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Whitney type inequalities for local anisotropic polynomial approximation

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Abstract

We prove a multivariate Whitney type theorem for the local anisotropic polynomial approximation in $L_p(Q)$ with $1 \le p \le \infty$. Here Q is a d-parallelepiped in $\mathbb{R}^d$ with sides parallel to the coordinate axes. We consider the error of best approximation of a function f by algebraic polynomials of fixed degree at most $r_i - 1$ in variable x_i, $i = 1, \ldots, d$, and relate it to a so-called total mixed modulus of smoothness appropriate to characterizing the convergence rate of the approximation error. This theorem is derived from a Johnen type theorem on equivalence between a certain K-functional and the total mixed modulus of smoothness which is proved in the present paper.

Keywords: Whitney type inequality; Anisotropic approximation by polynomials; Total mixed modulus of smoothness; Mixed K-functional; Sobolev space of mixed smoothness

1. Introduction and main results

The classical Whitney theorem establishes the equivalence between the modulus of smoothness $\omega_r(f, |I|)_{p,I}$ and the error of best approximation $E_r(f)_{p,I}$ of a function $f : I \to \mathbb{R}$ by algebraic polynomials of degree at most $r-1$, measured in L_p, $1 \le p \le \infty$, where $I := [a, b]$ is an interval in $\mathbb{R}$ and $|I| = b - a$ its length. Namely, the following inequalities

$$2^{-r}\omega_r(f, |I|)_{p,I} \le E_r(f)_{p,Q} \le C\omega_r(f, |I|)_{p,I} \tag{1.1}$$

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hold true with a constant C depending only on r. This result was first proved by Whitney [24] for $p = \infty$ and extended by Brudnyĭ [2] to $1 \le p < \infty$. The inequalities (1.1) provide, in particular, a convergence characterization for a local polynomial approximation when the degree $r - 1$ of polynomials is fixed and the interval I is small.

Several authors have dealt with this topic in order to extend and generalize the result in various directions. Let us briefly mention them. A multivariate (isotropic) generalization for functions on a coordinate d-cube Q in $\mathbb{R}^d$ was given by Brudnyĭ [3,4]. It turned out that the result is valid if one replaces the d-cube by a more general domain Ω. The case of a convex domain $\Omega \subset \mathbb{R}^d$ is already treated in [3]. Let us also refer to the recent contributions by Dekel and Leviatan [7] and Dekel [6] with focus on convex and Lipschitz domains and the improvement of the constants involved.

A reasonable question is also to ask for the case $0 < p < 1$. We refer to the works of Storozhenko [19], Storozhenko and Oswald [20], and in addition, to the appendix of the substantial paper by Hedberg and Netrusov [13] for a brief history and further references.

A natural question arises: Is there a Whitney type theorem for the anisotropic approximation of multivariate functions on a coordinate d-parallelepiped Q? Some work has been done in this direction; see for instance [12]. However, the present paper deals with a rather different setting, which is somehow related to the theory of function spaces with mixed smoothness properties [10,17,22,23]. We intend to approximate a multivariate function f by polynomials of fixed degree at most $r_i - 1$, in variable x_i, $i = 1, \ldots, d$, on a small d-parallelepiped Q. A total mixed modulus of smoothness is defined which turns out to be a suitable convergence characterization to this approximation. The classical Whitney inequality can be derived as a corollary of Johnen's theorem [14] on the equivalence of the rth Peetre K-functional $K_r(f, t^r)_{p,I}$ (see [16]) and the modulus of smoothness $\omega_r(f, t)_{p,I}$. A proof was given by Johnen and Scherer in [15]. Following this approach to Whitney type theorems, we will introduce the notion of a mixed K-functional and prove its equivalence to the total mixed modulus of smoothness by generalizing the technique of Johnen and Scherer to the multivariate mixed situation.

1.1. Notation

In order to give an exact setting of the problem and formulate the main results, let us preliminarily introduce some necessary notations. As usual, $\mathbb{N}$ is reserved for the natural numbers, by $\mathbb{Z}$ we denote the set of all integers, and by $\mathbb{R}$ the real numbers. Furthermore, $\mathbb{Z}_+$ and $\mathbb{R}_+$ denote the set of non-negative integers and real numbers, respectively. Elements x of $\mathbb{R}^d$ will be denoted by $x = (x_1, \ldots, x_d)$. For a vector $r \in \mathbb{Z}_+^d$ and $x \in \mathbb{R}^d$, we will further write

$$x^r := (x_1^{r_1}, \ldots, x_d^{r_d}).$$

Moreover, if $x, y \in \mathbb{R}^d$, the inequality $x \le y$ $(x < y)$ means that $x_i \le y_i$ $(x_i < y_i)$, $i = 1, \ldots, d$. As usual, the notation $A \ll B$ indicates that there is a constant $c > 0$ (independent of the parameters which are relevant in the context) such that $A \le cB$, whereas $A \asymp B$ is used if $A \ll B$ and $B \ll A$, respectively.

If $r \in \mathbb{N}^d$, let $\mathcal{P}_r$ be the set of algebraic polynomials of degree at most $r_i - 1$ at variable x_i, $i \in [d]$, where $[d]$ denotes the set of all natural numbers from 1 to d. We intend to approximate a function f defined on a d-parallelepiped

$$Q := [a_1, b_1] \times \cdots \times [a_d, b_d]$$

by polynomials from the class $\mathcal{P}_r$. If $D \subset \mathbb{R}^d$ is a domain in $\mathbb{R}^d$, we denote by $L_p(D)$, $0 < p \le \infty$, the quasi-normed space of Lebesgue measurable functions on D with the usual pth integral quasi-norm $\|\cdot\|_{p,D}$ to be finite, whereas, we use the ess sup norm if $p=\infty$. The error of best approximation of $f \in L_p(Q)$ by polynomials from $\mathcal{P}_r$ is measured by

$$E_r(f)_{p,Q} := \inf_{\varphi\in\mathcal{P}_r} \|f-\varphi\|_{p,Q}.$$

For $r \in \mathbb{Z}_+,\ h \in \mathbb{R}$, and a univariate functions f, the rth difference operator Δ_h^r is defined by

$$\Delta_h^r(f,x) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(x+jh), \qquad \Delta_h^0 f(x) := f(x),$$

whereas for $r \in \mathbb{Z}_+^d,\ h \in \mathbb{R}^d$ and a d-variate function $f:\mathbb{R}^d \to \mathbb{R}$, the mixed rth difference operator Δ_h^r is defined by

$$\Delta_h^r := \prod_{i=1}^{d} \Delta_{h_i,i}^{r_i}.$$

Here, the univariate operator $\Delta_{h_i,i}^{r_i}$ is applied to the univariate function f by considering f as a function of variable x_i with the other variables fixed. Let

$$\omega_r(f,t)_{p,Q} := \sup_{|h_i|\le t_i, i\in[d]} \|\Delta_h^r(f)\|_{p,Q_{rh}}, \quad t \in \mathbb{R}_+^d,$$

be the mixed rth modulus of smoothness of f, where for $y,h \in \mathbb{R}^d$, we write $yh := (y_1h_1,\dots,y_dh_d)$ and $Q_y := \{x \in Q : x_i, x_i+y_i \in [a_i,b_i],\ i \in [d]\}$. For $r \in \mathbb{Z}_+^d$ and $e \subset [d]$, denote by $r(e) \in \mathbb{Z}_+^d$ the vector with $r(e)_i = r_i, i \in e$ and $r(e)_i = 0, i \notin e$ $(r(\emptyset)=0)$. If $r \in \mathbb{N}^d$, we define the *total mixed modulus of smoothness of order r by*

$$\Omega_r(f,t)_{p,Q} := \sum_{e\subset[d], e\neq\emptyset} \omega_{r(e)}(f,t)_{p,Q}, \quad t \in \mathbb{R}_+^d.$$

This particular modulus of smoothness is not new. In the periodic context, the total mixed modulus of smoothness $\Omega_r(f,\cdot)_{\infty,Q}$ has been used in [5] for estimations of the convergence rate of the approximation of continuous periodic functions by rectangular Fourier sums. Moreover, $\Omega_r(f,\cdot)_{p,Q}$ is related to mixed moduli of smoothness necessary for characterizing function spaces with dominating mixed smoothness properties; see [10,17] and the recent contributions [22,23,21,11].

1.2. Main results

In the present paper, we generalize the Whitney inequality (1.1) to the error of best local anisotropic approximation $E_r(f)_{p,Q}$ by polynomials from $\mathcal{P}_r$ and the total mixed modulus of smoothness $\Omega_r(f,t)_{p,Q}$. More precisely, we prove the following Whitney type inequalities.

Theorem 1.1. *Let $1 \le p \le \infty,\ r \in \mathbb{N}^d$. Then there is a constant C depending only on r, d such that for every $f \in L_p(Q)$*

$$\left(\sum_{e\subset[d]} \prod_{i\in e} 2^{r_i}\right)^{-1} \Omega_r(f,\delta)_{p,Q} \le E_r(f)_{p,Q} \le C\,\Omega_r(f,\delta)_{p,Q}, \tag{1.2}$$

where $\delta = \delta(Q) := (b_1-a_1,\dots,b_d-a_d)$ is the size of Q.

Theorem 1.1 shows that the total mixed modulus of smoothness $\Omega_r(f,t)_{p,Q}$ gives a sharp convergence characterization of the best anisotropic polynomial approximation when r is fixed and the size $\delta(Q)$ of the d-parallelepiped Q is small. This may have applications in the approximation of functions with mixed smoothness by piecewise polynomials or splines.

So far we focus on the case $1 \le p \le \infty$. This makes it possible to apply a technique developed by Johnen and Scherer [15]. As mentioned above, they showed the equivalence of the Peetre K-functional of order r with respect to a classical Sobolev space W_p^r and the modulus of smoothness of order r for the univariate case. The question of a K-functional suitable for mixed Sobolev spaces has been often considered in the past. We refer, for instance, to [18,9]. By introducing a mixed K-functional $K_r(f,t)_{p,Q},\ t \in \mathbb{R}_+^d$ (see the definition in Section 3), such an equivalence between $K_r(f,t^r)_{p,Q}$ and the total mixed modulus of smoothness $\Omega_r(f,t)_{p,Q}$ can be established as well. Namely, we prove the following

Theorem 1.2. *Let $1 \le p \le \infty$ and $r \in \mathbb{N}^d$. Then for any $f \in L_p(Q)$, the following inequalities*

$$\left(\sum_{e\subset[d]}\prod_{i\in e}2^{r_i}\right)^{-1}\Omega_r(f,t)_{p,Q} \le K_r(f,t^r)_{p,Q} \le C\,\Omega_r(f,t)_{p,Q}, \quad t \in \mathbb{R}_+^d, \tag{1.3}$$

hold true with a constant C depending on r, p, d only.

The paper is organized as follows. In Section 2, we establish an error estimate for the anisotropic polynomial approximation for functions from Sobolev spaces of mixed smoothness. Section 3 is devoted to the equivalence of the total mixed modulus of smoothness and the mixed K-functional (Theorem 1.2) which is applied in Section 4 to derive the Whitney type inequality for the local anisotropic polynomial approximation (Theorem 1.1).

2. Anisotropic polynomial approximation in Sobolev spaces of mixed smoothness

By $f^{(k)},\ k \in \mathbb{Z}_+^d$, we denote the kth order generalized mixed derivative of a locally integrable function f, i.e.,

$$\int_Q f^{(k)}(x)\varphi(x)\,\mathrm{d}x = (-1)^{k_1+\cdots+k_d}\int_Q f(x)\frac{\partial^{k_1+\cdots+k_d}\varphi}{\partial x_1^{k_1}\cdots\partial x_d^{k_d}}(x)\,\mathrm{d}x$$

for all test functions $\varphi \in C_0^\infty(Q)$, where $C_0^\infty(Q)$ is the space of infinitely differentiable functions on Q with compact support, which is interior to Q. If a function f possesses sth locally integrable classical partial derivatives for all $s \le k$ on Q, then the kth generalized derivative of f coincides with the kth classical partial derivative. In this case, we identify both and use the same notation $f^{(k)}$.

For $r \in \mathbb{Z}_+^d$ and $1 \le p \le \infty$, the Sobolev space $W_p^r(Q)$ of mixed smoothness r is defined as the set of functions $f \in L_p(Q)$, for which the generalized derivative $f^{(r(e))}$ exists as a locally integrable function for all $e \subset [d]$, and the following norm is finite

$$\|f\|_{W_p^r(Q)} := \sum_{e\subset[d]}\|f^{(r(e))}\|_{p,Q}.$$

We aim at giving an upper bound of the error of best approximation of $f \in W_p^r(Q)$ by polynomials of degree $r_i - 1$ with respect to the variable $x_i,\ i = 1,\ldots,d$. For this purpose, we need some auxiliary lemmas. To begin with, we deal with univariate functions. The following lemma is proven in [8, page 38].

Lemma 2.1. *Let $1 \le p \le \infty$, $r \ge 1$ and $Q = [a, b]$. Then there exist constants C_1, C_2 depending only on r such that for $k = 0, \ldots, r-1$ and $0 \le t \le b-a$ the inequality*

$$t^k \|f^{(k)}\|_{p,Q} \le C_1(\|f\|_{p,Q} + t^r \|f^{(r)}\|_{p,Q}), \tag{2.1}$$

holds true for any $f \in W^r_p(Q)$.

Lemma 2.2. *Let $r \in \mathbb{Z}^d_+$, $1 \le p \le \infty$, and $Q = [0, b_1] \times \cdots \times [0, b_d]$ where $b_i > 0$, $i = 1, \ldots, d$. For fixed $f \in W^r_p(Q)$, $k \le r$, and $j \in [d]$ the univariate function*

$$g := f^{(k-k_j e_j)}(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_d)$$

belongs to $W^{r_j}_p([0, b_j])$ for almost all $x_i \in [0, b_i]$, $i \in [d] \setminus \{j\}$.

Proof. Let $\varphi_i \in C^\infty_0(0, b_i)$, $i = 1, \ldots, d$, be arbitrary smooth compactly supported functions. Clearly, the tensor product $\Phi(x_1, \ldots, x_d) := \prod_{i \in [d]} \varphi_i(x_i)$ belongs to $C^\infty_0(Q)$. Then, for $0 \le \ell_j \le r_j$

$$\begin{aligned}
&\int_0^{b_1} \cdots \int_0^{b_{j-1}} \int_0^{b_{j+1}} \cdots \int_0^{b_d} \prod_{\substack{i \in [d] \\ i \ne j}} \varphi_i(x_i) \\
&\qquad \times \left(\int_0^{b_j} f^{(k-k_j e_j)}(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_d) \varphi_j^{(\ell_j)}(t) \mathrm{d}t \right) \prod_{\substack{i \in [d] \\ i \ne j}} \mathrm{d}x_i \\
&= \int_0^{b_1} \cdots \int_0^{b_d} f^{(k-k_j e_j)}(x_1, \ldots, x_d)\, \Phi^{(0,\ldots,\ell_j,0,\ldots,0)}(x_1, \ldots, x_d) \mathrm{d}x_1, \ldots, \mathrm{d}x_d \\
&= (-1)^{\ell_j} \int_0^{b_1} \cdots \int_0^{b_d} f^{(k+e_j(\ell_j-k_j))}(x_1, \ldots, x_d)\, \Phi(x_1, \ldots, x_d) \mathrm{d}x_1, \ldots, \mathrm{d}x_d \\
&= \int_0^{b_1} \cdots \int_0^{b_{j-1}} \int_0^{b_{j+1}} \cdots \int_0^{b_d} \prod_{\substack{i \in [d] \\ i \ne j}} \varphi_i(x_i) \\
&\qquad \times \left(\int_0^{b_j} f^{(k+e_j(\ell_j-k_j))}(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_d) \varphi_j(t) \mathrm{d}t \right) \prod_{\substack{i \in [d] \\ i \ne j}} \mathrm{d}x_i.
\end{aligned}$$

This implies the coincidence of the dt-integrals in the first and last line almost everywhere (with respect to x_i, $i \in [d] \setminus \{j\}$). Therefore, the generalized derivatives of order ℓ_j exist as a locally integrable function, in fact, they coincide with $f^{(k+e_j(\ell_j-k_j))}(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_d)$. This is a function from $L_p([0, b_j])$ (almost everywhere with respect to x_i) since f belongs to $W^r_p(Q)$. Therefore, we have $g \in W^{r_j}_p([0, b_j])$. $\square$

The following result is interesting on its own. It generalizes the content of [8, Theorem 5.3] to the multivariate situation. The statement is not very surprising and probably known. However, since we did not find a proper reference in the literature, a proof is provided.

Lemma 2.3. *Let $r \in \mathbb{N}^d$ and $Q = [0, b_1] \times \cdots \times [0, b_d]$. Let further $f \in L_1(Q)$ such that $f^{(r(e))} = 0$ for all non-empty subsets $e \subset [d]$. Then f coincides almost everywhere with a polynomial P of degree $r-1$, i.e., $f \in \mathcal{P}_r$.*

Proof. For simplicity reasons, we give a proof for $d = 2$, so let $Q = [0, b_1] \times [0, b_2]$. We follow the inductive argument in the proof of the corresponding one-dimensional statement [8, Theorem 5.3]. The latter and Lemma 2.2 imply the statement in case $r = (1, 1)$. Assume now that it is proven for some $r \in \mathbb{N}^2$. Put $\bar{r} = (r_1 + 1, r_2)$ without loss of generality. We will prove that the assumption

$$f^{(\bar{r}(e))} = 0 \text{ for all non-empty subsets } e \subset [d] \tag{2.2}$$

implies that f coincides almost everywhere with a polynomial $P \in \mathcal{P}_{\bar{r}}$. To do this, we need to construct special test functions. Choose a function $\psi \in C_0^\infty(Q)$ arbitrarily and let $h \in C_0^\infty([0, b_1])$ be a univariate function such that $\int_0^{b_1} h(t)\mathrm{d}t = 1$. We define the functions

$$\begin{aligned} \varphi(x_1, x_2) &:= \psi(x_1, x_2) - h(x_1) \int_0^{b_1} \psi(s, x_2)\mathrm{d}s, \\ \Phi(x_1, x_2) &:= \int_0^{x_1} \varphi(s, x_2)\mathrm{d}s. \end{aligned} \tag{2.3}$$

This construction gives immediately $\Phi \in C_0^\infty(Q)$. By our assumption (2.2), we have in particular

$$\begin{aligned} 0 &= \int_Q \Phi^{(r_1+1,0)} f \mathrm{d}x_1\mathrm{d}x_2 \\ &= \int_Q \psi^{(r_1,0)} f \mathrm{d}x_1\mathrm{d}x_2 - \int_Q f(x_1, x_2)h^{(r_1)}(x_1) \int_{s=0}^{b_1} \psi(s, x_2)\mathrm{d}s\mathrm{d}x_1\mathrm{d}x_2 \\ &= \int_Q \psi^{(r_1,0)} f \mathrm{d}x_1\mathrm{d}x_2 - \int_Q f(x_1, x_2)h^{(r_1)}(x_1) \int_{s=0}^{b_1} \psi^{(r_1,0)}(s, x_2)\frac{s^{r_1}}{r_1!}\mathrm{d}s\mathrm{d}x_1\mathrm{d}x_2 \\ &= \int_{s=0}^{b_1}\int_{x_2=0}^{b_2} \psi^{(r_1,0)}(s, x_2) \cdot \left(f(s, x_2) - \frac{s^{r_1}}{r_1!} \int_{x_1=0}^{b_1} f(x_1, x_2)h^{(r_1)}(x_1)\mathrm{d}x_1 \right)\mathrm{d}x_2\mathrm{d}s \\ &= \int_Q \psi^{(r_1,0)}(s, x_2) \cdot \left(f(s, x_2) - \frac{s^{r_1}}{r_1!} \int_{x_1=0}^{b_1} f(x_1, x_2)h^{(r_1)}(x_1)\mathrm{d}x_1 \right)\mathrm{d}s\mathrm{d}x_2. \end{aligned} \tag{2.4}$$

Analogously we see

$$\begin{aligned} 0 &= \int_Q \Phi^{(r_1+1,r_2)} f \mathrm{d}x_1\mathrm{d}x_2 \\ &= \int_Q \psi^{(r_1,r_2)}(s, x_2) \cdot \left(f(s, x_2) - \frac{s^{r_1}}{r_1!} \int_{x_1=0}^{b_1} f(x_1, x_2)h^{(r_1)}(x_1)\mathrm{d}x_1 \right)\mathrm{d}s\mathrm{d}x_2. \end{aligned} \tag{2.5}$$

Using (2.2) once more, we get for any $s \in [0, b_1]$

$$\int_0^{b_1}\int_0^{b_2} h^{r_1}(x_1)\psi^{(0,r_2)}(s, x_2) f(x_1, x_2)\mathrm{d}x_2\mathrm{d}x_1 = 0$$

which implies

$$0 = \int_Q \psi^{(0,r_2)}(s, x_2) \cdot \left(f(s, x_2) - \frac{s^{r_1}}{r_1!} \int_0^{b_1} f(x_1, x_2)h^{(r_1)}(x_1)\mathrm{d}x_1 \right)\mathrm{d}s\mathrm{d}x_2. \tag{2.6}$$

Since ψ was chosen arbitrarily, our induction hypothesis together with (2.4) and (2.5), (2.6) implies that the function

$$g(s,x_2) := f(s,x_2) - \frac{s^{r_1}}{r_1!}\int_0^{b_1} f(x_1,x_2)h^{(r_1)}(x_1)\mathrm{d}x_1$$

is a bivariate polynomial from $\mathcal{P}_r$. If we show that the univariate function

$$p(t) = \int_0^{b_1} f(x_1,t)h^{(r_1)}(x_1)\mathrm{d}x_1$$

is a polynomial of degree at most $r_2 - 1$, we prove that $f \in \mathcal{P}_{\bar{r}}$. Indeed, let $\varphi \in C_0^\infty([0,b_2])$ arbitrary, then

$$\int_0^{b_2} \varphi^{(r_2)}(t)p(t)\mathrm{d}t = \int_Q f(x_1,x_2)h^{(r_1)}(x_1)\varphi^{(r_2)}(x_2)\mathrm{d}x_1\mathrm{d}x_2 = 0 \tag{2.7}$$

by using (2.2) once more. This, together with [8, Theorem 5.3] imply that p is a univariate polynomial of degree at most $r_2 - 1$. The proof is finished in case $d = 2$. For $d > 2$, the argument is essentially the same. Note that in this situation, one needs an additional inductive step with respect to d to adapt the argument after (2.6). □

By using the previous result, we are now able to define a Taylor type polynomial via its integral representation. For simplicity, we restrict again to the case $d = 2$. A corresponding statement holds true in case $d > 2$, too. See Remark 2.6.

Lemma 2.4. *Let $r \in \mathbb{N}^2$, $1 \le p \le \infty$, and $f \in W_p^r(Q)$ for $Q = [0,b_1]\times[0,b_2]$. Then the function $P_r f$ defined by*

$$\begin{aligned} P_r f(x_1,x_2) := {} & f(x_1,x_2) - \int_0^{x_2} f^{(0,r_2)}(x_1,t)\frac{(x_2-t)^{r_2-1}}{(r_2-1)!}\mathrm{d}t \\ & - \int_0^{x_1} f^{(r_1,0)}(s,x_2)\frac{(x_1-s)^{r_1-1}}{(r_1-1)!}\mathrm{d}s \\ & + \int_0^{x_1}\int_0^{x_2} f^{(r_1,r_2)}(s,t)\frac{(x_1-s)^{r_1-1}}{(r_1-1)!}\frac{(x_2-t)^{r_2-1}}{(r_2-1)!}\mathrm{d}t\mathrm{d}s \end{aligned} \tag{2.8}$$

is well defined and coincides almost everywhere with a polynomial from $\mathcal{P}_r$.

Proof. Since f is from $W_p^r(Q)$, i.e., all the derivatives belong to $L_p(Q) \subset L_1(Q)$, the function $P_r f$ is well defined. We intend to apply Lemma 2.3 in order to obtain $P_r f \in \mathcal{P}_r$. Let us compute the derivatives $(P_r f)^{(r_1,0)}$, $(P_r f)^{(0,r_2)}$, and $(P_r f)^{(r_1,r_2)}$. Choose $\varphi \in C_0^\infty(Q)$ arbitrarily. We start with $(P_r f)^{(r_1,0)}$. By changing the order of integration, we get

$$\begin{aligned} \int_Q P_r f(x_1,x_2)\varphi^{(r_1,0)}\mathrm{d}x_1\mathrm{d}x_2 = {} & \int_Q f(x_1,x_2)\varphi^{(r_1,0)}(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 \\ & - \int_{t=0}^{b_2}\int_{x_1=0}^{b_1} f^{(0,r_2)}(x_1,t)\int_{x_2=t}^{b_2}\frac{(x_2-t)^{r_2-1}}{(r_2-1)!}\varphi^{(r_1,0)}(x_1,x_2)\mathrm{d}x_2\mathrm{d}x_1\mathrm{d}t \\ & - \int_{x_2=0}^{b_2}\int_{s=0}^{b_1} f^{(r_1,0)}(s,x_2)\int_{x_1=s}^{b_1}\frac{(x_1-s)^{r_1-1}}{(r_1-1)!}\varphi^{(r_1,0)}(x_1,x_2)\mathrm{d}x_1\mathrm{d}s\mathrm{d}x_2 \end{aligned}$$

$$+\int_{s=0}^{b_1}\int_{t=0}^{b_2} f^{(r_1,r_2)}(s,t)\int_{x_1=s}^{b_1}\int_{x_2=t}^{b_2}\frac{(x_1-s)^{r_1-1}}{(r_1-1)!}\frac{(x_2-t)^{r_2-1}}{(r_2-1)!}\varphi^{(r_1,0)}$$
$$\times(x_1,x_2)\mathrm{d}x_2\mathrm{d}x_1\mathrm{d}t\mathrm{d}s. \tag{2.9}$$

Integration by parts shows that

$$\int_{x_1=s}^{b_1}\frac{(x_1-s)^{r_1-1}}{(r_1-1)!}\varphi^{(r_1,0)}(x_1,x_2)\mathrm{d}x_1=(-1)^{r_1}\varphi(s,x_2) \tag{2.10}$$

and the third summand on the right-hand side of (2.9) can therefore be rewritten to

$$\begin{aligned}
&-\int_{x_2=0}^{b_2}\int_{s=0}^{b_1} f^{(r_1,0)}(s,x_2)\int_{x_1=s}^{b_1}\frac{(x_1-s)^{r_1-1}}{(r_1-1)!}\varphi^{(r_1,0)}(x_1,x_2)\mathrm{d}x_1\mathrm{d}s\mathrm{d}x_2\\
&=-(-1)^{r_1}\int_{x_2=0}^{b_2}\int_{s=0}^{b_1} f^{(r_1,0)}(s,x_2)\varphi(s,x_2)\mathrm{d}s\mathrm{d}x_2\\
&=-\int_{x_2=0}^{b_2}\int_{s=0}^{b_1} f(s,x_2)\varphi^{(r_1,0)}(s,x_2)\mathrm{d}s\mathrm{d}x_2
\end{aligned} \tag{2.11}$$

which cancels the first summand. Using (2.10) once more we can rewrite the last summand in (2.9) to

$$\begin{aligned}
&(-1)^{r_1}\int_{s=0}^{b_1}\int_{t=0}^{b_2} f^{(r_1,r_2)}(s,t)\int_{x_2=t}^{b_2}\frac{(x_2-t)^{r_2-1}}{(r_2-1)!}\varphi(s,x_2)\mathrm{d}x_2\mathrm{d}t\mathrm{d}s\\
&=\int_{s=0}^{b_1}\int_{t=0}^{b_2} f^{(0,r_2)}(s,t)\int_{x_2=t}^{b_2}\frac{(x_2-t)^{r_2-1}}{(r_2-1)!}\varphi^{(r_1,0)}(s,x_2)\mathrm{d}x_2\mathrm{d}t\mathrm{d}s
\end{aligned} \tag{2.12}$$

which cancels the second summand. Hence, we obtain $(P_r f)^{(r_1,0)} = 0$ since φ was chosen arbitrarily. A similar effect occurs if we deal with $\int_Q P_r f(x_1,x_2)\varphi^{(0,r_2)}\mathrm{d}x_1\mathrm{d}x_2$ which gives that also $(P_r f)^{(0,r_2)} = 0$. In case of $\int_Q P_r f(x_1,x_2)\varphi^{(r_1,r_2)}\mathrm{d}x_1\mathrm{d}x_2$, we easily see that both the (modified) second and third summand in (2.9) can be rewritten to the negative of the first summand. However, the (modified) last summand can be rewritten to the first summand itself. Finally, all four summands sum up to zero. □

Remark 2.5. The polynomial $P_r f$ in (2.8) can be identified with the bivariate Taylor polynomial

$$T_r f(x_1,x_2) := \sum_{k_2=0}^{r_2-1}\sum_{k_1=0}^{r_1-1} f^{(k_1,k_2)}(0,0)\frac{x_1^{k_1}}{k_1!}\frac{x_2^{k_2}}{k_2!} \tag{2.13}$$

in the following sense. If $r\in\mathbb{N}^2$, $1\le p\le\infty$, $Q=[0,b_1]\times[0,b_2]$, and $f\in W_p^r(Q)$, then f has continuous derivatives of order $k<r$. This result is implicitly contained in the book [1]. Indeed, it is a combination of multiparameter Sobolev averaging using product kernels in Section [1, 2.7.10] and [1, 3.13] with the estimates in [1, 3.10], especially [1, Theorem 3.10.4]. The condition involving r and k there, has to be replaced by the componentwise condition $k<r$. We omit the details. Consequently, it makes sense to define the Taylor polynomial (2.13). Integration by parts shows that $T_r f$ coincides almost everywhere with $P_r f$ in (2.8). Hence, for functions from $W_p^r(Q)$, we have the Taylor formula

$$T_r f(x_1,x_2) = f(x_1,x_2) - \int_0^{x_2} f^{(0,r_2)}(x_1,t)\frac{(x_2-t)^{r_2-1}}{(r_2-1)!}\mathrm{d}t$$

$$- \int_0^{x_1} f^{(r_1,0)}(s,x_2)\frac{(x_1-s)^{r_1-1}}{(r_1-1)!}\mathrm{d}s$$
$$+ \int_0^{x_1}\int_0^{x_2} f^{(r_1,r_2)}(s,t)\frac{(x_1-s)^{r_1-1}}{(r_1-1)!}\frac{(x_2-t)^{r_2-1}}{(r_2-1)!}\mathrm{d}t\mathrm{d}s. \tag{2.14}$$

Remark 2.6. Lemma 2.4 and the Taylor formula (2.14) have an obvious counterpart in d dimensions. Note that the sum in (2.14) is twice the iteration (componentwise) of the one-dimensional integral

$$T_r f(x) := f(x) - \int_0^x f^{(r)}(s)\frac{(x-s)^{r-1}}{(r-1)!}\mathrm{d}s. \tag{2.15}$$

The d-times iteration of this procedure results in a sum of iterated integrals where the number of integrals in every summand corresponds to a unique subset $e \subset [d]$. The sign in front is given by $(-1)^{|e|}$.

The following theorem states an upper bound for the error of best approximation of multivariate mixed Sobolev functions with respect to anisotropic polynomials. It turns out that $P_r f$ from (2.8) provides a good approximation of $f \in W^r_p(Q)$.

Theorem 2.7. *Let $1 \le p \le \infty$, $r \in \mathbb{N}^d$. Then there is a constant C depending only on r, d such that for every $f \in W^r_p(Q)$*

$$E_r(f)_{p,Q} \le C \sum_{e\subset[d], e\neq\emptyset} \prod_{i\in e} \delta_i^{r_i} \|f^{(r(e))}\|_{p,Q},$$

where $\delta = \delta(Q)$ is given as in Theorem 1.1.

Proof. For simplicity, we prove the theorem for the case $d = 2$ and $Q = [0, b_1] \times [0, b_2]$. Let now $f \in W^r_p(Q)$ be a bivariate function. By Hölder's and triangle inequality we obtain from (2.8) the following estimate

$$\|f - P_r f\|_{p,Q} \ll b_2^{r_2}\|f^{(0,r_2)}\|_{p,Q} + b_1^{r_1}\|f^{(r_1,0)}\|_{p,Q} + b_1^{r_1} b_2^{r_2}\|f^{(r)}\|_{p,Q}. \tag{2.16}$$

For the general case ($d > 2$) one has to take Remark 2.6 into account. □

3. Johnen type inequalities for mixed K-functionals

For $r \in \mathbb{N}^d$, the *mixed K-functional* $K_r(f,t)_{p,Q}$ is defined for functions $f \in L_p(Q)$ and $t \in \mathbb{R}^d_+$ by

$$K_r(f,t)_{p,Q} := \inf_{g\in W^r_p(Q)} \left\{ \|f-g\|_{p,Q} + \sum_{e\subset[d], e\neq\emptyset} \left(\prod_{i\in e} t_i\right) \|g^{(r(e))}\|_{p,Q} \right\}.$$

The following technical lemma needs a further notation. Let us assume $a_i \le c_i < d_i \le b_i$ for $i \in [d]$. We put $I^i = [a_i, b_i]$, $I^i_1 = [a_i, d_i]$, and $I^i_0 = [c_i, b_i]$ and further

$$Q_e := \prod_{i=1}^d I^i_{\chi_e(i)}, \tag{3.1}$$

where χ_e denotes the characteristic function of the set $e \subset [d]$.

Lemma 3.1. *Let $1 \le p \le \infty$ and $r \in \mathbb{N}^d$. Then for any $f \in L_p(Q)$, the inequality*

$$K_r(f, t^r)_{p,Q} \le C \sum_{e \subset [d]} K_r(f, t^r)_{p,Q_e}$$

holds true for all $t \in \mathbb{R}^d_+$ with $t_i \le d_i - c_i$, $i \in [d]$. The constant C only depends on r and d.

Proof. The proof is based on an iterative argument. The first step is to observe

$$\begin{aligned} Q &= Q_1 \cup Q_0 \\ &= \left(I^1_1 \times \prod_{i \in [d] \setminus \{1\}} I^i \right) \cup \left(I^1_0 \times \prod_{i \in [d] \setminus \{1\}} I^i \right) \end{aligned}$$

and to show that

$$K_r(f, t^r)_{p,Q} \ll K_r(f, t^r)_{p,Q_1} + K_r(f, t^r)_{p,Q_0}. \tag{3.2}$$

We start with an increasing function $\varphi \in C^\infty(\mathbb{R})$ such that

$$\varphi(s) = \begin{cases} 0 & \text{if } s < 0 \\ 1 & \text{if } s > 1. \end{cases}$$

Putting $h = d_1 - c_1$ and

$$\lambda(s) = \varphi\left(\frac{s - c_1}{h} \right), \quad s \in \mathbb{R},$$

we obtain a $C^\infty(\mathbb{R})$-function λ that equals zero on $[a_1, c_1]$, equals one on $[d_1, b_1]$, and is increasing on $[c_1, d_1]$. As a direct consequence, we get

$$\|\lambda^{(k)}\|_{\infty,\mathbb{R}} \le h^{-k} \|\varphi^{(k)}\|_{\infty,\mathbb{R}}, \quad k \in \mathbb{N}.$$

Let now $f \in W^r_p(Q)$ and $t \in \mathbb{R}^d_+$ with $t_i \le d_i - c_i$, $i \in [d]$. For arbitrary $g_1 \in W^r_p(Q_1)$ and $g_0 \in W^r_p(Q_0)$, put

$$\begin{aligned} g(x) &= \lambda(x_1) g_0(x) + (1 - \lambda(x_1)) g_1(x) \\ &= g_1(x) + \lambda(x_1)(g_0(x) - g_1(x)). \end{aligned}$$

First of all, the function g is defined on $Q_0 \cap Q_1 \subset Q$. We extend g by g_0 on $Q_0 \setminus Q_1$ and by g_1 on $Q_1 \setminus Q_0$ and denote the result also by g. By the construction of λ, this g belongs to $W^r_p(Q)$ and we have

$$\begin{aligned} \|f - g\|_{p,Q} &\le \|\lambda(x_1) f(x) - \lambda(x_1) g_0(x) + (1 - \lambda(x_1)) f(x) - (1 - \lambda(x_1)) g_1(x)\|_{p,Q} \\ &\le \|f - g_0\|_{p,Q_0} + \|f - g_1\|_{p,Q_1}. \end{aligned} \tag{3.3}$$

Furthermore, for any non-empty fixed subset $e \subset [d]$, we have

$$g^{(r(e))}(x) = g_1^{(r(e))}(x) + \sum_{k=0}^{r_1} \binom{r_1}{k} \lambda^{(r_1 - k)}(x_1)(g_1^{(k,\tilde{r}(e))}(x) - g_0^{(k,\tilde{r}(e))}(x))$$

on $Q_0 \cap Q_1$, where $\tilde{r}(e)$ denotes the vector $r(e \setminus \{1\})$. Hence, for any non-empty fixed subset $e \subset [d]$, we obtain

$$\left(\prod_{i \in e} t_i^{r_i} \right) \|g^{(r(e))}\|_{p,Q_0 \cap Q_1}$$

$$\ll \left(\prod_{i\in e} t_i^{r_i}\right)\left(\|g_1^{(r(e))}\|_{p,Q_0\cap Q_1} + \max_{0\le k\le r_1} h^{-(r_1-k)}\|g_1^{(k,\tilde r(e))} - g_0^{(k,\tilde r(e))}\|_{p,Q_1\cap Q_0}\right)$$

$$\ll \left(\prod_{i\in e\setminus\{1\}} t_i^{r_i}\right)\left(t_1^{r_1\chi_e(1)}\|g_1^{(r(e))}\|_{p,Q_0\cap Q_1}\right.$$

$$\left. + \max_{0\le k\le r_1}\left(\frac{t_1}{h}\right)^{r_1-k} t_1^k\|g_1^{(k,\tilde r(e))} - g_0^{(k,\tilde r(e))}\|_{p,Q_1\cap Q_0}\right). \tag{3.4}$$

We apply Lemma 2.1 together with Lemma 2.2 to obtain

$$t_1^k\|g_1^{(k,\tilde r(e))} - g_0^{(k,\tilde r(e))}\|_{p,Q_0\cap Q_1}$$
$$\ll \|g_1^{(0,\tilde r(e))} - g_0^{(0,\tilde r(e))}\|_{p,Q_0\cap Q_1} + t_1^{r_1}\|g_1^{(r(e))} - g_0^{(r(e))}\|_{p,Q_0\cap Q_1}.$$

Plugging this into (3.4) and taking $t_1 \le h$ into account gives in case $\tilde r(e) \ne 0$

$$\left(\prod_{i\in e} t_i^{r_i}\right)\|g^{(r(e))}\|_{p,Q_0\cap Q_1} \ll \left(\prod_{i\in e} t_i^{r_i}\right)\|g_0^{(r(e))}\|_{p,Q_0} + \left(\prod_{i\in e\setminus\{1\}} t_i^{r_i}\right)\|g_0^{(0,\tilde r(e))}\|_{p,Q_0}$$
$$+ \left(\prod_{i\in e} t_i^{r_i}\right)\|g_1^{(r(e))}\|_{p,Q_1} + \left(\prod_{i\in e\setminus\{1\}} t_i^{r_i}\right)\|g_1^{(0,\tilde r(e))}\|_{p,Q_1}, \tag{3.5}$$

and in case $\tilde r(e) = 0$, i.e., $e = \{1\}$,

$$\left(\prod_{i\in e} t_i^{r_i}\right)\|g^{(r(e))}\|_{p,Q_0\cap Q_1} \ll \left(\prod_{i\in e} t_i^{r_i}\right)\|g_0^{(r(e))}\|_{p,Q_0} + \|f - g_0\|_{p,Q_0}$$
$$+ \left(\prod_{i\in e} t_i^{r_i}\right)\|g_1^{(r(e))}\|_{p,Q_1} + \|f - g_1\|_{p,Q_1}. \tag{3.6}$$

Using that

$$\|g^{(r(e))}\|_{p,Q} \le \|g^{(r(e))}\|_{p,Q_0\cap Q_1} + \|g_0^{(r(e))}\|_{p,Q_0} + \|g_1^{(r(e))}\|_{p,Q_1},$$

we obtain together with (3.3), (3.5) and (3.6) the relation

$$K_r(f, t^r)_{p,Q} \ll K_r(f, t^r)_{p,Q_0} + K_r(f, t^r)_{p,Q_1}$$

which is (3.2). We continue with the same procedure, this time with Q_1 and Q_0 instead of Q, proving that (analogously for Q_1)

$$K_r(f, t^r)_{p,Q_0} \ll K_r(f, t^r)_{p,Q_{01}} + K_r(f, t^r)_{p,Q_{00}},$$

where

$$Q_{00} = \left(I_0^1 \times I_0^2 \times \prod_{i\in[d]\setminus\{1,2\}} I^i\right) \quad \text{and} \quad Q_{01} = \left(I_0^1 \times I_1^2 \times \prod_{i\in[d]\setminus\{1,2\}} I^i\right),$$

and so forth. An iteration of this argument finishes the proof. $\square$

3.1. Proof of Theorem 1.2

Proof. The first inequality in (1.3) follows from the definition. Namely, if $f \in L_p(Q)$, for any non-empty $e \subset [d]$ and any $g \in W_p^r(Q)$, we have

$$\begin{aligned}\omega_{r(e)}(f,t)_{p,Q} &\le \omega_{r(e)}(f-g,t)_{p,Q} + \omega_{r(e)}(g,t)_{p,Q}\\ &\le \left(\prod_{i\in e} 2^{r_i}\right)\left\{\|f-g\|_{p,Q} + \left(\prod_{i\in e} t_i^{r_i}\right)\|g^{(r(e))}\|_{p,Q}\right\}.\end{aligned}$$

Indeed, the last inequality follows from the well-known relation

$$\|\Delta_h^m g\|_{p,I} \le 2^m |h|^m \|g^{(m)}\|_{p,I}$$

for univariate functions $g \in W_p^m(I)$, which is a simple consequence of the univariate Taylor formula (2.15) and the fact that $\Delta_h^m p \equiv 0$ for a univariate polynomial of degree less than m. We iterate this relation for any index in $i \in e$ using that for frozen variables $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d$, the univariate trace function $f(x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_d)$ belongs to the Sobolev space $W_p^{r_i}(I_i)$; see Lemma 2.2. This proves the first inequality in (1.3).

Let us prove the second one. For simplicity, we prove it for $d = 2$ and $t \in \mathbb{R}_+^2,\ t > 0$. If k is a natural number, then for univariate functions φ on the interval $[a, b]$, we define the operator $P_t^k,\ t \ge 0$, by

$$P_t^k(\varphi, x) := \varphi(x) + (-1)^{k+1} \int_{-\infty}^{\infty} \Delta_{th}^k(\varphi, x) M_k(h) \mathrm{d}h,$$

where M_k is the B-spline of order k with knots at the integer points $0, \dots, k$, and support $[0, k]$. The function $P_t^k(\varphi)$ is defined on $[a, b - h/4]$ for $t \le \bar{t} := h/4k^2$, where $h := b - a$. We have (see [8, page 177])

$$\{P_t^k(\varphi)\}^{(k)}(x) = t^{-k} \sum_{j=1}^{k} (-1)^{j+1} j^{-k} \Delta_{jt}^k(\varphi, x). \tag{3.7}$$

Put $h_i := b_i - a_i$ and $c_i := a_i + h_i/4,\ d_i := b_i - h_i/4, i \in [2]$. It holds $a_i < c_i < d_i < b_i$, and we will use the notation Q_e given in (3.1) for any $e \subset [d]$. In particular, we have $Q_{[2]} = [a_1, d_1] \times [a_2, d_2]$. For functions f on the parallelepiped $Q = [a_1, b_1] \times [a_2, b_2]$ the operator $P_t^r,\ t \in \mathbb{R}_+^2$, is defined by

$$P_t^r(f) := \prod_{i=1}^{2} P_{t_i,i}^{r_i}(f),$$

where the univariate operator $P_{t_i,i}^{r_i}$ is applied to the univariate function f by considering f as a function of variable x_i with the remaining variables fixed. The function $P_t^k(f)$ is defined on $Q_{[2]}$ for $t \le \bar{t}$, where $\bar{t}_i := h_i/4r_i^2$. We have

$$\begin{aligned}P_t^r(f, x) = {} & f(x) + (-1)^{r_1+1} \int_{-\infty}^{\infty} \Delta_{th}^{r^1}(f, x) M_{r_1}(h_1) \mathrm{d}h_1\\ &+ (-1)^{r_2+1} \int_{-\infty}^{\infty} \Delta_{th}^{r^2}(f, x) M_{r_2}(h_2) \mathrm{d}h_2\\ &+ (-1)^{r_1+r_2+2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Delta_{th}^{r}(f, x) M_{r_1}(h_1) M_{r_2}(h_2) \mathrm{d}h_1 \mathrm{d}h_2,\end{aligned}$$

where $r^1 := (r_1, 0)$ and $r^2 := (0, r_2)$. Let us define the function $g_t = P_t^r(f)$. If $f \in L_p(Q)$, by Minkowski's inequality and properties of the B-spline M_{r_i}, we get

$$\begin{aligned}\|f - g_t\|_{p,Q_{[2]}} &\ll \omega_{r^1}(f,t)_{p,Q_{[2]}} + \omega_{r^2}(f,t)_{p,Q_{[2]}} + \omega_r(f,t)_{p,Q_{[2]}} \\ &= \Omega_r(f,t)_{p,Q_{[2]}}.\end{aligned} \tag{3.8}$$

Further, by (3.7) we obtain

$$g_t^{(r^1)} = P_{t_2,2}^{r_2}(\{P_{t_1,1}^{r_1}(f)\}^{(r^1)}) = P_{t_2,2}^{r_2}\left\{t_1^{-r_1}\sum_{j_1=1}^{r_1}(-1)^{j_1+1} j_1^{-r_1}\binom{r_1}{j_1}\Delta_{j_1t_1,1}^{r_1}(f)\right\}.$$

Since $P_{t_2,2}^{r_2}$ is a linear bounded operator from $L_p(Q_{[2]})$ into $L_p(Q_{[2]})$ and further $\|\Delta_{j_1t_1,1}^{r_1}(f)\|_{p,Q_{[2]}} \ll \omega_{r^1}(f,t)_{p,Q_{[2]}}$, we have

$$t_1^{r_1}\|g_t^{(r^1)}\|_{p,Q_{[2]}} \ll \omega_{r^1}(f,t)_{p,Q_{[2]}}. \tag{3.9}$$

Similarly, we can prove that

$$t_2^{r_2}\|g_t^{(r^2)}\|_{p,Q_{[2]}} \ll \omega_{r^2}(f,t)_{p,Q_{[2]}}.$$

Again, by (3.7) we get

$$g_t^{(r)} = t_1^{-r_1}t_2^{-r_2}\sum_{j_1=1}^{r_1}\sum_{j_2=1}^{r_2}(-1)^{j_1+j_2+2} j_1^{-r_1} j_2^{-r_2}\binom{r_1}{j_1}\binom{r_2}{j_2}\Delta_{jt}^{r}(f).$$

From the inequality $\|\Delta_{jt}^r(f)\|_{p,Q} \ll \omega_r(f,t)_{p,Q_{[2]}}$ it follows that

$$t_1^{r_1}t_2^{r_2}\|g_t^{(r)}\|_{p,Q_{[2]}} \ll \omega_r(f,t)_{p,Q_{[2]}}. \tag{3.10}$$

Combining (3.8)–(3.10) gives

$$\|f - g_t\|_{p,Q_{[2]}} + t_1^{r_1}\|g_t^{(r^1)}\|_{p,Q_{[2]}} + t_2^{r_2}\|g_t^{(r^2)}\|_{p,Q_{[2]}} + t_1^{r_1}t_2^{r_2}\|g_t^{(r)}\|_{p,Q_{[2]}} \ll \Omega_r(f,t)_{p,Q}.$$

Therefore, we get

$$K_r(f,t^r)_{p,Q_{[2]}} \ll \Omega_r(f,t)_{p,Q},$$

and in a similar way

$$K_r(f,t^r)_{p,Q_e} \ll \Omega_r(f,t)_{p,Q}$$

for any subset $e \subset [2]$, where Q_e is given by (3.1). The last inequality and Lemma 3.1 prove (1.3) for $t \le \bar{t}$. Now take a function $\bar{g} \in W_p^r(Q)$ such that

$$\|f - \bar{g}\|_{p,Q} + \bar{t}_1^{r_1}\|\bar{g}^{(r^1)}\|_{p,Q} + \bar{t}_2^{r_2}\|\bar{g}^{(r^2)}\|_{p,Q} + \bar{t}_1^{r_1}\bar{t}_2^{r_2}\|\bar{g}^{(r)}\|_{p,Q} \ll \Omega_r(f,\bar{t})_{p,Q}. \tag{3.11}$$

By Theorem 2.7 we have

$$\|\bar{g} - T_r(\bar{g})\|_{p,Q} \ll \bar{t}_1^{r_1}\|\bar{g}^{(r^1)}\|_{p,Q} + \bar{t}_2^{r_2}\|\bar{g}^{(r^2)}\|_{p,Q} + \bar{t}_1^{r_1}\bar{t}_2^{r_2}\|\bar{g}^{(r)}\|_{p,Q}. \tag{3.12}$$

Since $T_r(\bar{g}) \in W_p^r(Q)$ and $(T_r(\bar{g}))^{(r(e))} = 0$ for every non-empty subset $e \subset [d]$, it holds for all $t > \bar{t}$

$$K_r(f,t^r)_{p,Q} \le \|f - T_r(\bar{g})\|_{p,Q}$$

$$\begin{aligned} &\le \|f - \bar{g}\|_{p,Q} + \|\bar{g} - T_r(\bar{g})\|_{p,Q} \\ &\ll \Omega_r(f, \bar{t})_{p,Q} \le \Omega_r(f, t)_{p,Q}, \end{aligned}$$

where the third step combines (3.11) and (3.12). Therefore, (1.3) has been proved for arbitrary $t > 0$. □

4. Whitney type inequalities

Using the results from Section 3 we are now able to prove Theorem 1.1.

Proof. The first inequality in (1.2) is trivial. Indeed, if $f \in L_p(Q)$ then for any non-empty $e \subset [d]$ and any $\varphi \in \mathcal{P}_r$ we have

$$\begin{aligned} \omega_{r(e)}(f, \delta)_{p,Q} &= \omega_{r(e)}(f - \varphi, \delta)_{p,Q} \\ &\le \left(\prod_{i \in e} 2^{r_i}\right) \|f - \varphi\|_{p,Q}. \end{aligned}$$

Hence, we obtain the first inequality in (1.2). On the other hand, from Theorem 2.7 it follows that for any $g \in W^r_p(Q)$

$$\begin{aligned} E_r(f)_{p,Q} &\le \|f - g\|_{p,Q} + E_r(g)_{p,Q} \\ &\le \|f - g\|_{p,Q} + \|g - T_r(g)\|_{p,Q} \\ &\ll \|f - g\|_{p,Q} + \left(\prod_{i \in e} \delta_i^{r_i}\right) \|g^{(r(e))}\|_{p,Q}. \end{aligned}$$

Hence, we get

$$E_r(f)_{p,Q} \ll K_r(f, \delta^r)_{p,Q}.$$

By Theorem 1.2 we have proved the second inequality in (1.2). □

The result in Theorem 1.1 can be slightly modified. For $r \in \mathbb{Z}^d_+, h \in \mathbb{R}^d, e \subset [d]$ and a d-variate function $f : \mathbb{R}^d \to \mathbb{R}$, the mixed p-mean modulus of smoothness of order $r(e)$ is given by

$$w_{r(e)}(f, t)_p := \left(\left(\prod_{i \in e} t_i^{-1}\right) \int_{U(t)} \int_{Q_{r(e)h}} |\Delta_h^{r(e)}(f, x)|^p \, \mathrm{d}x \, \mathrm{d}h\right)^{1/p}, \quad t \in \mathbb{R}^d_+,$$

where $U(t) := \{h \in \mathbb{R}^d : |h_i| \le t_i,\ i \in [d]\}$, with the usual change of the outer mean integral to sup if $p = \infty$. This leads to the definition of the *total mixed p-mean modulus of smoothness of order* $r \in \mathbb{N}^d$ by

$$W_r(f, t)_{p,Q} := \sum_{e \subset [d], e \ne \emptyset} w_{r(e)}(f, t)_{p,Q}, \quad t \in \mathbb{R}^d_+.$$

Note that $W_r(f, t)_{p,Q}$ coincides with $\Omega_r(f, t)_{p,Q}$ when $p = \infty$. In a way similar to the proof of Theorem 1.1, we can prove the following result.

Theorem 4.1. *Let $1 \le p \le \infty$, $r \in \mathbb{N}^d$. Then there are constants C, C' depending only on r, d such that for every $f \in L_p(Q)$*

$$C W_r(f, \delta)_{p,Q} \le E_r(f)_{p,Q} \le C' W_r(f, \delta)_{p,Q}$$

where $\delta = \delta(Q)$.

Remark 4.2. A corresponding inequality in the case $0 < p < 1$ is so far left open for subsequent contributions. It seems that the modulus $W_r(f, t)_{p,Q}$ is suitable to treat this case, cf. the appendix of [13].

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