreducible matrix A, $\beta \subset N$, and $\alpha = N - \beta$, the Perron complement of $A[\beta]$ in A is defined to be

$$\mathcal{P}(A/A[\beta]) = A[\alpha] + A[\alpha, \beta](\rho(A)I - A[\beta])^{-1}A[\beta, \alpha].$$

Among other things, it was shown that $\rho(\mathcal{P}(A/A[\beta])) = \rho(A)$ and that if A is row stochastic, then so is $\mathcal{P}(A/A[\beta])$. These results were applied to obtain an algorithm for computing the stationary distribution vector for a Markov chain.

In [52] the following question was investigated: when are Perron complements primitive or just irreducible? Their answer settled some questions posed in [61,62].

In [67] it is established that if A is an irreducible $\mathcal{I}M$ matrix, then its Perron complements are also $\mathcal{I}M$. In fact,

Theorem 12.13 *Let A be an irreducible $\mathcal{IM}$ matrix, $\beta \subset N$, and $\alpha = N - \beta$. Then, for any $t \in [\rho(A), \infty)$, the matrix*

$$\mathcal{P}_t(A/A[\beta]) = A[\alpha] - A[\alpha, \beta](tI - A[\beta])^{-1}A[\beta, \alpha]$$

is invertible and is an $\mathcal{I}M$ matrix. In particular, the Perron complement $\mathcal{P}(A/A[\beta])$ $(= \mathcal{P}_1(A/A[\beta]))$ is $\mathcal{I}M$.

Also, for irreducible $\mathcal{I}M$ matrices whose inverses are tridiagonal, the following result was proved.

Theorem 12.14 *Let $A \in \mathcal{I}M$ with A^{-1} irreducible and tridiagonal. Then, for any subsets $\beta \subset N$ and $\alpha = N - \beta$, the Perron complement*

$$\mathcal{P}(A/A[\beta]) = A[\alpha] + A[\alpha, \beta](I - A[\beta])^{-1}A[\beta, \alpha]$$

is an $\mathcal{I}M$ matrix whose inverse is irreducible and tridiagonal and hence is (also) totally nonnegative, i.e., all minors are nonnegative.

Further, it was shown that for an n-by-n $\mathcal{I}M$ matrix A, the inverse of associated principal submatrices of A are sandwiched between the inverses of the Perron complements of A and the inverses of the corresponding Schur complements of A. Lastly, directed graphs of inverses of Perron complements of irreducible $\mathcal{I}M$ matrices were investigated.

12.6. Products of $\mathcal{IM}$ matrices

Let $A \in \mathbb{R}^{n \times n}$. In [21] the following question is investigated: when is the matrix A a product of M-matrices ($\mathcal{IM}$ matrices)? Let Π denote the set of $\mathcal{IM}$ matrices that are finite products of $\mathcal{IM}$ matrices. They prove that $A \in \Pi$ if and only if $A = (L_1U_1)(L_2U_2)\cdots(L_kU_k)$ in which L_i (U_i) is a lower (upper) triangular $\mathcal{IM}$ matrix, $i = 1, \ldots, n$ if and only if A is a product of elementary matrices that are $\mathcal{IM}$.

Let $\mathcal{R}(A) = \frac{\det A}{\prod_{i=1}^{n} a_{ii}}$ and let $\Pi_{\mathcal{R}}$ denote the set of $A \in \mathcal{IM}$ with $\mathcal{R}(A) = 1$. A necessary and sufficient combinatorial condition is given for a matrix to be in $\Pi_{\mathcal{R}}$. Toeplitz $\mathcal{IM}$ matrices are also studied.

More detailed analysis of products of $\mathcal{IM}$ matrices and of M-matrices, from a different perspective, is found in [43] which followed in [42].

12.7. Topological closure of $\mathcal{IM}$ matrices

A matrix A belongs to the closure of the $\mathcal{IM}$ matrices if A is the limit of a convergent sequence of $\mathcal{IM}$ matrices. If so, we write $A \in \overline{\mathcal{IM}}$. The following theorem was proved in [16].

Theorem 12.15 *Let A be a p-by-p $\mathcal{IM}$ matrix, and let Q be a p-by-n nonnegative matrix with exactly one nonzero entry in each column. Then*

$$Q^TAQ + D$$

is an $\mathcal{IM}$ matrix for any n-by-n positive diagonal matrix D.

Several important facts about $\mathcal{IM}$ matrices follow as special cases.

The following characterization of $\overline{\mathcal{IM}}$ was obtained in [16].

Theorem 12.16 *Suppose A is a nonnegative n-by-n matrix. Then the folowing statements are equivalent:*

(i) $A \in \overline{\mathcal{IM}}$,
(ii) $(A + D)^{-1} \leq D^{-1}$ *for each positive diagonal matrix D,*
(iii) $(A + D)^{-1}$ *belongs to $\mathcal{M}$ for each positive diagonal matrix D,*
(iv) $(A + \alpha I)^{-1}$ *belongs to $\mathcal{M}$ for all $\alpha > 0$,*
(v) $(A + D)^{-1}A \geq 0$ *for each positive diagonal matrix D,*
(vi) $(I + cA)^{-1} \leq I$ *for all $c > 0$, and*
(vii) $cA^2(I + cA)^{-1} \leq A$ *for alll $c > 0$.*

The theorem allows the authors to characterize nilpotent matrices on the boundary of $\overline{\mathcal{IM}}$.

Denote by $\mathcal{L}_{rn}$ the set of all nonnegative r-by-n matrices, $r \leq n$, which contain exactly one nonzero entry in each column. If the dimensions are not specified, write simply $\mathcal{L}$. In [19] a matrix A belonging to $\overline{\mathcal{IM}}$ was shown to have an explicit form. Specifically, they prove

Theorem 12.17 *An n-by-n matrix A is in $\overline{\mathcal{IM}}$ if and only if there exists a permutation matrix P, a diagonal matrix D with positive diagonal entries, a matrix $B \in \mathcal{IM}$, and a matrix $Q \in \mathcal{L}$ without a zero row such that*

$$\mathbf{D}^{-1}PAP^TD = \begin{bmatrix} 0 & UBQ & UBV + W \\ 0 & Q^TBQ & Q^TBV \\ 0 & 0 & 0 \end{bmatrix}$$

for some nonnegative matrices U, V, and W.

In the partitioning, any one or two of the three block rows (and their corresponding block columns) can be void.

The above theorem allows the characterization of singular matrices in $\overline{\mathcal{IM}}$.

If A is a square matrix, the smallest integer k for which rank (A^k) = rank (A^{k+1}) is called the *index* of A, denoted by index $(A) = k$. The *Drazin inverse* of A is the unique matrix A^D such that

(i) $A^{k+1}A^D = A^k$;
(ii) $A^DAA^D = A^D$;
(iii) $AA^D = A^DA$.

In [20] the Drazin inverse of a matrix belonging to $\overline{\mathcal{IM}}$ is determined and, for such a matrix A, the arrangement of the nonzero entries of the powers $A^2, A^3, \ldots$ is shown to be invariant.

12.8. Tridiagonal, triangular, and reducible $\mathcal{IM}$ Matrices

In [31] $\mathcal{IM}$ matrices whose nonzero entries have particular patterns were studied. Firstly, tridiagonal $\mathcal{IM}$ matrices were characterized as follows.

Theorem 12.18 *Let A be a nonnegative, nonsingular, tridiagonal n-by-n matrix. Then the following statements are equivalent:*

(i) *A is an $\mathcal{IM}$ matrix.*
(ii) *All principal minors of A are positive, and A is the direct sum of matrices of the following types: (a) diagonal matrices, (b) 2-by-2 positive matrices, or (c) matrices of the form*

$$\mathbf{A} = \begin{bmatrix} d_1 & a_1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & d_2 & 0 & 0 & & & & & & \cdot \\ 0 & b_1 & d_3 & a_2 & 0 & & & & & \cdot \\ \cdot & 0 & 0 & d_4 & 0 & 0 & & & & \cdot \\ \cdot & & 0 & b_2 & d_5 & a_3 & 0 & & & \cdot \\ \cdot & & & 0 & 0 & d_6 & 0 & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & & & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & & & & \cdot & \cdot & a_t \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & b_u & d_s \end{bmatrix}$$

where $a_t = 0$ and $u = \frac{s-1}{s}$ when s is odd, and $b_u = 0$ and $t = \frac{s}{2}$ when s is even.

A nonempty subset K of $\mathbb{R}^n$ which is closed under addition and nonnegative scalar multiplication is called a (convex) *cone* in $\mathbb{R}^n$. If a cone K is also closed topologically, has a nonempty interior, and satisfies $K \cap (-K) = \phi$, then K is called a *proper cone*. If A is a real matrix, then the set $K(A) = \{Ax : x \geq 0\}$ is a *polyhedral cone*, i.e., a set of nonnegative linear combinations of a finite set S of vectors in $\mathbb{R}^n$. Using these notions, the following geometric characterization of upper triangular $\mathcal{IM}$ matrices was given [31].

Theorem 12.19 *Let A be a nonnegative upper triangular n-by-n matrix with 1's along the diagonal. Then the following statements are equivalent.*

(i) *A is an $\mathcal{IM}$ matrix.*
(ii) *$Ae_k - e_k \in K(Ae_1, Ae_2, \ldots, Ae_{k-1})$ for $k = 2, \ldots, n$.*

Also, for certain types of reducible matrices, both necessary conditions and sufficient conditions for the reducible matrix to be $\mathcal{IM}$ are provided.

12.9. Ultrametric matrices

$A = (a_{ij})$ is a *strictly ultrametric* matrix [63] if A is real symmetric and (entrywise) nonnegative and satisfies

(i) $a_{ik} \geq \min(a_{ij}, a_{jk})$ for all i, j, k; and
(ii) $a_{ii} > \max_{k \neq i} a_{ik}$ for all i.

In [63] it was shown that if $A = (a_{ij})$ is strictly ultrametric, then A is nonsingular and $A^{-1} = (\alpha_{ij})$ is a strictly diagonally dominant Stieltjes matrix satisfying $\alpha_{ij} = 0$ if and only if $a_{ij} = 0$. The proof utilized tools from topology and real analysis. Then, a simpler proof that relied on tools from linear algebra was given in [68]. Such statements, and some following, also follow easily from Theorem 9.1, giving a yet simpler proof.

In [13] new characterizations of matrices whose inverse is a weakly diagonally dominant symmetric M-matrix are obtained. The result in [63] is shown to follow as a corollary. In [13] connections between ultrametric matrices and Euclidean point spaces are explored. See also [12].

There has been a great deal of work devoted to generalizations of ultrametric matrices. In [69,64], nonsymmetric ultrametric (called *generalized ultrametric*) matrices were independently defined and characterized. Necessary and sufficient conditions are given for regularity and, in the case of regularity, the inverse is shown to be a row and column diagonally dominant M-matrix. In [65] nonnegative matrices whose inverses are M-matrices with unipathic graphs (a digraph is *unipathic* if there is at most one simple path from any vertex i to any vertex k) are characterized. A symmetric ultrametric matrix A is *special* [15] if, for all i,

$$a_{ii} = \max_{k \neq i} a_{ik}.$$

In [15] graphical characterizations of special ultrametric matrices are obtained. In [66], by using dyadic trees, a new class of nonnegative matrices is introduced and it is shown that their inverses are column diagonally dominant M-matrices.

In [75] a polynomial time spectral decomposition algorithm to determine whether a specified, real symmetric matrix is a member of several closely related classes is obtained. One of these classes is the class of strictly ultrametric matrices.

In [76] it is shown that a classical inequality due to Hardy can be interpreted in terms of symmetric ultrametric matrices and then a generalization of this inequality can be derived for a particular case.

Remark. Other papers that investigate $\mathcal{IM}$ matrices are [2,3,10,54,72].

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