Existence of Global Classical Solutions of the Initial-Boundary Value Problem for Some Nonlinear Wave Equations*

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0. INTRODUCTION

In this paper we are concerned with the existence of global classical solutions of the initial-boundary value problem for the following nonlinear wave equations

\[ u_{tt} - Au + \sigma(x, u_t) + g(x, u) = f(x, t) \quad \text{on} \quad \Omega \times [0, \infty) \quad (0.1) \]

\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \text{and} \quad u(x, t)|_{\partial \Omega} = 0, \quad (0.2) \]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), \( N = 3 \), with sufficiently smooth boundary \( \partial \Omega \), and \( \sigma(x, v) \) and \( g(x, u) \) are smooth functions satisfying

\[ \sigma(x, 0) = g(x, 0) = 0 \quad \text{and} \quad \sigma(x, v)v \geq 0. \quad (0.3) \]

In fact we shall assume \( \sigma(x, v) \) is strictly increasing in \( v \) and also impose a certain smallness condition on the initial data \((u_0, u_1)\) and the forcing term \( f \), but it is weaker compared with earlier works (see Theorem 1.1).

For a moment we assume further \( g(x, u)u \geq 0 \) for simplicity. When \( N = 1 \) the global existence of the smooth solutions for (0.1)–(0.2) is easily proved by a standard energy method (cf. Strauss [12], Nakao [7]). When \( N = 2 \) we can also prove a similar result if \( \sigma(x, v) \) satisfies the conditions:

\[ \frac{\partial \sigma(x, v)}{\partial v} \geq 0, \]

\[ |\sigma(x, v)| \leq C(1 + |v|^3) \]

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and

\[
\left| \frac{\partial^2}{\partial v^2} \sigma(x, v) \right| ^2 \leq C \left( 1 + \left| \frac{\partial}{\partial v} \sigma(x, v) \right| \right),
\]

though such a result seems not to have been published (cf. Sather [9]). When \( N \geq 3 \) and \( \sigma(x, v) \) is genuinely nonlinear, however, we have very little result on the global existence of smooth solutions to the problem (0.1)–(0.2).

The object of this paper is to derive an existence theorem of global smooth (classical) solutions for a set \( \mathcal{S} \) of initial data \((u_0, u_1)\) under a strictly monotonicity condition on \( \sigma(x, v) \):

\[
\frac{\partial}{\partial v} \sigma(x, v) \geq \varepsilon_0(L) > 0 \quad \text{if} \quad |v| \leq L (3L > 0).
\]

We would emphasize that \( \mathcal{S} \) is small but unbounded in a certain sense and also our result is new even for the case that \( \sigma(x, v) \) is linear in \( v \), that is, \( \sigma(x, v) = a(x)v, a(x) \geq \varepsilon_0 > 0 \).

It is true that we can show rather easily, by applying a standard contraction principle, that under the condition (0.5) the problem (0.1)–(0.2) admits a global classical solution for each smooth initial datum satisfying certain compatibility condition and a “smallness” condition in a strong norm, say, \( H_4 \times H_5 \) norm (when \( N = 3 \)). In fact, similar results concerning small amplitude solutions are known even for stronger or fully nonlinear equations (cf. Shibata [11], Shatah [10], Matsumura [4], etc.) But, we emphasize, our set \( \mathcal{S} \) of initial data which allow the global existence of classical solutions is unbounded in the space \( H_4 \times H_5 \). For a typical case \( \sigma(x, v) = v + v^3 \) (\( N = 3 \)), \( \mathcal{S} \) is unbounded even in the space \( H_3 \times H_2 \) (see Corollary 1.1). We note also that we make no growth conditions on \( \sigma(x, v) \) and \( g(x, u) \) in \( v \) and \( u \), respectively.

Recently, Ebihara [2] proved a related result for the case \( g(x, u) = f(x, t) = 0 \) by use of a penalty method. There, it is shown that there exists an unbounded set \( \mathcal{W} \subset H_3 \times H_4 \) (if \( N = 3 \)) such that for \( (u_0, u_1) \in \mathcal{W} \) the problem (0.1)–(0.2) (with \( g = f = 0 \)) admits a global classical solution. The set \( \mathcal{W} \) in [2] is defined through a qualitative behaviour of certain modified (approximate) solutions and hence the definition of \( \mathcal{W} \) is ambiguous in the sense that we have no means to characterize the unboundedness of \( \mathcal{W} \). While, we employ here a more natural energy method (cf. Amerio and Prouse [1], Nakao [5]) to show the global existence and boundedness (or exponential decay) of a classical solution of (0.1)–(0.2) for \( (u_0, u_1) \in \mathcal{S}, \mathcal{S} \) being clearly defined in a quantitative way. Since our set \( \mathcal{S} \) is unbounded in \( H_4 \times H_5 \) (possibly in \( H_3 \times H_2 \)) it is trivially unbounded in \( H_5 \times H_4 \). Our
method goes well even for the case $g(x, u) \neq 0$ and we can treat, for a typical example, the case that $\sigma(x, v) = v$ and $g(x, u) = \pm u^5$ (see Corollary 1.2. See also our forthcoming paper [8], where the case $\sigma(x, v) = a(x)v, a(x) \geq 0,$ is considered.) It may be also worth mentioning that since our method does not utilize the decay property of solutions, the forcing term $f$ is not required to decay as $t \to \infty$ and hence, periodic functions in $t,$ for example, are allowed.

Although our method could be applied to the equations in any dimension $N$ we restrict ourselves to the typical and most important case $N = 3$ in order to make the essential feature clear. The readers can consult [14, 2, 8], etc., on the generalization of our result to the higher dimensional cases $N \geq 4.$ Moreover, we assume in what follows that $\sigma(x, v) = \sigma(v)$ and $g(x, u) = g(u)$ (independent of $x$) to simplify the notation (the dependence of $\sigma, g$ on $x$ causes no difficulty in our argument).

1. Preliminaries and Result

The functions considered are all real valued. The function spaces we use are all familiar and we omit the definitions of them. Let us begin with the following well known and powerful lemma.

**Lemma 1.1 (Gagliardo and Nirenberg).** It holds that

$$\|u\|_q \leq \text{const.} \|u\|_{H^m}^{\theta} \|u\|_{L^r}^{-\theta} \quad \text{for} \quad u \in H^m(\Omega)$$

with $\theta = (1/r - 1/q)(m/N + 1/r - 1/2)^{-1},$ where $m$ is a positive integer and $r, q$ should satisfy the following conditions: $1 < r < q < 2N/(N - 2m)$ if $N > 2m,$ $1 \leq r < q < \infty$ if $N = 2m,$ and $1 \leq r < q < \infty$ if $1 \leq N < 2m.$ ($\| \cdot \|_r$ denotes $L^r(\Omega)$ norm.)

We use the following elementary lemma to show the exponential decay of the solutions.

**Lemma 1.2 (Cf. [5]).** Let $\phi(t)$ be a nonnegative function on $R^+ = [0, \infty),$ satisfying

$$\sup_{t \leq s \leq t + 1} \phi(s) \leq C_0(\phi(t) - \phi(t + 1)) + C_1 e^{-\lambda t}, \quad t > 0,$$

with $C_0 > 0, C_1 \geq 0$ and $\lambda \geq 0.$

Then, it holds that

$$\phi(t) \leq \max\{ \sup_{0 \leq s \leq 1} \phi(s), C_1 \} \quad \text{for} \quad t \geq 0$$
where we set \( \nu = \log \left( \frac{1 + C_0}{C_1} \right) > 0 \) and

\[
p(t) = \begin{cases} 
\frac{C_1}{(1 + C_0)(\nu - \lambda)} \cdot e^{-\lambda t} & \text{if } \nu > \lambda, \\
\frac{C_1}{(1 + C_0)} \cdot te^{-\lambda t} & \text{if } \nu = \lambda, \\
\frac{C_1}{(1 + C_0)(\lambda - \nu)} \cdot e^{\lambda - \nu} e^{-\lambda t} & \text{if } \nu < \lambda.
\end{cases}
\]

**Proof.** If \( \phi(t) \leq \phi(t + 1) \) for some \( t \), then (1.2) implies

\[
\phi(t + 1) \leq C_1 e^{-\lambda t} \leq C_1.
\]

From this we conclude (1.3). From (1.2) we see

\[
\phi(t + 1) \leq \frac{C_0}{(C_0 + 1)} \cdot \phi(t) + \frac{C_1}{(C_0 + 1)} \cdot e^{-\lambda t} \quad \text{for } t \geq 0
\]

which gives (1.4) easily.

Now, we state our hypotheses on \( \sigma(v), g(u), \) and \( f(x, t) \).

**HYPOTHESIS I.** \( \sigma(v) \) is a smooth (\( C^3 \)-class is sufficient) function on \( \mathbb{R} \), and there exists \( L > 0 \) (possibly \( L = \infty \)) such that the following conditions hold:

(i) \( \sigma(0) = 0 \),

(ii) \( k_0 \searrow \sigma'(v) \) and \( \sigma'(v) \leq \overline{k}_0 \sigma(v) \) if \( |v| < L \),

(iii) \( \sigma(v)^2 \leq k_1 (\sigma(v)v)^{2(r+1)/r+2} + \overline{k}_1 |v|^2 \) if \( |v| < L \) with some \( 0 \leq r \leq 2 \), and

(iv) \( \sigma^{(i)}(v)^2 \leq k_i \sigma'(v) \), \( i = 2, 3 \), if \( |v| < L \), where \( k_i, i = 0, 1, 2, 3 \), and \( \overline{k}_i, i = 0, 1 \), are nonnegative constants (\( k_0 > 0 \)) which may depend on \( L \).

Note that we may assume \( k_0 \geq 1 \) in (ii) and also that we have from (ii) and (iii)

\[
|\sigma(v)|^2 \leq k_1^{r+2} |v|^{2(r+1)} + \overline{k}_1 |v|^2 \tag{1.5}
\]

and

\[
|\sigma'(v)|^2 \leq \overline{k}_0^2 (k_1^{r+2} |v|^{2r} + \overline{k}_1) \tag{1.6}
\]

if \( |v| < L \).

**HYPOTHESIS II.** \( g(u) \) is a smooth (\( C^3 \)-class is sufficient) function on \( \mathbb{R} \), and there exists \( M > 0 \) (possibly \( M = \infty \)) such that the following conditions hold:
(i) \( g(0) = 0 \),

(ii) \( C_0^{-1} \| \nabla u \|_2^2 \leq \frac{1}{2} \| \nabla u \|_2^2 + \int_{\Omega} \int_{\Omega} g(\eta) \, d\eta \, dx \leq C_0 \| \nabla u \|_2^2 + \int_{\Omega} g(u) \, u \, dx \) for \( u \in H_0^1 \cap L^\infty(\Omega) \) with \( |u| < M \),

(iii) \( |g(u)| \leq C_1 |u| (1 + |u|^{2+1}) \) if \( |u| < M \), with some \( \alpha, 0 \leq \alpha < 1 \), and

(iv) \( |g^{(i)}(u)| \leq C_1 (1 + |u|^{2+2-i}) \), \( i = 1, 2, 3 \), if \( |u| < M \)

with the same \( \alpha \) as in (iii), where in the above \( C_i, i = 0, 1 \), are positive constants which may depend on \( M \).

**HYPOTHESIS III.** \( f(\cdot) \) belongs to
\[ C^3([0, \infty); L^2(\Omega)) \cap C^1([0, \infty); H_0^1) \cap C^0([0, \infty); H_2) \]
and satisfies
\[ \| D_i^i f(t) \|_2 \leq d_i e^{-\lambda_i t}, \quad i = 0, 1, 2, 3, \quad (D_i^i = \partial^i / \partial t^i) \]
with some \( d_i \geq 0 \) and \( \lambda_i \geq 0 \).

Under the Hypotheses I, II, and III we consider the problem

\[ (1.7) \begin{align*}
(u_{tt} - \Delta u + \sigma(u_t) + g(u) &= f(x, t) \quad \text{on} \quad \Omega \times [0, \infty) \\
(\sigma, u_0(x), u_1(x, 0) &= u_0(x), \quad u(x, t)|_{\partial \Omega} = 0.
\end{align*} \]

When the time interval \([0, \infty)\) is replaced by \([0, T)\) or \([0, T]\) we also call the problem as (1.7) or (1Bv).

For a smooth solution of (1Bv) to exist the initial datum \((u_0, u_1)\) or \((u_0, u_1, f(0))\) must satisfy an appropriate compatibility condition with the boundary condition \(u|_{\partial \Omega} = 0\). Let us state such a one under the assumption \((u_0, u_1) \in H_4 \times H_3\). Noting that

\[ H_4 \times H_3 \subset C^2(\bar{\Omega}) \times C^1(\bar{\Omega}) \quad \text{(recall \( N = 3 \))} \]

we define the functions \( u_i, i = 2, 3, 4 \), on \( \Omega \) as follows

\[ u_2 = \Delta u_0 - \sigma(u_1) - g(u_0) + f(x, 0), \]

\[ u_3 = \Delta u_1 - \sigma'(u_1)u_2 - g'(u_0)u_1 + f_t(x, 0) \]

and

\[ u_4 = \Delta u_2 - \sigma''(u_1)(u_2)^2 - \sigma'(u_1)u_3 - g'(u_0)u_2 - g''(u_0)(u_1)^2 + f_{tt}(x, 0). \]
It is easy to see that if \( u(t) \) is a smooth solution of (IBV), then
\[
D_i^j u(t) = u_i, \quad i = 0, 1, 2, 3, 4. \tag{1.9}
\]

**DEFINITION 1.1.** We say the initial datum \((u_0, u_1)\) satisfies the compatibility condition of order \( m(\leq 4) \) if \( u_i, \ i = 0, \ldots, m, \) belong(s) to \( H_i^0 \).

It is also easy to see that if \( f \equiv 0, (u_0, u_1) \in H_4 \times H_3 \) satisfies the compatibility condition of order 3 if and only if
\[
(u_0, u_1) \in D(A^2) \times D(A^{3/2}),
\]
where \( A \) is the selfadjoint operator in \( L^2(\Omega) \) defined by
\[
A = -\Delta \quad \text{with} \quad D(A) = H_2(\Omega) \cap H_1^0(\Omega).
\]

The following existence theorem of local smooth solutions is standard (cf. Inou [3], v. Wahl [15], Pecher [14], etc.).

**PROPOSITION 1.1 (Existence of Local Solution).** Suppose that the Hypotheses I, II, and III are fulfilled. Let \((u_0, u_1)\) belong to \( H_4 \times H_3 \) and satisfy the compatibility condition of order 3. We assume moreover the inequalities
\[
\|u_0\|_\infty < M \quad \text{and} \quad \|u_1\|_\infty < L \tag{1.10}
\]
hold.

Then, there exists \( T = T(\|u_0\|_{H_4}', \|u_1\|_{H_3}, M - \|u_0\|_\infty, L - \|u_1\|_\infty) > 0 \) such that the problem (IBV) admits a unique solution \( u \) in the class
\[
C^4([0, T); L^2) \cap \bigcap_{i=0}^3 C^{3-i}([0, T); H_{i+1} \cap H_1^0). \tag{1.11}
\]

By a standard argument on the basis of Proposition 1.1, it suffices for the existence of a global solution in the class (1.11) with \( T = \infty \) to show that for each \( T > 0 \) there exist \( M_0 = M_0(T) < M, L_0 = L(T) < L \) and \( K = K(T) < \infty \) such that
\[
\sup_{0 \leq t \leq T} \|u(t)\|_\infty < M_0, \quad \sup_{0 \leq t \leq T} \|u_1(t)\|_\infty < L_0 \tag{1.12}
\]
and
\[
\sup_{0 \leq t \leq T} \sum_{i=0}^4 \|D_i^j u(t)\|_{H_{4-i}} \leq K(T) \quad (H_0 \equiv L^2) \tag{1.13}
\]
hold for assumed smooth solutions \( u(t) \) on \([0, T + \varepsilon), \varepsilon > 0\).
To state our result we introduce the following notations or quantities.

\[ E(u(t)) = \frac{1}{2} \{ \| u_i(t) \|_2^2 + \| \nabla u(t) \|_2^2 \} + \int_{\Omega} \int_{0}^{\tau} g(\eta) \, d\eta \, dx, \]

\[ E(D_i^j u(t)) = \frac{1}{2} \{ \| D_i^j u(t) \|_2^2 + \| \nabla D_i^j u(t) \|_2^2 \}, \quad i = 1, 2, 3, \]

\[ I_i^2 = E(D_i^j u(0)) = \frac{1}{2} \{ \| u_{i+1} \|_2^2 + \| \nabla u_i \|_2^2 \}, \quad i = 1, 2, 3, \]

\[ I_0^2 = E(u(0)) = \frac{1}{2} \{ \| u_1 \|_2^2 + \| \nabla u_0 \|_2^2 \} + \int_{\Omega} \int_{0}^{\tau} g(\eta) \, d\eta \, dx, \]

\[ Q_0^2 = I_0^2 + k_0 d_0^2, \]

\[ B_0^2 = \max \{ Q_0^2, C^3(k^{2(r+2)} d_0^{2(r+1)} + k_0^2 (1 + k_0^4) d_0^2) \}, \]

\[ Q_1^2 - I_1^2 + C_1 (M)^4(1-x) (1 + C_0 Q_0^2)2^{(x+1)/(1-x)} Q_0^2 + d_1^2, \]

\[ (B_1^2)^2 = k_0 C_0 B_0^2 + C_1 (M)^4(1-x) (1 + C_0 B_0^2)2^{(x+1)/(1-x)} B_0^2 + d_1^2, \]

\[ B_1^2 = \max \{ Q_1^2, (B_1^2)^2 \}, \]

\[ B_i^2 = B_i^2 + k_i^2 + 2B_i^2 + k_i B_0^2 + C_1 (C_0 B_0^2 + C_0^{2x} B_0^{2x} + 4), \]

\[ Q_2^2 = I_2^2 + \{ C_1 (1 + C_0 Q_0^2)^2 + (1/2) \}^{8/(1-x)} Q_0^2 + d_2^2, \]

\[ (B_2^2)^2 = B_2^2 + \{ C_1 (1 + C_0 B_0^2)^2 + (1/2) \}^{8/(1-x)} B_1^2 + d_2^2, \]

\[ B_2^2 = \max \{ Q_2^2, (B_2^2)^2 \} \]

and

\[ B_2^2 = B_2^2 + k_0^2 (k_i^2 + 2B_i^2 + k_i B_i^2) + C_1^2 (N_0^2 + B_0^{2(x+1)} B_0^2) + d_2^2. \]

We note that if \( f \equiv 0 \), the quantities introduced in the above become much simpler (cf. Corollary 1.1).

Our result reads as follows.

**Theorem 1.1.** In addition to the assumptions of Proposition 1.1 suppose that

\[ CB_1^{1/2} B_1^{1/2} < M \quad \text{and} \quad CB_0^{1/4} B_2^{3/4} < L \] \hspace{0.5cm} (1.14)

with certain constant \( C > 0 \) independent of \( M, L \) and \( (u_0, u_1) \).

Then, there exists \( \varepsilon_0 > 0 \) independent of \( L, M \) and \( (u_0, u_1) \) such that if

\[ k_2 B_1 B_2^2 < \varepsilon_0 \] \hspace{0.5cm} (1.15)

the problem (IBV) admits a unique solution \( u \) belonging to

\[ C^4 (R^+; L^2) \bigcap \bigcap_{i=0}^{3} C^3-t (R^+; H_{i+1} \cap H_0^t) \] \hspace{0.5cm} (1.16)
\((R^+ \equiv [0, \infty))\), and we have the estimates

\[
\sup_{t \geq 0} E(u(t)) \leq B_0^2, \quad (1.17)
\]

\[
\sup_{t \geq 0} E(D_i^j u(t)) \leq C B_i^2, \quad i = 1, 2, 3, \quad (1.18)
\]

\[
\sup_{t \geq 0} \|u(t)\|_\infty \leq C B_0^{1/2} B_1^{1/2}, \quad (1.19)
\]

and

\[
\sup_{t \geq 0} \|u_i(t)\|_\infty \leq C B_0^{1/4} B_2^{1/4}, \quad (1.20)
\]

where \(B_3 \) is a positive constant depending on \(\|u_0\|_{H_4}\) and \(\|u\|_{H_3}\). Moreover, if \(\lambda_0, \lambda_1, \lambda_2, \lambda_3 > 0\), we have

\[
E(D_i^j u(t)) \leq \tilde{Q}_i^2 e^{-\mu_i t}, \quad i = 0, 1, 2, 3, \quad (1.21)
\]

for some \(\tilde{Q}_i^2 = \tilde{Q}_i^2 (\|u_0\|_{H_4}, \|u_1\|_{H_3}) > 0\) and \(\mu_i = \mu_i (\lambda_0, \lambda_1, \lambda_2, \lambda_3) > 0\).

Remark. Let us denote by \(\mathcal{S}_{M,L}\) the set of \((u_0, u_1) \in H_4 \times H_3\) satisfying (1.14) and (1.15) together with the compatibility condition of order 3. Then, the set \(\mathcal{S}\) in the introduction is defined by \(\mathcal{S} = \bigcup_{M,L > 0} \mathcal{S}_{M,L}\).

Let us consider some special cases.

**Corollary 1.1.** We assume \(g(u) \equiv f \equiv 0\) and the Hypothesis I is fulfilled with \(L = \infty\). Let \((u_0, u_1) \in H_4 \times H_3\) satisfy the compatibility condition of order 3. Then, there exists a positive constant \(\varepsilon_1\) independent of \((u_0, u_1)\) such that if

\[
(I_0 + I_1)(I_0 + I_1 + I_2) < \varepsilon_1, \quad (1.22)
\]

the problem (IBV) (with \(g = f = 0\)) admits a unique (classical) solution \(u\) in the class (1.16). Moreover \(E(D_i^j u(t)), i = 0, 1, 2, 3,\) decay exponentially as \(t \to \infty\).

**Proof.** In this situation we can take \(C_1 = d_i = 0, \ i = 0, 1, \ldots\), and \(M = L = \infty\). Thus, the conditions (1.14) are satisfied automatically. Moreover we see

\[
Q_0^2 = I_0^2, \quad Q_1^2 = I_1^2, \quad B_1^2 = \max\{I_0^2, I_1^2\},
\]

and

\[
B_2^2 = \max\{I_0^2, I_1^2, I_2^2\}.
\]

Therefore, the condition (1.15) is equivalent to (1.21). Q.E.D.
Remark. It is easy to see that for the solution $u$ in Corollary 1.1, 
$\|u(t)\|_{C^2(\Omega)}$, $\|u_t(t)\|_{C^1(\Omega)}$, and $\|u_{tt}(t)\|_{C(\Omega)}$ decay exponentially as $t \to \infty$. The same result holds for the solutions in Theorem 1.1 if $f(t)$ tends to 0 exponentially in stronger norms.

**Corollary 1.2.** Consider the case

$$\sigma(u_i) = u_i, \quad g(u) = u^{2m+1},\quad \text{and} \quad f(x, t) \equiv 0,$$

with $m > 1$. We set

$$\mathcal{S}_M = \{(u_0, u_1) \in H_4 \times H_3 \mid (u_0, u_1) \text{ satisfies the compatibility condition of order } 3 \text{ and the inequality } C I_0 \{I_0 + I_1 + I_0(1 + I_0)^{M^{2m-1}} + I_0(1 + I_0)^2 M^{4m-2} \} < M^2 \} \quad (1.22)$$

for $M > 0$, where $C$ is a certain positive constant independent of $(u_0, u_1)$. Then, if $(u_0, u_1) \in \mathcal{S}_M$ for some $M > 0$ the problem (IBV) admits a unique solution $u$ in the class (1.16) with $\|u(t)\|_{\infty} < M$.

*Proof.* In this case Hypothesis I is satisfied with $k_0 = k_0 = k_1 = 1, \quad \bar{k}_1 = 0, \quad r = 0, \quad k_i = 0, \quad i = 2, 3, \quad \text{and} \quad L = \infty$. Hypothesis II is satisfied with $C_0 = 1, \quad \alpha = 0, \quad \text{and} \quad C_1(M) = C M^{2m-1} \quad (1.23)$

for certain $C > 0$. Thus, the second inequality of (1.14) and the condition (1.15) are fulfilled automatically. It is easy to see the first inequality of (1.14) is equivalent to (1.22). Q.E.D.

Remark. Needless to say Theorem 1.1 is applicable even if $g(u)$ is not monotonically increasing. For example, consider $g(u) = -u^{2m+1}$ in Corollary 1.1 instead of $u^{2m+1}$. Then, the conclusion is still valid for $0 < M < M_0$, where $M_0$ is a certain positive constant independent of $(u_0, u_1)$.

### 2. A Priori Estimates (I)

For the proof of Theorem 1.1 it suffices to derive a priori estimates for assumed smooth solutions (cf. (1.12), (1.13)). We begin with the following estimate.

**Proposition 2.1.** Let $T > 0$ and $u(t)$ be a solution as in Proposition 1.1, satisfying

$$\|u(t)\|_{\infty} < M \quad \text{and} \quad \|u_t(t)\|_{\infty} < L \quad \text{on } [0, T).$$
Then, we have
\[
\sup_{0 \leq s < \min(1, T)} E(u(s)) \leq Q_0^2 \quad \text{and} \quad E(u(t)) \leq C B_0^2 \quad \text{for} \quad 0 \leq t < T.
\] (2.1)

Moreover, if \( \lambda_0 > 0 \) we have
\[
E(u(t)) \leq \bar{Q}_0^2 e^{\mu_0 t} \quad \text{for} \quad 0 < t < T
\] (2.2)
with certain \( \mu_0 > 0 \) and \( \bar{Q}_0^2 = \bar{Q}_0^2(I_0, M, L) > 0. \)

**Proof.** The proof of (2.1) or (2.2) is essentially included in our previous papers (cf. [6, 7]). However, we need precise information on the quantities \( Q_0^2, B_0^2 \), etc., and we reproduce the proof briefly, which will be also convenient for subsequent further estimations of solutions. For simplicity we use the notation \( \| \cdot \| \) for \( \| \cdot \|_2 \) in what follows.

Multiplying the equation in (IBV) (see (1.17)) by \( u_t \) and integrating we have
\[
\frac{d}{dt} E(u(t)) + \int_{\Omega} \sigma(u_t) u_t \, dx = \int_{\Omega} f u_t \, dx
\] (2.3)
and hence, by the assumption on \( \sigma \),
\[
k_0^{-1} \int_{t}^{t+1} \| u_t(s) \|^2 \, ds \leq \int_{t}^{t+1} \sigma(u_t) u_t \, dx \, ds
\] 
\[
= E(u(t)) - E(u(t+1)) + \int_{t}^{t+1} \int_{\Omega} f u_t \, dx \, ds
\] 
\[
\equiv D(t). \quad (2.4)
\]

Therefore, there exist \( t_i \in [t, t + \frac{1}{4}] \), \( t_5 \in [t + \frac{3}{4}, t + 1] \) such that
\[
\| u_i(t_i) \| \leq 2 \sqrt{k_0} D(t), \quad i = 1, 2. \quad (2.5)
\]
Next, multiplying the equation by \( u(t) \) and integrating over \( \Omega \times [t_1, t_2] \) we find
\[
\int_{t_1}^{t_2} \left\{ \| \nabla u(s) \|^2 + \int_{\Omega} g(u) u \, dx \right\} \, ds
\] 
\[
= -(u_t(t_2), u(t_2)) + (u_t(t_1), u(t_1)) + \int_{t_1}^{t_2} \| u_t(s) \|^2 \, ds
\] 
\[
- \int_{t_1}^{t_2} \int_{\Omega} \sigma(u_t) u \, dx \, ds + \int_{t_1}^{t_2} \int_{\Omega} f u \, dx \, ds
\]
\[ 4 \sqrt{k_0} D(t) \sup_{t \leq s \leq t + 1} \| u(s) \| + k_0 D(t)^2 \]
\[ + k_1 \left\{ \left( \int_{t_1}^{t_2} \left( \int_{\Omega} \sigma(u_t) u_t \, dx \right) (r + 1) / (r + 2) \right) \left( \int_{t_1}^{t_2} \| u \|_{r + 2}^{r + 2} \right)^{1/(r + 2)} \, ds \right\} \]
\[ + \left( \int_{t_1}^{t_2} \| u_t \|^2 \, ds \right)^{1/2} \left( \int_{t_1}^{t_2} \| u(s) \|^2 \, ds \right)^{1/2} \]
\[ + d_0 e^{-\mu_0 t} \left( \int_{t_1}^{t_2} \| u(s) \|^2 \, ds \right)^{1/2} \]
\[ \leq (4 \sqrt{k_0} D(t) + d_0 e^{-\mu_0 t}) \sup_{t \leq s \leq t + 1} \| u(s) \| + k_0 D(t)^2 \]
\[ + k_1 D(t)^{2(r + 1)/(r + 2)} \sup_{t \leq s \leq t + 1} \| u(s) \|_{r + 2} \]
\[ + \kappa_1 \sqrt{k_0} D(t) \sup_{t \leq s \leq t + 1} \| u(s) \| \]
\[ \equiv A(t)^2, \]  
(2.6)

where we have used Hypothesis I.

It follows from (2.4) and (2.6) that (see Hyp. II)

\[ \int_{t_1}^{t_2} E(u(s)) \, ds \leq C_0 A(t)^2 + k_0 D(t)^2, \]
(2.7)

and hence, there exists \( t^* \in [t_1, t_2] \) such that

\[ E(u(t^*)) \leq 2(C_0 A(t)^2 + k_0 D(t)^2). \]  
(2.8)

By the energy identity as in (2.4) we have further

\[ \sup_{t \leq s \leq t + 1} E(u(s)) \leq E(u(t^*)) + \int_{t}^{t + 1} \int_{\Omega} \sigma(u_t) u_t \, dx \, ds \]
\[ \leq 2(C_0 A(t)^2 + k_0 D(t)^2) + D(t)^2. \]  
(2.9)

Since

\[ \| u(s) \|_{r + 2} \leq \text{const.} \| \nabla u \| \quad \text{and} \quad \| u \| \leq \text{const.} \| \nabla u \| \]

we have from (2.6) and (2.9) that

\[ \sup_{t \leq s \leq t + 1} E(u(s)) \leq \frac{k_0^2 D(t)^4(r + 1)/(r + 2)}{2} + k_0^2 (1 + \kappa_1^2) D(t)^2 + d_0^2 e^{-2\mu_0 t} \]  
(2.10)

for a certain constant \( C > 0 \) independent of \( M, L, \) and \( u. \)
Here we also note that (2.3) implies immediately
\[
\sup_{0 \leq s \leq 1} E(u(s)) \leq E(u(0)) + k_0 d_0^2. \tag{2.11}
\]

From (2.10) we shall first derive the boundedness of \( E(u(t)) \). For this suppose that \( E(u(t)) \leq E(u(t + 1)) \) for some \( t \). Then, we have from (2.10) (recall (2.4))
\[
\sup_{t \leq s \leq t + 1} E(u(s)) \leq C_0^3 \left\{ k_1^2 \left( \int_t^{t+1} \int_\Omega |f_{u_i}| \, dx \, ds \right)^{2(r+1)/(r+2)} + k_0 (1 + \tilde{k}_1^2) \left( \int_t^{t+1} \int_\Omega |f_{u_i}| \, dx \, ds + d_0^2 e^{-2\tilde{q}t} \right) \right\}
\leq C_0^3 \left\{ k_1^2 d_0^{2(r+1)/(r+2)} \sup_{t \leq s \leq t + 1} \|u(s)\|^{2(r+1)/(r+2)} + k_0 (1 + \tilde{k}_1^2) d_0 \sup_{t \leq s \leq t + 1} \|u(s)\| \right\}. \tag{2.12}
\]

Thus, if \( E(u(t)) \leq E(u(t + 1)) \) for some \( t \) we have
\[
E(u(t + 1)) \leq \sup_{t \leq s \leq t + 1} E(u(s)) \leq C_0^3 \{ k_1^2 d_0^{2(r+1)/(r+2)} + k_0 (1 + \tilde{k}_1^2) d_0^2 \} = (B_0')^2. \tag{2.13}
\]

From (2.11) and (2.13) we conclude
\[
\sup_{t \geq 0} E(u(t)) \leq \max \left\{ \sup_{0 \leq s \leq 1} E(u(s)), (B_0')^2 \right\} \leq \max \{ I_0^2 + k_0^2 d_0^2, (B_0')^2 \} = B_0^2. \tag{2.14}
\]

Moreover, from (2.10) and (2.14) we obtain
\[
\sup_{t \leq s \leq t + 1} E(u(s)) \leq C_0^3 \left\{ (k_1^2 B_0^{2r/(r+2)} + k_0 (1 + \tilde{k}_1^2))(E(u(t)) - E(u(t + 1))) \right\} + (k_1^2 B_0^{2r/(r+2)} + k_0 (1 + \tilde{k}_1^2))^2 d_0^2 e^{-2\tilde{q}t}. \tag{2.15}
\]

Applying Lemma 1.2 to (2.15) we can get the estimate (2.2). Q.E.D.

3. A PRIORI ESTIMATES (II)

For further estimations of the (assumed) solutions of the problem (IBV) we prepare the following:
**Proposition 3.1.** Let $T > 1$ and $u(t)$ be a solution of (IBV) on $[0, T)$ in the sense of Proposition 1.1 with

$$
\|u(t)\|_\infty < M \quad \text{and} \quad \|u_r(t)\|_\infty < L \quad \text{on} \ [0, T).
$$

Consider a solution $U \in C([0, T); H^1_0) \cap C^1([0, T); L^2)$ of the linear equation

$$
U_{tt} - AU + \sigma'(u_t) U_t = F(x, t) \quad \text{on} \quad \Omega \times [0, T)
$$

$$
U(0) = U_0 \in H^1_0, \quad U_t(0) = U_1 \in L^2, \quad \text{and} \quad U|_{\partial \Omega} = 0,
$$

where $F \in L^2([0, T); L^2(\Omega))$. Then, it holds that

$$
\int_t^{t+1} \int_{\Omega} \sigma'(u_t) U_t^2 \, dx \, ds
$$

$$
= E(U(t)) - E(U(t+1)) + \int_t^{t+1} \int_{\Omega} F U_t \, dx \, ds \quad (3.2)
$$

and

$$
\sup_{t \leq s \leq t+1} E(U(s)) \leq C k_0 \{ E(U(t)) - E(U(t+1)) \}
$$

$$
+ \int_t^{t+1} \int_{\Omega} \sigma'(u_t) |U|^2 \, dx \, ds
$$

$$
+ \int_t^{t+1} \int_{\Omega} |F| \left( |U| + k_0 |U_t| \right) \, dx \, ds \quad (3.3)
$$

for $0 < t < T - 1$. ($C$ denotes constants independent of $M$, $L$, $u$ and $U$.)

**Proof.** The proof is quite similar (and simpler) to the proof of Proposition 2.1 and we sketch it briefly.

Multiplying the equation by $U_t$ and integrating we have

$$
k_0^{-1} \int_t^{t+1} \|U_t\|^2 \, ds \leq \int_t^{t+1} \int_{\Omega} \sigma'(u_t) |U_t|^2 \, dx \, ds
$$

$$
= E(U(t)) - E(U(t+1)) + \int_t^{t+1} \int_{\Omega} F U_t \, dx \, ds
$$

$$
= D(t)^2. \quad (3.4)
$$

($(3.2)$ is thus trivial.)

On the other hand, multiplying the equation by $U$ and integrating over $\Omega \times [t_1, t_2]$, $t \leq t_1 < t_2 \leq t + 1$, we find
\[ \int_{t_1}^{t_2} \| \nabla U(s) \|^2 \, ds = -(U(t_2), U(t_1)) + (U(t_1), U(t_1)) + \int_{t_1}^{t_2} |U_i(s)|^2 \, ds - \int_{t_1}^{t_2} \sigma'(u_i) U_i U \, dx \, ds \]
\[+ \int_{t_1}^{t_2} \int_{\Omega} F U \, dx \, ds \]
\[\leq \sum_{i=1}^{n} |U_i(t_i)| |U(t_i)| + \int_{t_1}^{t_2} \| U_i(s) \|^2 \, ds \]
\[+ \frac{1}{2} \left\{ \int_{t_1}^{t_2} \int_{\Omega} \sigma'(u_i) |U_i|^2 \, dx \, ds \right\} \]
\[+ \int_{t_1}^{t_2} \int_{\Omega} F U \, dx \, ds. \quad (3.5) \]

Combining (3.4) and (3.5) we can derive
\[ \sup_{t \leq s \leq t+1} E(U(s)) \leq C \left\{ k_0 D(t)^2 + \int_{t}^{t+1} \int_{\Omega} F U \, dx \, ds \right\} \]
\[+ \int_{t}^{t+1} \int_{\Omega} \sigma'(u_i) |U|^2 \, dx \, ds \]
which implies (3.3) immediately. Q.E.D.

Now, differentiating Eq. (1.7) in \( t \) we have
\[ \begin{array}{c}
\frac{d}{dt} \left[ u_{,t} - A u_{,t} + \sigma'(u_i) u_{,t} + g'(u) u_t = f_i(x, t) \right] \\
\text{on } \Omega \times [0, \infty) \end{array} \]
\[ u_t(0) = u_1, \quad u_{,t}(0) = u_2, \quad \text{and} \quad u_{,t}|_{\partial\Omega} = 0. \quad (3.6) \]

Applying Propositions 2.1 and 3.1 to (3.6) we can show the following:

PROPOSITION 3.2. Let \( u(t) \) be a smooth solution on \([0, T), T > 0\), of the Problem (IBV) with
\[ \| u(t) \|_{\infty} < M \quad \text{and} \quad \| u_i(t) \|_{\infty} < L \quad \text{on } [0, T). \]

Then,
\[ \sup_{0 \leq t \leq \min(1, T)} E(u_i(s)) \leq C Q_i^2, \quad \sup_{t \geq 0} E(u_i(t)) \leq C B_i^2 \quad (3.7) \]
and moreover, if $\lambda_0 \lambda_1 > 0$,

$$E(u_i(t)) \leq \tilde{Q}^2_1 e^{-\mu_i t} \quad \text{for} \quad 1 < t < T$$

(3.8)

with certain $\mu_i > 0$ and $\tilde{Q}^2_1 = \tilde{Q}^2_1(\lambda_0, \lambda_1, M, L) > 0$.

Proof. Applying Proposition 3.1 with $U = u_i$ to (3.6) we see easily

$$\sup_{1 \leq r \leq s \leq t + 1} E(u_i(s)) \leq C_{k_0}\{ E(u_i(t)) - E(u_i(t + 1)) \}$$

$$+ C \int_t^{t+1} \int_{\Omega} \sigma'(u_i) |u_i|^2 \, dx \, ds$$

$$+ C \int_t^{t+1} \int_{\Omega} k_0^2 \{ |f_i| + |g'(u)u_i| \}$$

$$\times \{ |u_i| + |u_i| \} \, dx \, ds.$$  

(3.9)

Here,

$$\int_t^{t+1} \int_{\Omega} \sigma'(u_i) |u_i|^2 \, dx \, ds \leq k_0 \int_t^{t+1} \int_{\Omega} \sigma(u_i) u_i \leq C_{k_0} B_0^2$$

(3.10)

by (2.1) and (2.4). Moreover, since $|g'(u)| < C_1(M)(1 + |u|^{2+\varepsilon})$ with $0 < \varepsilon < 1$ we find

$$\int_t^{t+1} \int_{\Omega} \{ |g'(u)u_i| + |f_i| \} \{ |u_i| + |u_i| \} \, dx \, ds$$

$$\leq C \int_t^{t+1} \{ C_1(M)(1 + \|u\|^{2+\varepsilon}_{H^1}) \|u_i\|_{H^{2-\varepsilon}} + \|f_i\| \} \{ \|u_i\| + \|u_i\| \} \, ds$$

$$\leq C \{ C_1(M)(1 + \sup_{1 \leq s \leq t+1} \|u(s)\|^{2+\varepsilon}_{H^1}) \sup_{1 \leq s \leq t+1} \|u_i(s)\|^{1-\theta} \|\nabla u_i(s)\|^\theta + d_1 e^{-\lambda_1 t} \}$$

(3.11)

$$\times \sup_{1 \leq s \leq t+1} (C_0 E(u_i(s)))^{(\varepsilon + 1)/2}$$

$$+ d_1 e^{-\lambda_1 t} \sup_{1 \leq s \leq t+1} \sqrt{E(u_i(s))}.$$
Thus, we obtain, for $0 < t < T - 1$,

$$
\sup_{t \leq s \leq t + 1} E(u,(s)) \leq C k_0 \left[ E(u,(t)) - E(u,(t + 1)) \right] + C k_0 C_0 B_0^2 + C_1 C_1(M)^{4/(1 - \sigma)} \times (1 + C_0^{(2 + 1)/2} B_0^{x + 1})^{4/(1 - x)} C_0 B_0^2
$$

$$
+ d_1^2 e^{-2\xi_1 t}. \tag{3.11}
$$

Also, from the inequality

$$
E(u,(t)) \leq I_1^2 + \int_0^t \int_\Omega \left( |g'(u)| |u_t| + |f_i| |u_{ix}| \right) dx \, ds
$$

we have

$$
\sup_{0 \leq s \leq \min(1, T)} E(u,(s)) \leq C Q_1^2, \tag{3.12}
$$

where we recall

$$
Q_1^2 = I_1^2 + C_1(m)^{4/(1 - \sigma)} (1 + C_0^{(x + 1)/2} Q_0^{x + 1})^{4/(1 - x)} C_0 Q_0^2 + d_1^2
$$

(compare with the last term of (3.11)).

From (3.11) and (3.12) we obtain

$$
\sup_{0 \leq s \leq T} E(u,(t)) \leq \max \{ C Q_1^2, C(B'_i)^2 \} = CB_i^2, \tag{3.13}
$$

where we recall

$$
(B'_i)^2 = C_1(M)^{4/(1 - \sigma)} (1 + C_0^{(x + 1)/2} B_0^{x + 1})^{4/(1 - x)} C_0 B_0^2 + d_1^2.
$$

We can replace $B_0^2$ in (3.10) and (3.11) by $\hat{Q}_0^2 e^{-\mu t}$ (see (2.2)) and hence, applying Lemma 1.2 we obtain the decay estimate (3.8). Q.E.D.

**Proposition 3.3** Under the same assumptions of Proposition 3.2 we have further

$$
\|u(t)\|_{L_2}^2 \leq CB_i^2 \quad \text{and} \quad \|u(t)\|_{L_\infty}^2 \leq C \sqrt{C_0} B_0 \tilde{B}_1. \tag{3.14}
$$

**Proof.** By the theory of elliptic boundary value problem we see

$$
\|u(t)\|_{L_2} \leq C \|\Delta u(t)\|
$$

$$
\leq C \left\{ \|u_{ss}(t)\| + \|\sigma(u,(t))\| + \|g(u(t))\| + \|f(t)\| \right\}. \tag{3.15}
$$
Here, our assumptions on $\sigma(u_i)$ and $g(u)$ imply

$$\|\sigma(u_i(t))\|^2 \leq k_1^{r+2} \|u_i(t)\|^{2(r+1)} + \overline{K} _1 \|u_i(t)\|^2$$

(by (1.5))

$$ \leq C k_1^{r+2} \|u_i(t)\|^{2(r+1)} + \overline{K} _1 \|u_i(t)\|^2$$

and

$$|g(u(t))| \leq CC_1(M)\left\{\|u(t)\| + \|u(t)\|^{\frac{r+2}{r}}\right\}.$$ (3.16)

Thus, the estimate for $\|u(t)\|_{H^2}$ follows immediately from (2.1) and (3.7). The second inequality of (4.14) follows from the inequality

$$\|u(t)\|_{H^2} \leq C \|u(t)\|_{H^2} \leq C \sqrt{C_0 E(u(t))} \|u(t)\|_{H^2}. \quad \text{Q.E.D.}$$

4. A PRIORI ESTIMATES (III)

In this section we want to derive a priori estimates for $E(u, u(t))$, which will be a key for the proof of Theorem 1.1 when $\sigma(v)$ is nonlinear. Differentiating Eq. (1.7) twice with respect to $t$ we have

$$D^4_t u - \Delta D^2_t u + \sigma'(u_i) D^3_t u + \sigma''(u_i) (D^2_t u)^2$$

$$+ g''(u)(u_i)^2 + g'(u) D^2_t u = D^2_t f.$$ (4.1)

Applying Proposition 3.1 to (4.1) with $U = D^2_t u$ we see (if $T > 1$)

$$\sup_{t \leq s \leq t + 1} E(D^2_t u(s)) \leq C k_0 \{E(D^2_t u(t) - E(D^2_t u(t + 1))\}$$

$$\int_t^{t+1} \int_{\Omega} \left( |\sigma''(u_i)| |D^2_t u|^2 + |g''(u)| |u_i|^2 \right.$$

$$+ |g'(u)| |D^2_t u| + |D^2_t f|)(|D^2_t u| + |D^2_t u|) \, dx \, ds$$

$$+ C \int_t^{t+1} \int_{\Omega} \sigma(u_i) |D^2_t u|^2 \, dx \, ds$$

for $0 \leq t < T - 1$.

Here, we see

$$\int_t^{t+1} \int_{\Omega} \sigma''(u_i) |D^2_t u|^2 (|D^2_t u| + |D^2_t u|) \, dx \, ds$$

$$\leq \int_t^{t+1} \int_{\Omega} \sigma'(u_i)(|D^2_t u|^2 + |D^2_t u|^2) \, dx \, ds$$

$$+ C k_2 \int_t^{t+1} \|D^2_t u\|_{L^2}^2 \, ds \quad \text{(by Hypothesis I).} \quad (4.2)$$
\[
\leq \int_t^{t+1} \int_\Omega \sigma'(u_s) |D_s^3 u|^2 \, dx \, ds + CB_t^2 \\
+ Ck_2 \sup_{t \leq s \leq t+1} \|D_s^7 u(s)\|_2 \|D_s^2 u(s)\|^3_{L^\infty},
\]
where we have used the inequality
\[
\int_t^{t+1} \int_\Omega \sigma'(u_s) |u_{tt}|^2 \, dx \, ds \leq CB_t^2
\]
which follows easily from (3.4) with \( U = u_t \) and \( F = g'(u)u_t + f_t \) and the estimates (3.11) and (3.13).

Moreover we see
\[
\int_t^{t+1} \int_\Omega |g''(u)| |u_t|^2 (|D_s^3 u| + |D_s^2 u|) \, dx \, ds
\]
\[
\leq C_1(M) \int_t^{t+1} \int_\Omega (1 + |u|^\gamma) |u_t|^2 (|D_s^3 u| + |D_s^2 u|) \, dx \, ds
\]
(by Hypothesis II)
\[
\leq CC_1(M) \int_t^{t+1} \left( 1 + \|u\|^{2}_{H_t^1}\right) \|u_t\|^{2}_{L^\infty} (\|D_s^3 u\| + \|D_s^2 u\|) \, ds
\]
\[
\leq \varepsilon \sup_{t \leq s \leq t+1} E(D_s^2 u(s)) + C_\varepsilon C_1(M)^2
\]
\[
\times \sup_{t \leq s \leq t+1} (1 + \|u(s)\|^{2}_{H_t^1})^2 \|u_t(s)\|^4_{H_t^1}
\]
(4.5)
and
\[
\int_t^{t+1} \int_\Omega (|g'(u)| |D_s^2 u| + |D_s^2 f|)(D_s^3 u| + |D_s^2 u|) \, dx \, ds
\]
\[
\leq C \int_t^{t+1} \left\{ C_1(M)(1 + \|u\|^{2+1}_{H_t^0}) \|D_s^2 u\|_{L^1_{\varepsilon(2)}} + \|D_s^2 f\| \right\}
\]
\[
\times (\|D_s^3 u\| + \|D_s^2 u\|) \, ds
\]
\[
\leq CC_1(M)(1 + \sup_{t \leq s \leq t+1} \|u(s)\|^{2+1}_{H_t^0})
\]
\[
\times \sup_{t \leq s \leq t+1} \|D_s^2 u(s)\|^{1-\theta} E(D_s^2 u(s))^{(1+\theta)/2}
\]
\[
+ Cd_2 e^{-\lambda_2 t} \sup_{t \leq s \leq t+1} E(D_s^2 u(s))
\]
(4.6)
Here, we also note that (see (3.2))

\[ \int_{t}^{t+1} \int_{\Omega} \sigma'(u,)|D_{t}^{3}u|^{2} \, dx \, ds \leq E(D_{t}^{2}u(t)) - E(D_{t}^{2}u(t+1)) \]

\[ + \int_{t}^{t+1} \int_{\Omega} |\sigma''(u,)|u_{t}^{2} + |g'(u)|u_{t}^{2} \times |g'(u)|u_{t}^{2} + |D_{t}^{2}f| \, |D_{t}^{3}u| \, dx \, ds. \]  

(4.7)

It follows from (4.2)-(4.6) that

\[ \sup_{t \leq s \leq t+1} E(D_{t}^{2}u(s)) \leq C k_{0} \{ E(D_{t}^{2}u(t)) - E(D_{t}^{2}u(t+1)) \} \]

\[ + C B_{1}^{2} + C k_{2} \sup_{t \leq s \leq t+1} E(u_{t}(s)) E(D_{t}^{2}u(s))^{3/2} \]

\[ + C C_{1}(M)^{2} \sup_{t \leq s \leq t+1} (1 + C_{0} E(u(s)))^{2} E(u_{t}(s))^{2} \]

\[ + C(C_{1}(M) \{ 1 + (C_{0} E(u(s))}^{(\rho+1)/2} \}^{8/(1-\rho)} \times E(u_{t}(s)) + C d_{2}^{2} e^{-\lambda_{2} t} \]

\[ < C k_{0} \{ E(D_{t}^{2}u(t)) - E(D_{t}^{2}u(t+1)) \} \]

\[ + C B_{1}^{2} + C k_{2} B_{1} \sup_{t \leq s \leq t+1} E(D_{t}^{2}u(s))^{3/2} \]

\[ + C(C_{1}(M) \{ 1 + (C_{0} B_{0}^{2})^{(\rho+1)/2} \}^{8/(1-\rho)} B_{1}^{2} \]

\[ + C d_{2}^{2} e^{-2\lambda_{2} t} \]

\[ = A(t)^{2}, \]  

(4.8)

and also from (4.7),

\[ \int_{t}^{t+1} \int_{\Omega} \sigma'(u,)|D_{t}^{3}u|^{2} \, dx \, ds \leq A(t)^{2}. \]  

(4.9)

Similarly we have

\[ \sup_{0 \leq s \leq \min(1, T)} E(D_{t}^{2}u(s)) \]

\[ \leq C Q_{2}^{2} + C k_{2} Q_{1} \sup_{0 \leq s \leq \min(1, T)} E(D_{t}^{2}u(s))^{3/2}, \]  

(4.10)

where we recall

\[ Q_{2}^{2} = I_{2}^{2} + (C_{1}(M) \{ 1 + (C_{0} Q_{0}^{2})^{(\rho+1)/2} \}^{8/(1-\rho)} Q_{1}^{2} + d_{2}^{2}. \]
Let us suppose for a moment that
\[ C k_2 B_1 \sqrt{E(D_i^2 u(t))} \leq \frac{1}{2} \quad \text{for } 0 \leq t \leq T' \] (4.11)
with some \( T' < T \). Then, we have from (4.8)
\[ \sup_{t \leq s \leq t + 1} E(D_i^2 u(s)) \leq C k_0 \left\{ E(D_i^2 u(t)) - E(D_i^2 u(t + 1)) \right\} + C(B_2')^2 \] (4.12)
if \( 0 \leq t \leq T' - 1 \), where we recall
\[ (B_2')^2 = B_1^2 + (C_1(M) \{ 1 + (C_0 B_0^2)^{8/(1 - \alpha)} \}) B_1^2 + d_2^2. \]
Also, from (4.10) (note that \( Q_1 \leq B_1 \)) we see
\[ \sup_{0 \leq s < \min(1, T')} E(D_i^2 u(s)) \leq C Q_2^2. \] (4.13)
As is usual, it follows from (4.12) and (4.13)
\[ E(D_i^2 u(t)) \leq C \max\{ Q_2^2, (B_2')^2 \} \equiv CB_2^2 \quad \text{for } 0 \leq t \leq T'. \] (4.14)
Now, let us make the assumption
\[ k_2 B_1 B_2 < \varepsilon_0 \] (4.15)
for some \( \varepsilon_0 > 0 \). If we take \( \varepsilon_0 > 0 \) sufficiently small the assumption (4.15) implies that the inequality (4.11) holds as long as the (smooth) solution \( u \) exists. Consequently, we arrive at the estimates (4.13) and (4.14) with any \( T' < T \) under the assumption (4.15) with a certain \( \varepsilon_0 > 0 \). Moreover, from (4.7)
\[ \int_0^{t - 1} \int_\Omega \sigma'(u_i) \left| D_i^3 u \right|^2 \, dx \, ds \leq CB_2^2 \] (4.16)
for \( 0 \leq t < T - 1 \).
When \( \lambda_0 \lambda_1 \lambda_2 > 0 \) we can obtain, instead of (4.14) and (4.16), the exponential decay estimates as is usual. Let us summarize the above arguments in the following

**Proposition 4.1.** Let \( (u_0, u_1) \in H_3 \times H_3 \) satisfy the compatibility condition of order 3 and let \( u(t) \) be the solution of the problem (IBV) on \( \Omega \times [0, T), T > 0 \). Assume that
\[ \| u(t) \|_\infty < M \quad \text{and} \quad \| u_i(t) \|_\infty < L \quad \text{on } [0, T). \]
Then, there exists $c_0 > 0$ independent of $M, L, (u_0, u_1)$ and $T$ such that if (4.15) holds, we have the estimates

$$E(D_t^2 u(t)) \leq CB_2^2 \quad \text{for} \quad 0 \leq t < T$$

(4.17)

and (4.16).

Moreover, if $\lambda_0 \lambda_1 \lambda_2 > 0$ we have the decay estimate

$$E(D_t^2 u(t)) + \int_{t-1}^{t} \int_{\Omega} \sigma'(u(s)) D_t^3 u(s) \, dx \, ds$$

$$\leq \tilde{Q}_2 e^{-\mu t}, \quad 1 \leq t < T,$$

(4.18)

with some $\tilde{Q}_2 = \tilde{Q}_2^2(I_0, I_1, I_2, M, L)$ and $\mu = \mu_2(M, L, \lambda_0, \lambda_1, \lambda_2) > 0$.

Proposition 4.1 in the above is used to get the estimate for $\|u_i(t)\|_\infty$.

PROPOSITION 4.2. Under the same assumptions of Proposition 4.1 we have

$$\|u_i(t)\|_{H^2} \leq CB_2, \quad 0 \leq t < T,$$

(4.19)

and

$$\|u_i(t)\|_{\infty} \leq CB_0^{1/4} B_2^{3/4}, \quad 0 \leq t < T.$$  

(4.20)

Proof. By the regularity theory of elliptic equations we have

$$\|u_i(t)\|_{H^2} \leq C \|Au_i\| = C \|D_t^2 u + \sigma'(u) u_t + g'(u) u_i - f_i\|$$

$$\leq C \{ \|D_t^2 u(t)\| + \|\sigma'(u(t)) u_t(t)\| + \|g'(u(t)) u_i(t)\|$$

$$+ \|f_i(t)\| \}.$$

Here, since $|\sigma'(v)|^2 \leq \tilde{k}_0^2 \{ k_i^2 + 2|v|^{2r} + k_1 \}$, we have

$$\|\sigma'(u) u_t(t)\|^2 \leq \tilde{k}_0^2 \int_{\Omega} (k_i^2 + 2|u_t|^2 + \tilde{k}_1) |u_t|^2 \, dx$$

$$\leq C \tilde{k}_0^2 k_i^2 + 2 \|u_t(t)\|_{H^1}^2 \|u_{tt}(t)\|_{L^2}^2$$

and also

$$\|g'(u) u_i(t)\|^2 \leq C_1(M)^{2r} \|u_i(t)\|^2 + C \|u_{2(r+1)}^2 u_i(t)\|_{H^1}^2$$

$$\leq C_1(M)^2 (B_0^2 + CB_0^{2(r+1)} B_1^2).$$

Thus, we obtain the estimate (4.19). Inequality (4.20) follows immediately from (4.19) and the inequality

$$\|u_i(t)\|_{\infty} \leq C \|u_i(t)\|_{H^2}^{1/4} \|u_i(t)\|_{H^2}^{3/4}.$$

Q.E.D.
Remark. It is clear that if $\lambda_0 \lambda_1 \lambda_2 > 0$ we can get the exponential decay for $\|u_i(t)\|_H^2$.

5. A PRIORI ESTIMATES (IV) AND COMPLETION OF THE PROOF OF THEOREM 1.1

Let $u(t)$ be a solution in the sense of Proposition 1.1. Then, we have derived in the previous sections (see Propositions 3.3 and 4.2) that

$$\|u(t)\| \leq CB_0^{1/2}B_1^{1/2}$$
and
$$\|u_r(t)\|_\infty \leq CB_0^{1/4}B_2^{3/4} \quad (5.1)$$

for $0 \leq t < T$ with certain constant $C > 0$ independent of $M$, $L$, $(u_0, u_1)$ and $T$ under the assumptions (4.15) and

$$\|u(t)\|_\infty < M \quad \text{and} \quad \|u_r(t)\|_\infty < L \quad \text{in } [0, T). \quad (5.2)$$

Now, let us make the further assumption:

$$CB_0^{1/2}B_1^{1/2} < M \quad \text{and} \quad CB_0^{1/4}B_2^{3/4} < L \quad (5.3)$$

with $C$ in (5.1).

Then, by (5.1) we can conclude that the estimates (5.2) hold on $[0, T)$ automatically and consequently (5.1) under the assumption (4.15). Of course, the conclusions of Propositions 2.1, 3.2, and 4.1 hold under the conditions (4.15) and (5.3). Taking these into account we shall proceed to further estimations of the (assumed) smooth solution $u(t)$ on $[0, T)$.

Differentiating Eq. (1.7) three times with respect to $t$ we have

$$D^5_t u - \Delta D^3_t u + \sigma'(u_2) D^4_t u + 3\sigma''(u_2) u_2 D^3_t u + \sigma'''(u_3)(u_3) D^2_t u + g'(u) D^2_t u + g''(u) u_2 u_2 + g'''(u_4)(u_4)^3 = D^3_t f \quad (5.4)$$

with $D^3_t u(0) = u_3 \in H^1_0$ and $D^4_t u(0) = u_4 \in L^2$.

Applying Proposition 3.1 to (5.4) we have, if $T > 1$,

$$\sup_{t \leq s \leq t + 1} E(D^3_t u(s)) \leq Ck_0 \left\{ E(D^3_t u(t)) - E(D^3_t u(t + 1)) \right\}$$

\[+ q_0 \int_{t}^{t+1} \int_{\Omega} \left( |u_{tt}| |D^3_t u| + |u_{tt}|^3 \right. \]
\[+ |D^3_t u| + |u_2| |u_2| + |u_3| \]
\[\times (|D^3_t u| + |D^4_t u|) \, dx \, ds \]
\[
\leq Ck_0 \left\{ E(D^3_t u(t)) - E(D^3_t u(t+1)) \right\} \\
+ q_0 \sup_{1 < s < t+1} \{ \|u_t\|_{H^1} \|D^3_t u\|_3 \\
+ \|u_t\|_{H^1}^3 + \|D^3_t u\| + \|u_t\|_{H^1} \|u_t\|_3 \\
+ \|u_t\|_{H^1}^3 + d_3 e^{\lambda_3 t} \sqrt{E(D^3_t u(s))} \}
\]

for \(0 \leq t < T - 1\) with some \(q_0 = q_0(I_0, I_1, I_2) > 0\), and hence

\[
E(D^3_t u(t)) \leq \tilde{Q}_3^2 e^{-\mu_3 t}, \quad 0 < t < T,
\]  

(5.5)

with some \(\mu_3 > 0(\mu_3 = 0\) if \(\lambda_0 \lambda_1 \lambda_2 \lambda_3 = 0\) and \(\tilde{Q}_3^2 = \tilde{Q}_3^2(I_0, I_1, I_2, I_3) > 0\).

The a priori estimates obtained by now show that the local solution \(u\) in Proposition 1.1 can be continued on \([0, \infty)\) under the assumptions (4.15) and (5.3), and the extended solution, which we denote again by \(u\), belongs to

\[
W^{4, \infty}([0, \infty) ; L^2) \cap W^{3, \infty}([0, \infty) ; H^0) \cap W^{1, \infty}([0, \infty) ; H^2 \cap H^0)
\]

and hence (cf. Strauss [13])

\[
C^4([0, \infty) ; L^2) \cap C^3([0, \infty) ; H^0) \cap C^1([0, \infty) ; H^2 \cap H^0).
\]   

(5.6)

Needless to say, the concluding estimates of Propositions 2.1, 3.2, 4.1, 4.2, and the estimate (5.5) are valid for \(u\) on \([0, \infty)\).

Now, by Eq. (4.1) we have

\[
\Delta u_t = D^4_t u + \sigma'(u_t) D^4_t u + \sigma''(u_t)(u_t)^2 \\
+ g''(u)(u_t)^2 + g'(u) u_t - f_t \in C([0, \infty) ; L^2)
\]

and hence

\[
u_t \in C([0, \infty) ; H_2 \cap H^0) \text{ or } u \in C^2([0, \infty) ; H_2 \cap H^0).
\]   

(5.7)

Moreover, we can use the equation

\[
\Delta u_t = D^3_t u + \sigma'(u_t) u_t + g'(u) u_t - f_t \in C(R^+ ; H^0)
\]

to get

\[
u_t \in C(R^+ ; H_2 \cap H^0) \quad \text{or} \quad u \in C^1(R^+ ; H_2 \cap H^0).
\]   

(5.8)

Finally, we can check easily with the use of (5.7) that

\[
\Delta u = u_t + \sigma(u_t) + g(u) + f \in C([0, \infty) ; H_2)
\]

and hence

\[
u \in C([0, \infty) ; H_4 \cap H^0).
\]
The proof of Theorem 1.1 is now completed.

**Remark.** It is clear from the proof of the theorem that we can get the exponential decay estimate for $u$ in Theorem 1.1 in the following way:

$$\sum_{i=0}^{4} \|D^i u(t)\|_{H^{4-i}} \leq \tilde{Q}_4 e^{-\mu_4 t}$$

with some $\tilde{Q}_4 = \tilde{Q}_4(\|u_0\|_{H_4}, \|u_1\|_{H_4}) > 0$ and $\mu_4 > 0$ provided that

$$\sum_{i=0}^{2} \|D^i f(t)\|_{H^{2-i}} + \|D^3 f(t)\|_{L^2} \leq d_4 e^{-\lambda_4 t}$$

for some $\lambda_4 > 0$.

**REFERENCES**